Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 27, 2002, 173–182

REVERSE MARKOV INEQUALITY

Norman Levenberg and Evgeny A. Poletsky

University of Auckland, Department of Mathematics
Private Bag 92019, Auckland, New Zealand

Syracuse University, Department of Mathematics
215 Carnegie Hall, Syracuse, NY 13244, U.S.A.; eapolets@mailbox.syr.edu

Abstract. Let K be a compact convex set in $\mathbf{C}$. For each point $z_0 \in \partial K$ and each holomorphic polynomial $p = p(z)$ having all of its zeros in K, we prove that there exists a point $z \in K$ with $|z - z_0| \le 20 \operatorname{diam} K/\sqrt{\deg p}$ such that

$$|p'(z)| \ge \frac{(\deg p)^{1/2}}{20(\operatorname{diam} K)} |p(z_0)|;$$

i.e., we have a pointwise reverse Markov inequality. In particular,

$$\|p'\|_K \ge \frac{(\deg p)^{1/2}}{20(\operatorname{diam} K)} \|p\|_K.$$

1. Introduction

Let K be a compact set in the complex plane $\mathbf{C}$ and let $V_K(z)$ be the extremal function of K, i.e.,

$$V_K(z) = \max\left[0, \frac{1}{\deg p} \sup \log|p(z)|\right], \tag{1}$$

where the supremum is taken over all nonconstant holomorphic polynomials $p = p(z)$ with supremum norm $\|p\|_K = \sup_{z\in K} |p(z)| \le 1$. Suppose that the function V_K is Hölder continuous with exponent $0 < a \le 1$. Then from the Bernstein–Walsh inequality

$$|p(z)| \le \|p\|_K \exp[\deg p V_K(z)], \tag{2}$$

which follows from the definition of $V_K(z)$ in (1) (cf., [R]), and the Cauchy estimates, one obtains a *Markov inequality*: this estimates the size of the derivative

2000 Mathematics Subject Classification: Primary 41A17, 30C10; Secondary 26C05, 26D05.

E.A. Poletsky was partially supported by NSF Grant DMS-9804755.

of a polynomial p on K via the sup-norm $\|p\|_K$ of p, its degree, and the Hölder exponent a of V_K; namely,

$$\|p'\|_K \le C(K)(\deg p)^{1/a}\|p\|_K, \tag{3}$$

where $C(K)$ is a constant depending only on K. We outline a proof of (3); cf., [PP, Remark 3.2]. Given a polynomial p of degree n, take $z_0 \in K$ with $|p'(z_0)| = \|p'\|_K$. Apply the Cauchy estimates on the disk of radius $r = n^{-1/a}$ centered at z_0 to obtain

$$\|p'\|_K \le n^{1/a}\|p\|_{K_r}$$

where $K_r := \{z \in \mathbf{C} : \operatorname{dist}(z, K) \le r\}$. Since V_K is Hölder continuous with exponent a, (2) implies that

$$\|p\|_{K_r} \le \|p\|_K(1 + Mr^a)^n$$

for some constant $M = M(K)$. The choice of $r = n^{-1/a}$ then yields the result.

For example, if K is the unit disk, then (S.N. Bernstein)

$$\|p'\|_K \le (\deg p)\|p\|_K$$

and if K is the interval $[-1, 1]$, then (A.A. Markov)

$$\|p'\|_K \le (\deg p)^2\|p\|_K$$

(cf., [Lo]). Note that in these two examples we have precise knowledge of the constant: $C(K) = 1$, and this is sharp. In general, controlling $C(K)$ is difficult.

It is natural to ask whether one can improve the Markov inequality for some natural subclasses of polynomials. Indeed, P. Lax [L] proved a conjecture of P. Erdös that if all the zeros of p lie *outside* the open unit disk Δ, then

$$\|p'\|_{\bar\Delta} \le \tfrac{1}{2}\deg p\|p\|_{\bar\Delta}.$$

An example of $p(z) = z^n + 1$ shows that this is best possible.

However, in 1939, P. Turán [T] showed that if all the zeros of p lie in $\bar\Delta$, then a *reverse Markov inequality* holds:

$$\|p'\|_{\bar\Delta} \ge \tfrac{1}{2}\deg p\|p\|_{\bar\Delta}. \tag{4}$$

Again the same example of $p(z) = z^n + 1$ shows that it is sharp. Thus for this class the Markov inequality cannot be significantly improved.

Turán also proved in [T] that for the interval $I = [-1, 1]$ and polynomials with all their zeros in I, the reverse Markov inequality occurs in the following form:

$$\|p'\|_I \ge \tfrac{1}{6}(\deg p)^{1/2}\|p\|_I. \tag{5}$$

In this note, we are interested in a general form of the reverse Markov inequality for a polynomial p having all of its zeros in K:

$$\|p'\|_K \geq C(K)(\deg p)^b \|p\|_K. \tag{6}$$

Note that any polynomial $p(z) = cz^n + 1$, where $|c|$ is sufficiently small, provides a counterexample to any form of the reverse Markov inequality for polynomials with zeros *outside* K.

The inequality (6) contains two parameters: $C(K)$ and b. Since the holomorphic polynomials p with zeros in K are in one-to-one correspondence with the holomorphic polynomials $\tilde{p}$ with zeros in $aK := \{az \in \mathbf{C} : z \in K\}$ via $\tilde{p}(z) = p(z/a)$, $C(K)$ is inversely proportional to the diameter of K. We are interested in classes of compact sets K for which there is no other dependence on K for some value of b. Indeed, we show that for the class of R-*circular sets*, to be defined in the next section, which includes disks as well as circular arcs, the reverse Markov inequality holds with $b = 1$. For general *convex* sets we show that $b = \frac{1}{2}$. In both cases the value of b is best possible provided that $C(K)$ depends only on the diameter of K.

Moreover, we prove a *pointwise* version of (6):

$$|p'(z)| \geq C(K)(\deg p)^b |p(z)|,$$

where, in the case of R-circular sets, such an inequality is valid at *every* point $z \in \partial K$; while, in the case of general convex sets, given any point $z_0 \in \partial K$, we can find $z \in K$ lying within a distance $20 \operatorname{diam} K/\sqrt{\deg p}$ from z_0 where the above inequality holds.

Throughout the paper the following function will play a major role: given a polynomial $p(z) = c(z - z_1) \cdots (z - z_n)$, we define

$$\phi_p(z) = \phi(z) = \sum_{j=1}^{n} (z - z_j)^{-1}. \tag{7}$$

In the polynomial p (and thus in (7)), we allow repeated roots. Note that

$$p'(z) = p(z)\phi(z).$$

2. Reverse Markov for R-circular sets

We begin with the following result.

Proposition 2.1. *Let $K \subset \mathbf{C}$ be compact. Suppose that $z \in \partial K$ has the property that there exists a circle of radius $R = R(z, K)$ passing through z such* *that the closed disk it bounds contains K. Then for each complex polynomial p having all of its zeros in K,*

$$|p'(z)| \geq \frac{\deg p}{2R}|p(z)|.$$

Proof. Let p have degree n and zeros $z_1, \ldots, z_n$ in K (with repeated zeros listed as often as they occur). Since $p'(z) = p(z)\phi(z)$, it suffices to show that $|\phi(z)| \geq n/(2R)$ for $z \in \partial K$. Rotating and translating the z-variable (which does not affect the absolute value of the derivative), we may assume that $z = R$ and that the closed disk Δ_R of radius R centered at zero contains K. Then

$$|\phi(z)| = \left|\sum_{j=1}^{n}(R - z_j)^{-1}\right|$$

and the conformal mapping $w = 1/(R-z)$ maps Δ_R onto the half-plane $\operatorname{Re} w \geq 1/(2R)$. Thus

$$|\phi(z)| \geq \sum_{j=1}^{n} \operatorname{Re} \frac{1}{R - z_j} \geq \frac{n}{2R}$$

and we have

$$|p'(z)| \geq \frac{n}{2R}|p(z)|$$

for $z \in \partial K$. □

Motivated by the proposition, we make the following definition. A compact set $K \subset \mathbf{C}$ will be called R*-circular* if for every point $z \in \partial K$ there is a circle of radius R passing through z such that the closed disk it bounds contains K.

Theorem 2.2. *If K is a compact R-circular set in $\mathbf{C}$, then for each complex polynomial p having all of its zeros in K, and for every point $z \in \partial K$*

$$|p'(z)| \geq \frac{\deg p}{2R}|p(z)|.$$

As a corollary we obtain the first result of Turán.

Corollary 2.3. *For a complex polynomial p having all of its zeros in $\bar{\Delta}$, and for every point $z \in \partial\Delta$,*

$$|p'(z)| \geq \tfrac{1}{2}\deg p|p(z)|.$$

In particular,

$$\|p'\|_{\bar{\Delta}} \geq \tfrac{1}{2}\deg p\|p\|_{\bar{\Delta}}.$$

Remark. This inequality is not valid when the zeros lie very close to $\partial\Delta$ but outside $\bar{\Delta}$. Indeed, take $p(z) = z^n - n^2$. The zeros of p are the n-th roots of n^2 and hence all have modulus $n^{2/n}$ which tends to 1 as $n \to \infty$. However, $\|p\|_{\bar{\Delta}} = n^2 + 1$ while $\|p'\|_{\bar{\Delta}} = n$.

Remark. There exist non-convex compact sets satisfying the hypothesis of the corollary. For example, let K be an arc of a circle. But as we will see right now this estimate fails on an interval.

3. Reverse Markov inequality for convex sets

What happens for general *convex* compact sets K? First of all, we simplify the proof of Turan's second result (5) concerning the real interval $I := [-1, 1]$.

Proposition 3.1. *Let $p = p(x)$ be a real polynomial of degree n with real zeros $x_1, \dots, x_n \in I$. Then*

$$\|p'\|_I \geq \frac{\sqrt{n}}{2\sqrt{e}} \|p\|_I. \tag{8}$$

Proof. Order the zeros $-1 \leq x_1 \leq \cdots \leq x_n \leq 1$. Fix $x_0 \in I$ with $|p(x_0)| = \|p\|_I$. We may assume that $p(x_0) > 0$. For some $m \in \{1, \dots, n-1\}$, $x_0 \in (x_m, x_{m+1})$ or else $x_0 \in [0, x_1) \cup (x_n, 1]$.

In this last case, Proposition 2.1 tells us that

$$|p'(x_0)| \geq \tfrac{1}{2} n p(x_0).$$

Thus we may assume that $x_0 \in (x_m, x_{m+1})$.

Since $p' = p\phi$, for $x \in (x_m, x_{m+1})$ we have

$$p(x) = p(x_0) \exp\left(\int_{x_0}^{x} \phi(t)\, dt \right).$$

Hence

$$p'(x) = p(x_0)\phi(x) \exp\left(\int_{x_0}^{x} \phi(t)\, dt \right)$$

or, integrating by parts,

$$p'(x) = p(x_0)\phi(x) \exp\left((x - x_0)\phi(x) - \int_{x_0}^{x} (t - x_0)\phi'(t)\, dt \right). \tag{9}$$

Now since $|x - x_j| \leq 2$,

$$\phi'(x) = -\sum_{j=1}^{n} \frac{1}{(x - x_j)^2} \leq -\frac{n}{4}. \tag{10}$$

Therefore, for $x_0 \leq x \leq x_{m+1}$ we have

$$\phi(x) \leq -\tfrac{1}{4} n (x - x_0) \tag{11}$$

because $\phi(x_0) = 0$.

Since $\phi(x)$ approaches $-\infty$ as x approaches x_{m+1} from the left, there exists some point $x \in (x_0, x_{m+1})$ at which $\phi(x) = -1/(x-x_0)$. From (11), we conclude that $1/(x-x_0) \geq \frac{1}{4}n(x-x_0)$ at this point; i.e.,

$$x - x_0 \leq 2/\sqrt{n}. \tag{12}$$

Thus, at this point x, since $\phi(x) = -1/(x-x_0)$, using (9) we obtain

$$p'(x) = -\frac{p(x_0)}{x-x_0}\exp\left(-1 - \int_{x_0}^{x}(t-x_0)\phi'(t)\,dt\right).$$

By (10) and (12) the integral

$$\int_{x_0}^{x}(t-x_0)\phi'(t)\,dt \leq -\frac{n}{4}\int_{x_0}^{x}(t-x_0)\,dt = -\frac{n(x-x_0)^2}{8}.$$

Thus

$$|p'(x)| \geq \frac{p(x_0)}{x-x_0}\exp\left(-1 + \frac{n(x-x_0)^2}{8}\right).$$

The function of x in the right side of the inequality above is decreasing when $0 < x - x_0 \leq 2/\sqrt{n}$ and attains its minimum when $x - x_0 = 2/\sqrt{n}$. Therefore

$$|p'(x)| \geq \frac{p(x_0)\sqrt{n}}{2\sqrt{e}}. \square$$

The polynomial $p(x) = (x^2-1)^{n/2}$ (n even) shows that the $\frac{1}{2}$ power of n is the correct exponent in this inequality. However, the constant is not sharp. Indeed, Erod [E] proved a sharp result for each degree.

We proceed to show that a reverse Markov inequality with exponent $\frac{1}{2}$ is valid for general convex compact sets.

Theorem 3.2. *For any convex compact set K in $\mathbf{C}$, any polynomial $p = p(z)$ of degree n with all of its zeros in K, and any point $z_0 \in \partial K$, there is a point $z \in K$ with $|z - z_0| \leq 20(\operatorname{diam} K)/\sqrt{n}$ such that*

$$|p'(z)| \geq \frac{\sqrt{n}}{20(\operatorname{diam} K)}|p(z_0)|.$$

In particular,

$$\|p'\|_K \geq \frac{\sqrt{n}}{20(\operatorname{diam} K)}\|p\|_K.$$

Proof. For simplicity, we assume the diameter of K is 1. Fix $p = p(z)$ of degree n with all of its zeros $z_1, \ldots, z_n$ in K and let $z_0 \in \partial K$. We may assume that $z_0 = 0$. We want to prove that there is a constant $C > 0$—we will see that $C = 1/20$ will work—such that

$$|p'(z)| \geq C\sqrt{n}\,|p(0)|$$

for some point $z \in K$ with $|z| \leq r_n := 1/\bigl(C\sqrt{n}\bigr)$. We may assume that the set K lies in the upper half plane H. Then the points $w_k := -1/z_k$ lie in H as well. We fix the angle $\alpha = \pi/12$; note that

$$\tfrac{1}{4} < \sin\alpha < \tfrac{1}{3}.$$

We divide H into the sector $S_1 := \{w \in H : \alpha < \arg w < \pi - \alpha\}$ and $S_2 = H \setminus S_1$. As before, we set

$$\phi(z) := \sum_{k=1}^{n} \frac{1}{z - z_k} = \phi_1(z) + \phi_2(z),$$

where

$$\phi_1(z) = \sum{}' \frac{1}{z - z_k} \quad \text{and} \quad \phi_2(z) = \sum{}'' \frac{1}{z - z_k}.$$

Here $\sum'$ denotes the sum over k with $w_k \in S_1$ and $\sum''$ denotes the sum over k with $w_k \in S_2$.

Let n_1 be the number of of points w_k in S_1 and let n_2 be the number of points w_k in S_2. Since $\operatorname{Im} w_k \geq 0$, $\operatorname{Im} w_k \geq |w_k| \sin\alpha$ when $w_k \in S_1$, and since the diameter of K is 1 implies $|w_k| \geq 1$, we see that

$$|\phi(0)| \geq \sum_{k=1}^{n} \operatorname{Im} w_k \geq \sin\alpha \sum{}' |w_k| \geq \tfrac{1}{4} a \geq \tfrac{1}{4} n_1, \tag{13}$$

where $a = \sum' |w_k|$.

We let d denote the distance from $z_0 = 0$ to the nearest zero of p; i.e., $d = \min\{|z_1|, \ldots, |z_n|\}$. Let $t = d \sin\alpha$. We want to estimate the values of

$$\phi'(z) = \phi_1'(z) + \phi_2'(z) = -\sum{}' \frac{1}{(z - z_k)^2} - \sum{}'' \frac{1}{(z - z_k)^2}$$

when $|z| \leq t$.

Note that for such z,

$$|z_k|(1 - \sin\alpha) \leq |z - z_k| \leq |z_k|(1 + \sin\alpha).$$

Hence

$$\begin{aligned} |\phi_1'(z)| &= \left| \sum{}' \frac{1}{(z - z_k)^2} \right| \leq \sum{}' \frac{1}{|z - z_k|^2} \\ &\leq \frac{1}{(1 - \sin\alpha)^2} \left(\sum{}' \frac{1}{|z_k|^2} \right) \leq \frac{1}{(1 - \sin\alpha)^2} \left(\sum{}' \frac{1}{|z_k|} \right)^2 \leq \frac{9a^2}{4}. \end{aligned} \tag{14}$$

Next we consider

$$-\operatorname{Re}\phi_2'(z) = \operatorname{Re}\sum\nolimits'' \frac{1}{(z-z_k)^2}.$$

If $|z| \le t$ and $w_k \in S_2$, then $|\arg 1/z_k^2|$ does not exceed 2α. Since $|zw_k| \le \sin\alpha$, we see that $|\arg(1+zw_k)| \le \alpha$ and

$$\beta_k(z) := \left|\arg\frac{1}{(z-z_k)^2}\right| = \left|\arg\frac{1}{z_k^2(1+zw_k)^2}\right| \le 4\alpha.$$

Now

$$\operatorname{Re}\frac{1}{(z_k-z)^2} = \frac{1}{|z_k-z|^2}\cos\beta_k(z) \ge \frac{1}{|z_k|^2(1+\sin\alpha)^2}\cos 4\alpha.$$

Thus

$$\operatorname{Re}\sum\nolimits''\frac{1}{(z-z_k)^2} \ge \frac{\cos 4\alpha}{(1+\sin\alpha)^2}\sum\nolimits''\frac{1}{|z_k|^2} \ge \frac{9}{32}\sum\nolimits''\frac{1}{|z_k|^2}. \tag{15}$$

Recall again that the diameter of K is 1 implies $|w_k|^2 \ge 1$, so that

$$\sum\nolimits''\frac{1}{|z_k|^2} \ge n_2. \tag{16}$$

Plugging (16) into (15) we obtain

$$-\operatorname{Re}\phi_2'(z) = \operatorname{Re}\sum\nolimits''\frac{1}{(z-z_k)^2} \ge \frac{9}{32}n_2. \tag{17}$$

Fixing a constant $C > 0$ which we will specify later, recall that our goal is to prove that $|p'(z)| \ge C\sqrt{n}\,|p(0)|$ for some point $z \in K$ with $|z| \le r_n = 1/\big(C\sqrt{n}\big)$ (the conclusion of the theorem is that we can take $C = 1/20$). First of all, we note that it suffices to consider the case when $d > r_n$. For if $d \le r_n$, then $|z_k| \le 1/\big(C\sqrt{n}\big)$ for some k and $|p'(z)| \ge C\sqrt{n}\,|p(0)|$ for some point $z \in [0, z_k]$; this interval lies in K by convexity of K.

If $|\phi(0)| > C\sqrt{n}$, then $|p'(0)| = |p(0)\phi(0)| > C\sqrt{n}\,|p(0)|$ and the desired inequality is true. Thus we may assume that $|\phi(0)| \le C\sqrt{n}$, and, by (13), that both $\frac{1}{4}a$ and $\frac{1}{4}n_1$ are less than $C\sqrt{n}$.

Let $t_n = r_n\sin\alpha$. For the sake of obtaining a contradiction, we suppose that

$$|p'(z)| < C\sqrt{n}\,|p(0)| \quad \text{for all } z \in K \text{ satisfying } |z| < r_n. \tag{18}$$

Then for $|z| < t_n$ we find that

$$|p(z)| \ge |p(0)| - \left|\int_0^z p'(\zeta)\,d\zeta\right| > |p(0)|(1-\sin\alpha) > \tfrac{2}{3}|p(0)|.$$

We now take any point $z' \in K$ with $|z'| = \sin\alpha/\big(C\sqrt{n}\,\big)$. By the convexity of K and the assumption $d > r_n > t_n = |z'|$, such points exist. Since $n_2 = n - n_1 \geq n - 4C\sqrt{n}$, it follows from (17) and the inequality $\sin\alpha > \frac{1}{4}$ that

$$\left|\int_0^{z'} \phi_2'(z)\,dz\right| \geq \frac{9}{32}\big(n - 4C\sqrt{n}\,\big)\frac{\sin\alpha}{C\sqrt{n}} = \frac{9}{32}\left(\frac{\sqrt{n}}{4C} - 1\right). \tag{19}$$

Equation (14) and the inequalities $a < 4C\sqrt{n}$, $|z'| < 1/[3C\sqrt{n}\,]$ give

$$\left|\int_0^{z'} \phi_1'(z)\,dz\right| \leq \frac{9a^2}{4}|z'| < 12C\sqrt{n}\,. \tag{20}$$

Now

$$|\phi(z')| \geq \left|\int_0^{z'} \phi_2'(z)\,dz\right| - \left|\int_0^{z'} \phi_1'(z)\,dz\right| - |\phi(0)|.$$

Plugging (19) and (20) and the assumption $|\phi(0)| < C\sqrt{n}$ (from (18) into the latter inequality we obtain

$$|\phi(z')| > \frac{9}{32}\left(\frac{\sqrt{n}}{4C} - 1\right) - 12C\sqrt{n} - C\sqrt{n} = \sqrt{n}\left(\frac{9}{128C} - 13C\right) - \frac{9}{32}$$

and

$$|p'(z')| = |p(z')|\cdot|\phi(z')| > \frac{2}{3}\left[\sqrt{n}\left(\frac{9}{128C} - 13C\right) - \frac{9}{32}\right]|p(0)|.$$

If $C = 1/20$ then

$$|p'(z')| > .05\sqrt{n}\,|p(0)| = C\sqrt{n}\,|p(0)|$$

and this contradicts our assumption (18) since

$$|z'| = \sin\alpha/\big(C\sqrt{n}\,\big) < 1/\big[3C\sqrt{n}\,\big] < 1/\big[C\sqrt{n}\,\big] = r_n. \ \square$$

We end this section (and the paper) with a family of examples to show that for a star-shaped compact set the exponent b in a reverse Markov inequality can be arbitrarily small.

Proposition 3.3. *Given $\varepsilon > 0$, there exists a compact, star-shaped set $K \subset \mathbf{C}$ such that there is no constant $C > 0$ with*

$$\|p'\|_K \geq C(\deg p)^{\varepsilon}\|p\|_K$$

for all polynomials p.

Proof. Fix a positive integer m which will be chosen appropriately later and let $z_1,\dots,z_m$ be the m-th roots of unity. Consider the set consisting of m "spokes of a wheel"

$$K_m := \{tz_j : 0 \le t \le 1,\ j = 1,\dots,m\}.$$

For a positive integer n, let $p(z) = (z^m - 1)^n$. Then $\deg p = nm$; $\|p\|_{K_m} = 1$; and a calculation shows that

$$\|p'\|_{K_m} = mn\left(\frac{m-1}{mn-1}\right)^{(m-1)/m}\left[\frac{m-1}{mn-1} - 1\right]^n.$$

For n large,

$$\|p'\|_{K_m} \asymp m^{2-1/m}e^{-1+1/m}n^{1/m}.$$

Thus, taking $m > 1/\varepsilon$ proves the proposition. ◻

References

[E] Erod, J.: Bizonyus polinomok maximumanak. - Mat. Fiz. Lapok. 46, 1939, 58–82.

[L] Lax, P.D.: Proof of a conjecture of P. Erdös on the derivative of a polynomial. - Bull. Amer. Math. Soc. 50, 1944, 509–513.

[Lo] Lorentz, G.G.: Approximation of Functions. - Holt, Rinehart and Winston, New York, 1966.

[PP] Pawlucki, W., and W. Pleśniak: Markov's inequality and C^∞ functions on sets with polynomial cusps. - Math. Ann. 275, 1986, 467–480.

[R] Ransford, T.: Potential Theory in the Complex Plane. - Cambridge University Press, Cambridge, 1995.

[T] Turán, P.: Über die Ableitung von Polynomen. - Comp. Math. 7, 1939, 89–95.

Received 21 November 2000