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ON LEBESGUE POINTS OF FUNCTIONS IN THE SOBOLEV CLASS $W^{1,N}$

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Abstract. We present a method which appears to be useful in identifying the Lebesgue points for a function in the Sobolev class $W^{1,n}$. The key argument is applied for obtaining the uniqueness of asymptotic values, characterizing the approximate continuity, and for showing that functions satisfying a weak minimum principle possess Lebesgue points everywhere.

1. Introduction

This note deals with the pointwise behaviour of functions in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$, $n \geq 2$. The main ingredient of the paper is a method based on a somewhat non-standard combination of Sobolev type inequalities. This method, among others, implies that any function $u \in W^{1,n}(B)$ defined in the unit ball $B \subset \mathbf{R}^n$ cannot have two different asymptotic values at the boundary point of B (Theorem 3.1). We have not been able to locate this general result in the existing literature. Of course, the question concerning asymptotic values has a long history and much stronger results are true if u is assumed to be, for example, monotone in the sense of Lebesgue, see e.g. [7], [8], [10] and the references therein.

We present two other applications of the key argument. Firstly, it is shown that for functions in $W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ the points of approximate continuity coincide with the Lebesgue points (Theorem 3.4). The new feature here is that the approximate continuity implies the Lebesgue point property. Secondly, we prove that functions in $W^{1,n}_{\mathrm{loc}}(\Omega)$ satisfying a weak minimum principle in an open set $\Omega \subset \mathbf{R}^n$ possess Lebesgue points everywhere in Ω.

An elementary example shows that results do not hold in the same form for Sobolev functions $u \in W^{1,p}(\mathbf{R}^n)$, $p < n$. Another example shows that approximate continuity does not imply the n-fine continuity even for functions satisfying a weak minimum principle.

One should keep in mind that functions in $W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ always possess Lebesgue points outside an n-polar set. Thus our results could be regarded as kind of minimal conclusions concerning the pointwise behaviour in the exceptional set.

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2. Definitions and auxiliary lemmas

Throughout $n \geq 2$ and we use the notation $B_r = B(x_0, r)$ for $x_0 \in \mathbf{R}^n$ and $r > 0$. The boundary of the ball $B(x_0, r)$ is denoted by $\partial B(x_0, r)$ and the integral average of $u \in L^1(B_r)$ over B_r is denoted by

$$u_{B_r} = \fint_{B_r} u\,dx = \frac{1}{|B_r|}\int_{B_r} u\,dx.$$

Here we integrate with respect to the n-dimensional Lebesgue measure $|\cdot|$. The linear measure is denoted by $|\cdot|_1$.

Let $\Omega \subset \mathbf{R}^n$ be open. Then the *Sobolev space* $W^{1,n}(\Omega)$ consists of functions from the Lebesgue space $L^n(\Omega)$ for which $|\nabla u| \in L^n(\Omega)$. Here and elsewhere ∇u is the weak gradient of u. We write $u \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ if $u \in W^{1,n}(B)$ for all balls $B \subset \mathbf{R}^n$.

Recall next the definitions of Lebesgue point and approximate continuity. Suppose that $u\colon B(x_0, r) \to [-\infty, \infty]$ is integrable. Then x_0 is called a *Lebesgue point* for u, if

$$\lim_{r\to 0} \fint_{B_r} |u - u(x_0)|\,dx = 0.$$

The function u is called *approximately continuous* at x_0, if there is a set $E \subset \mathbf{R}^n$ with

$$\lim_{r\to 0} \frac{|E \cap B_r|}{|B_r|} = 1$$

such that the restriction $u|_E$ is continuous at x_0.

The key argument is based on certain consequences of Poincaré and Sobolev inequalities. These consequences are well known for specialists but we give brief proofs for the reader's convenience. Our first auxiliary lemma deals with functions vanishing on a large set.

Lemma 2.1. *Let $B \subset \mathbf{R}^n$ be a ball and let $u \in W^{1,n}(B)$. Suppose that*

$$\big|\{x \in B : |u(x)| > 0\}\big| \leq \alpha |B|$$

for some $0 < \alpha < 1$. Then there is a constant c depending only on n and α such that

$$\fint_B |u|^n\,dx \leq c \int_B |\nabla u|^n\,dx.$$

Proof. The following argument is taken from [3, Lemma 2.8]. By the Minkowski and the Poincaré inequalities, we have

$$\biggl(\int_B |u|^n\,dx\biggr)^{1/n} \leq c\biggl(|B| \int_B |\nabla u|^n\,dx\biggr)^{1/n} + |u_B|\,|B|^{1/n}.$$

By Hölder's inequality,

$$|u_B|\,|B|^{1/n} \le |B|^{1/n-1} \int_{\{x\in B:|u(x)|>0\}} |u|\,dx \le \Big(\int_B |u|^n\,dx\Big)^{1/n} \alpha^{1-1/n},$$

and the assertion easily follows. □

The other lemma concerns n-quasicontinuous representatives of Sobolev functions. Recall that $u\colon \Omega \to [-\infty,+\infty]$ is *n-quasicontinuous* on an open set $\Omega \subset \mathbf{R}^n$ if for every $\varepsilon > 0$ there is a set $E \subset \Omega$ such that $C_{1,n}(E) < \varepsilon$ and the restriction of u to $\Omega \setminus E$ is continuous. In this paper $C_{1,n}$ refers to the *Sobolev n-capacity*, see e.g. [1].

Lemma 2.2. *Let $x_0 \in \mathbf{R}^n$ and suppose that $u \in W^{1,n}(\mathbf{R}^n)$ is n-quasicontinuous. Then for any $R > \varepsilon > 0$ there is $A_\varepsilon \subset [0,R]$ such that the linear measure of A_ε satisfies $|A_\varepsilon|_1 \ge R-\varepsilon$ and*

$$\big(\operatorname{osc} u\big(\partial B(x_0,r)\big)\big)^n \le \frac{cr}{\varepsilon} \int_{B(x_0,R)} |\nabla u|^n\,dx$$

for all $r \in A_\varepsilon$. The constant c depends only on n.

Proof. A version of the Gehring oscillation lemma implies that u is continuous on $\partial B_r = \partial B(x_0,r)$ and the estimate

$$\big(\operatorname{osc} u(\partial B_r)\big)^n \le cr \int_{\partial B_r} |\nabla u|^n\,d\sigma \tag{2.1}$$

holds for a.e. $r \in]0,R[$, see [4, Lemma 2.1]. Here c depends only on n. The assertion follows from this by elementary estimates. For $r \in]0,R]$, let $\varphi(r) = \int_{\partial B_r} |\nabla u|^n\,d\sigma$, where σ is the surface measure on the sphere ∂B_r. Recall that

$$\int_{B_R} |\nabla u|^n\,dx = \int_0^R \varphi(r)\,dr.$$

For $\lambda > 0$, we have

$$\big|\{t \in [0,R] : \varphi(t) \le \lambda\}\big|_1 \ge R - \frac{1}{\lambda}\int_0^R \varphi(r)\,dr.$$

Choosing $\lambda = (1/\varepsilon)\int_0^R \varphi(r)\,dr$, $A_\varepsilon = \{t \in [0,R] : \varphi(t) \le \lambda\}$, we have $|A_\varepsilon|_1 \ge R-\varepsilon$ and each $r \in A_\varepsilon$ satisfies

$$\int_{\partial B_r} |\nabla u|^n\,d\sigma = \varphi(r) \le \lambda = \frac{1}{\varepsilon}\int_0^R \varphi(t)\,dt$$

assuming that $\lambda > 0$. Since the case $\lambda = 0$ is trivial, we conclude the assertion from (2.1). □

3. Lebesgue points and approximate continuity

We show first that functions in $W^{1,n}(B)$ cannot have two distinct asymptotic values at the boundary point x_0 of B. Such a property is well known for quasiconformal functions and for monotone functions in the Sobolev space $W^{1,n}(B)$; see [10, p. 181 and p. 189]. We state the result in general terms as follows.

Theorem 3.1. *Let $u \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ and let $E \subset \mathbf{R}^n$ with*

$$\beta = \liminf_{R\to 0^+} \frac{|\{r \in (0,R) : \partial B(x_0,r) \cap E \neq \emptyset\}|_1}{R} > 0. \tag{3.1}$$

If u has a finite limit α as $x \to x_0$ in E, then

$$\lim_{r\to 0^+} \fint_{B(x_0,r)} |u-\alpha|^n \, dx = 0.$$

Proof. Define $v \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ by $v(x) = |u(x) - \alpha|$ and denote

$$E_\varepsilon = \{x \in \Omega : v(x) \le \varepsilon\}$$

for $\varepsilon > 0$. We divide the integral average of v^n over B_r into two parts by writing

$$\fint_{B_r} v^n \, dx = \frac{1}{|B_r|} \int_{B_r \cap E_\varepsilon} v^n \, dx + \frac{1}{|B_r|} \int_{B_r \setminus E_\varepsilon} v^n \, dx.$$

The first term on the right-hand side is trivially estimated as

$$\frac{1}{|B_r|} \int_{B_r \cap E_\varepsilon} v^n \, dx \le \varepsilon^n.$$

To estimate the second term we write

$$\begin{aligned} \frac{1}{|B_r|} \int_{B_r \setminus E_\varepsilon} v^n \, dx &\le \frac{2^{n-1}}{|B_r|} \int_{B_r \setminus E_\varepsilon} \big((v-\varepsilon)^n + \varepsilon^n\big) \, dx \\ &\le 2^{n-1} \fint_{B_r} \big((v-\varepsilon)^+\big)^n \, dx + 2^{n-1}\varepsilon^n. \end{aligned}$$

It is not hard to see that Lemma 2.2 implies together with (3.1) the inequality

$$|B_r \cap E_\varepsilon| \ge c|B_r|$$

for all r small enough. Here $0 < c < 1$ depends only on n and β. Hence we may apply Lemma 2.1 and conclude that

$$\fint_{B_r} \big((v-\varepsilon)^+\big)^n \, dx \le c \int_{B_r} |\nabla v|^n \, dx$$

for all r small enough. It follows that

$$\frac{1}{|B_r|}\int_{B_r\setminus E_\varepsilon} v^n\,dx \le c\int_{B_r}|\nabla v|^n\,dx + 2^{n-1}\varepsilon^n$$

for a constant c depending only on n and β. Consequently

$$\fint_{B_r} v^n\,dx \le \varepsilon^n(1+2^{n-1}) + c\int_{B_r}|\nabla u|^n\,dx.$$

Letting $r\to 0$, we obtain

$$\limsup_{r\to 0}\fint_{B_r} v^n\,dx \le \varepsilon^n(1+2^{n-1}),$$

and the claim

$$\lim_{r\to 0}\fint_{B_r}|u-\alpha|^n\,dx = 0$$

follows. □

It is known that Theorem 3.1 does not hold for $p<n$. The following example is taken from [6].

Example 3.2. Define u in $\mathbf{R}^2$ by

$$u(z)=u(re^{i\theta})=\begin{cases}\theta & \text{for } 0\le\theta\le\frac12\pi,\\ \frac12\pi & \text{for } \frac12\pi\le\theta\le\pi,\\ \frac32\pi-\theta & \text{for } \pi\le\theta\le\frac32\pi,\\ 0 & \text{for } \frac32\pi\le\theta\le 2\pi.\end{cases}$$

Then $u\in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^2)$ for $p<2$, but there is an infinite number of asymptotic values in the origin.

Theorem 3.1 easily implies that the points of approximate continuity coincide with the Lebesgue points for functions in $W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$.

Theorem 3.3. *Let $u\in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ and let $x_0\in\mathbf{R}^n$. Then the following are equivalent*:

(i) $\displaystyle\lim_{r\to 0}\fint_{B_r} u\,dx = u(x_0)<\infty$,

(ii) $\displaystyle\lim_{r\to 0}\fint_{B_r}|u-u(x_0)|^n\,dx = 0$,

(iii) $\displaystyle\operatorname*{app-lim}_{x\to x_0} u(x) = u(x_0)<\infty$.

Proof. It is a trivial consequence of Hölder's inequality that (ii) implies (i). Also, it is well known that (ii) implies (iii), see [11, p. 190]. We show first that (i) implies (ii). Suppose that

$$\lim_{r\to 0} u_{B_r} = \lim_{r\to 0} \fint_{B_r} u\,dx = u(x_0).$$

By the Poincaré inequality,

$$\begin{aligned}\fint_{B_r} |u-u(x_0)|^n\,dx &\le 2^{n-1}\fint_{B_r} |u-u_{B_r}|^n\,dx + 2^{n-1}\fint_{B_r} |u_{B_r}-u(x_0)|^n\,dx \\ &\le C\int_{B_r} |\nabla u|^n\,dx + 2^{n-1}|u_{B_r}-u(x_0)|^n,\end{aligned}$$

and the claim (ii) easily follows. We conclude the assertion of the theorem by showing that (iii) implies (ii). Assume that

$$\operatorname*{app-lim}_{x\to x_0} u(x) = u(x_0) < \infty.$$

By definition, there is a set $E\subset \mathbf{R}^n$ with the n-dimensional measure density 1 at x_0 such that $u\to u(x_0)$ as $x\to x_0$ along the set E. It follows that

$$\lim_{R\to 0} \frac{|\{r\in(0,R):\partial B_r\cap E\neq\emptyset\}|_1}{R} = 1,$$

and (ii) holds by Theorem 3.1. □

Remark 3.4. (a) Suppose that $u\in L^1_{\mathrm{loc}}\bigl(B(x_0,r)\bigr)$ is bounded. Then it is not hard to see that the approximate continuity implies the Lebesgue point property, see [9, Remark 6.7]. Vuorinen [9, Theorem 6.13] has given sufficient conditions for the case in which the approximate limit property implies the angular limit property for functions in the half-space $\mathbf{R}^n_+$.

(b) Example 3.2 also shows that Theorem 3.3 does not hold for $p<n$. In fact, for u as in Example 3.2, the limit

$$\lim_{r\to 0} \fint_{B(0,r)} u\,dx$$

exists as a finite number. However, u is not approximately continuous in the origin. In particular, the origin is not a Lebesgue point for u.

We construct an example showing that the approximate continuity does not in general imply the n-fine continuity for functions in $W^{1,n}(\mathbf{R}^n)$, $n>2$. For the notion of n-fine topology, see [1] and the references therein.

Example 3.5. For each $j = 1, 2, \dots$, let $x_j = (2^{-j}, 0, \dots, 0)$, $R_j = 2^{-2j-2}$, $r_j = 2^{-3j-3}$, and define

$$u_j(x) = \begin{cases} \dfrac{\log R_j - \log|x - x_j|}{\log R_j - \log r_j} & \text{for } r_j < |x - x_j| < R_j, \\ 1 & \text{for } 0 \le |x - x_j| \le r_j, \\ 0 & \text{for } |x - x_j| \ge R_j. \end{cases}$$

The functions u_j are indeed extremal for the n-capacity of the spherical condenser $\bigl(\overline{B}\,(x_j, r_j), B(x_j, R_j)\bigr)$, see [1, pp. 35–36]. Define

$$u = \sum_{j=1}^{\infty} u_j.$$

A computation then yields

$$\int_{B(x_j,R_j)} |\nabla u|^n \, dx = \omega_{n-1}\Bigl(\log \frac{R_j}{r_j}\Bigr)^{1-n} = \omega_{n-1}\bigl((j+1)\log 2\bigr)^{1-n},$$

and therefore

$$\int_{\mathbf{R}^n} |\nabla u|^n \, dx = \sum_{j=1}^{\infty} \int_{B(x_j,R_j)} |\nabla u_j|^n \, dx = \omega_{n-1}(\log 2)^{1-n} \sum_{j=2}^{\infty} \frac{1}{j^{n-1}} < \infty$$

if only $n \ge 3$. Here ω_{n-1} is the surface measure of $\partial B(0,1)$. Thus $u \in W^{1,n}(\mathbf{R}^n)$.

A straightforward calculation shows that the n-dimensional density of the set $\bigcup_{j=1}^{\infty} B(x_j, R_j)$ is equal to zero in the origin. Hence u has the approximate limit zero in the origin. However, u does not have the n-fine limit zero in the origin. To see this, it is enough to prove that the set

$$E = \bigcup_{j=1}^{\infty} B(x_j, r_j)$$

is not n-thin in the origin, [1, p. 234]. Equivalently, see [2, pp. 120–121], it is sufficient to show that

$$\sum_{k=1}^{\infty} C_{1,n}\bigl(E \cap B(0, 2^{-k})\bigr)^{1/n-1} = \infty.$$

This follows from the estimate

$$\begin{aligned} C_{1,n}\bigl(B(x_j, r_j)\bigr)^{1/n-1} &\ge c\biggl(\Bigl(\log \frac{2}{r_j}\Bigr)^{1-n}\biggr)^{1/n-1} = c(\log 2^{3j+4})^{-1} \\ &= \frac{c}{(3j+4)\log 2}, \end{aligned}$$

since the constant c depends only on n, see [1, 2.41].

The function u in Example 3.5 satisfies the minimum principle in the Sobolev sense [6]. Hence Example 3.5 shows as well that the weak minimum principle is not enough to guarantee the pointwise n-fine continuity. On the other hand, the weak minimum principle does imply the pointwise approximate continuity. We finish the paper by proving this assertion. We again use the argument of Theorem 3.1 but also some facts about the n-fine topology are needed.

Theorem 3.6. *Let $\Omega \subset \mathbf{R}^n$ be open and suppose that $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ is such that for each domain $\Omega' \Subset \Omega$ the inequality $m \le u$ holds a.e. in Ω' whenever $m \in \mathbf{R}$ and $(m-u)^+ \in W^{1,n}_0(\Omega')$. Then u possesses Lebesgue points everywhere in Ω.*

Proof. Take any n-quasicontinuous representative of u and define it pointwise in Ω by

$$u(z) = \text{n-fine-}\liminf_{x\to z} u(x). \tag{3.3}$$

Then the resulting function u remains to be n-quasicontinuous [5, Theorem 2.145]. Denote $v(x) = \big(u(x) - u(x_0)\big)^+$, $E_\varepsilon = \{x \in \Omega : v(x) \le \varepsilon\}$ for each $\varepsilon > 0$. We have the estimate

$$|B_r \cap E_\varepsilon| \ge \tfrac{1}{2}|B_r| \tag{3.4}$$

for all r small enough. This follows from Lemma 2.2 since the weak minimum principle implies the existence of a radius $R > 0$ such that

$$\min v(\partial B_r) < \tfrac{1}{2}\varepsilon$$

for all $r < R$. In fact, if there is a decreasing sequence $r_i \to 0$, $i = 0, 1, \ldots$, such that $\min v(\partial B_{r_i}) \ge \frac{1}{2}\varepsilon$, we may apply the weak minimum principle in annuli together with the n-quasicontinuity to conclude that $v \ge \frac{1}{2}\varepsilon$ in B_{r_0} outside an n-polar set. This contradicts (3.3). The estimate (3.4) allows us to apply Lemma 2.1 and proceed exactly the same way as in the proof of Theorem 3.1 in order to achieve

$$\lim_{r\to 0} \fint_{B_r} \big(u - u(x_0)\big)^+\, dx = 0.$$

Hence it is enough to prove that

$$\lim_{r\to 0} \fint_{B_r} \big(u(x_0) - u\big)^+\, dx = 0.$$

To do this, observe first that u is locally bounded from below by the weak minimum principle. In fact, since u is n-quasicontinuous, there are lots of spheres

$\partial B(x_0, R) \subset \Omega$ such that u is continuous on $\partial B(x_0, R)$; see the proof of Lemma 2.2. Fix a number $m < u(x_0)$ such that $u \geq m$ in a neighbourhood of x_0. By (3.3), for each $\varepsilon > 0$, there is an n-fine neighbourhood V_ε of x_0 such that $u \geq u(x_0) - \varepsilon$ in V_ε. We have

$$\lim_{r \to 0} \frac{|B_r \setminus V_\varepsilon|}{|B_r|} = 0,$$

see e.g. [1, p. 230], and therefore

$$\begin{aligned} \fint_{B_r} \bigl(u(x_0) - u\bigr)^+ \, dx &= \frac{1}{|B_r|} \int_{B_r \cap V_\varepsilon} \bigl(u(x_0) - u\bigr)^+ \, dx + \frac{1}{|B_r|} \int_{B_r \setminus V_\varepsilon} \bigl(u(x_0) - u\bigr)^+ \, dx \\ &\leq \varepsilon \frac{|B_r \cap V_\varepsilon|}{|B_r|} + \bigl(u(x_0) - m\bigr) \frac{|B_r \setminus V_\varepsilon|}{|B_r|} \leq 2\varepsilon \end{aligned}$$

for all sufficiently small radii r. Thus x_0 is a Lebesgue point for u. □

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