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THE LAW OF THE ITERATED LOGARITHM FOR LOCALLY UNIVALENT FUNCTIONS

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Abstract. In this paper we prove a sharp version of the Makarov law of the iterated logarithm. In particular, we show that the constant in the right side of this law depends on an asymptotic behaviour of the integral means of the derivative of an analytic function. Also, we establish that this constant is equal to the asymptotic variance for some domains with fractal type boundaries.

Let f be an analytic and univalent function in the unit disk $D = \{|z| < 1\}$. Makarov [5] proved that there exists a universal constant $C > 0$ such that

$$\limsup_{r\to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log\bigl(1/(1-r)\bigr)\log\log\log\bigl(1/(1-r)\bigr)}} \le C\|\log f'\|_{\mathbf{B}} \tag{1}$$

for almost all ζ on $|\zeta| = 1$, where

$$\|\log f'\|_{\mathbf{B}} = |\log f'(0)| + \sup_{|z|<1}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|$$

is the Bloch norm. Pommerenke [8, p. 186] showed that this inequality is true for $C = 1$ and there is a univalent function for which the inequality is false for $C \le 0.685$. Therefore, this result is not far from being the best possible. Przytycki, Urbański and Zdunik [9] established that for some classes of domains with fractal type boundaries the equality holds with $\sqrt{\sigma^2}$ in the right side of (1) where

$$\sigma^2 = \frac{1}{2\pi}\limsup_{r\to 1}\frac{\int|\log f'|^2\,\mathrm{d}\theta}{\log\bigl(1/(1-r)\bigr)}$$

is the asymptotic variance. In the paper [9] the authors used another definition of the asymptotic variance. But, in fact, their definition is equal to our definition in the "fractal" case.

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The goal of this paper is to obtain a sharp version of the Makarov law of the iterated logarithm for locally univalent functions, i.e. for functions f for which $f'(z) \neq 0$, $z \in D$.

Let f be a locally univalent function in the unit disk D and p be a complex number. Then for all $\delta > 0$ we define

$$\beta_\delta(p) = \sup_{r \in [0,1)} \frac{\log\left[\delta \int_{|z|=r} |f'(z)^p| \mathrm{d}\theta\right]}{\log\big(1/(1-r)\big)}.$$

In other words $\beta_\delta(p)$ is the minimal number for which

$$\int |f'^p| \,\mathrm{d}\theta \leq \frac{1}{\delta}\left(\frac{1}{1-r}\right)^{\beta_\delta(p)}, \qquad 0 \leq r < 1.$$

If p is a real number then

$$\beta_\delta(p) \to \beta(p) \qquad \text{as } \delta \to 0,$$

where

$$\beta(p) = \limsup_{r \to 1} \frac{\log \int_{|z|=r} |f'(z)|^p \,\mathrm{d}\theta}{\log\big(1/(1-r)\big)}$$

is the classical integral means spectrum [8, p. 176].

It follows from the integral means spectrum concept ([2], [7], [8]) that there is a connection between geometric properties of domains and the integral means spectrum. On the other hand, Makarov [5] established that the law of the iterated logarithm is closely related to the boundary properties of conformal maps. This leads to the following natural question: *Is there a simple relation between the law of the iterated logarithm and the integral means spectrum*? A possible answer for this question is the following result which we will prove later.

Suppose f is a locally univalent function in the unit disk and $\delta > 0$. Then

$$\limsup_{r \to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log\big(1/(1-r)\big) \log\log\log\big(1/(1-r)\big)}} \leq 2 \limsup_{p \to 0} \frac{\sqrt{\beta_\delta(p)}}{|p|}$$

for almost all ζ on $|\zeta| = 1$.

It is more convenient for us to formulate this result in the form of the following

Theorem 1. *Suppose f is a locally univalent function in the unit disk. Then the following inequality holds*

$$(2) \qquad \limsup_{r \to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log\big(1/(1-r)\big) \log\log\log\big(1/(1-r)\big)}} \leq \sqrt{\sigma^2(0+)}$$

for almost all ζ *on* $|\zeta| = 1$, *where*

$$\sigma^2(\delta) = 4 \limsup_{p \to 0} \frac{\beta_\delta(p)}{|p|^2}.$$

We remark that Pommerenke's result with the constant $C = 1$ easily follows from our theorem because it is known [3] that if $\log f'$ is a Bloch function then $\sigma^2(0+) \le \| \log f' \|^2_{\mathbf{B}}$. Moreover, we will see that in many cases

$$\sigma^2(0+) = \sigma^2.$$

Further, it is convenient to use the following abbreviation:

$$\int_0^{2\pi} h(re^{i\theta})\, \mathrm{d}\theta \equiv \int h \, \mathrm{d}\theta$$

The next lemma can be deduced from Makarov's proof of the law of the iterated logarithm [5].

Lemma 1. *Let* C_k *be a sequence of positive numbers and* $C_k^{1/k} \to 1$ *as* $k \to \infty$. *If*

$$\int |\log f'|^{2n} \, d\theta \le C_n n! A^{2n} \log^n \frac{1}{1-r}$$

for all natural n *and for all* $r \in \big[1 - \exp(-\exp e^n), 1\big)$, *then*

$$\limsup_{r \to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log\big(1/(1-r)\big) \log\log\log\big(1/(1-r)\big)}} \le A$$

for almost all ζ *on* $|\zeta| = 1$.

Proof. We use Pommerenke's version of Makarov's law [8, p. 186]. Our proof is almost the same as Pommerenke's proof. Let us only remark that instead of $\int_e^\infty$ in his proof we have to consider $\int_{\exp(e^n)}^\infty$.

Proof of Theorem 1. Fix $\delta > 0$ and $\varepsilon > 0$. Then there exists $p_0 = p_0(\delta, \varepsilon) > 0$ such that

$$\int |f'^p| \, \mathrm{d}\theta \le \frac{1}{\delta} \left(\frac{1}{1-r} \right)^{(\sigma^2(\delta)+\varepsilon)|p|^2/4}$$

for $|p| < p_0$. This implies

$$\int I_0(t|\log f'|) = \int \frac{1}{2\pi t} \int_{|p|=t} |f'^p| \, |\mathrm{d}p| \, \mathrm{d}\theta = \frac{1}{2\pi t} \int_{|p|=t} \int |f'^p| \, \mathrm{d}\,\theta |\mathrm{d}p|$$
$$\le \frac{1}{\delta} \left(\frac{1}{1-r} \right)^{(\sigma^2(\delta)+\varepsilon)t^2/4}, \qquad t \in (0, p_0),$$

where

$$I_0(x)=\sum_{k=0}^{\infty}\left(\frac{x^2}{4}\right)^k/k!^2=\frac{1}{2\pi}\int_0^{2\pi}e^{x\cos\theta}\,\mathrm{d}\theta$$

is the modified Bessel function of order zero [1]. At the same time,

$$\int|\log f'|^{2n}\,\mathrm{d}\theta\le\frac{4^n}{t^{2n}}n!^2\int I_0(t|\log f'|)\,\mathrm{d}\theta\le\frac{4^n}{t^{2n}}n!^2\frac{1}{\delta}\left(\frac{1}{1-r}\right)^{(\sigma^2(\delta)+\varepsilon)t^2/4}.$$

Setting

$$t^2=\frac{4n}{(\sigma^2(\delta)+\varepsilon)\log\bigl(1/(1-r)\bigr)},\qquad n\le\log\log\log\frac{1}{1-r},$$

and using the identity

$$e=\left(\frac{1}{1-r}\right)^{1/\log(1/(1-r))}$$

we obtain

$$\int|\log f'|^{2n}\,\mathrm{d}\theta\le\frac{1}{\delta}n!^2e^n\frac{1}{n^n}\bigl(\sigma^2(\delta)+\varepsilon\bigr)^n\left(\log\frac{1}{1-r}\right)^n.$$

Applying Lemma 1, we get

$$\limsup_{r\to1-}\frac{|\log f'(r\zeta)|}{\sqrt{\log\bigl(1/(1-r)\bigr)\log\log\log\bigl(1/(1-r)\bigr)}}\le\sqrt{\sigma^2(\delta)+\varepsilon}$$

for almost all ζ on $|\zeta|=1$ and for all $\delta>0$, $\varepsilon>0$. Hence, this result is true for $\delta=0+$ and $\varepsilon=0$. This completes the proof.

If f is a univalent function and $f(D)$ is a domain with rectifiable boundary then inequality (2) is trivially sharp. Non-trivial examples, which show that this inequality is sharp, can be obtained by using lacunary series.

Let $\log f'=\sum_{k=1}^{\infty}a_kz^{n_k}$ be a lacunary series with bounded coefficients and $n_{k+1}/n_k\ge q>1$. Since $\log f'$ is a Bloch function then $\sigma^2(0+)<+\infty$ as was mentioned above. In the other direction, Makarov [6] showed that if $n_k=2^k$ and $a_k=1$ for all k then $\sigma^2(0+)>0$. Rohde [8] improved his result in the following sense. Suppose q is an integer, $n_k=q^k$ and $a_k=a>0$, then $\sigma^2(0+)\ge a^2/\log q$.

In [4] it was shown that if $q\ge2$ then

$$\sigma^2(0+)=\limsup_{r\to1}\frac{B^2}{\log\bigl(1/(1-r)\bigr)},$$

where $B^2=\sum_{k=1}^{\infty}|a_k|^2r^{2n_k}$.

We want to extend this result for the case $q>1$. To do this we need the following

Definition ([10]). We say that a lacunary series satisfies condition (ϱ, R) if it consists of blocks of terms of length R, separated by empty blocks of length ϱ.

Weiss [10] proved the following

Lemma 2. *Let* $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ *be a lacunary series satisfying condition* (ϱ, R), *where*

$$(q^{\varrho/3} - 1)^{-1} \le \tfrac{1}{4}(q-1) \qquad \text{and} \qquad q^{-R/3}(R+1)^2 \le \tfrac{1}{2}.$$

Let $M = \sup |a_k|$, $B^2 = \sum_{n=1}^{\infty} |a_k|^2 r^{2n_k}$, $r \in [0,1)$. *Then*

$$\pi e^{(1-ctR^2)t^2B^2/4} \le \int e^{t \operatorname{Re} f(re^{i\theta})}\, d\theta \le 3\pi e^{(1+ctR^2)t^2B^2/4}.$$

Denote by ϱ_0 the minimal positive number for which

$$(q^{\varrho_0/3} - 1)^{-1} \le \tfrac{1}{4}(q-1) \qquad \text{and} \qquad q^{-\varrho_0/3}(\varrho_0+1)^2 \le \tfrac{1}{2}.$$

Now, we can prove the following

Theorem 2. *Let* $\log f' = \sum_{k=1}^{\infty} a_k z^{n_k}$ *be a lacunary series with bounded coefficiens for which* $n_{k+1}/n_k \ge q > 1$. *Then*

$$\sigma^2(0+) = \limsup_{r\to 1} \frac{B^2}{\log\bigl(1/(1-r)\bigr)},$$

where $B^2 = \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k}$.

Proof. It is clear that we can represent $\log f'$ as $f_1 + f_2$, where f_1 satisfies the (ϱ_0, R)-condition and f_2 satisfies the (R, ϱ_0)-condition. Let us estimate B_1^2 and B_2^2. In fact, it is enough to estimate only B_2 because $B^2 = B_1^2 + B_2^2$. We have

$$B_2 \le M^2 \sum_{j=1}^{\infty} \sum_{k=jR+(j-1)\varrho_0}^{j(R+\varrho_0)} r^{2q^k} \le M^2 \varrho_0 \sum_{j=1}^{\infty} r^{2q^{Rj}} \le C \log \frac{1}{1-r} / R.$$

Evidently, without loss of generality, we can assume that $p = t$ is a real positive number. Setting $R = t^{-2/5}$, we have

$$\int |f'|^t \,\mathrm{d}\theta = \int e^{t \operatorname{Re} f_1 + t \operatorname{Re} f_2}\,\mathrm{d}\theta \le \Bigl(\int e^{\alpha t \operatorname{Re} f_1}\,\mathrm{d}\theta\Bigr)^{1/\alpha} \Bigl(\int e^{\beta t \operatorname{Re} f_2}\,\mathrm{d}\theta\Bigr)^{1/\beta},$$

where $\beta = t^{-1/5}$ and $\alpha = (1 - t^{1/5})^{-1}$. Applying Lemma 2 with f_1 and f_2 and the estimate for B_2 we obtain

$$\int |f'|^t \,\mathrm{d}\theta \le C e^{B^2 t^2/4} \Bigl(\frac{1}{1-r}\Bigr)^{Ct^{2+1/5}}.$$

Analogously,

$$\int |f'|^t \,\mathrm{d}\theta \ge C e^{B^2 t^2/4} (1-r)^{Ct^{2+1/5}}.$$

This concludes the proof.

Applying the law of the iterated logarithm for lacunary series [10] it is easy to show that if

$$\lim_{r\to 1} \frac{B^2}{\log\bigl(1/(1-r)\bigr)}$$

exists then the equality in (2) holds. Other examples which show that (2) is sharp come from the theory of Julia sets. The idea for using Julia sets in the theory of univalent functions is due to Carleson and Jones [2]. They conjectured that the basin of attraction of infinity for an iteration z^2+c for some c maximizes $\beta_{0+}(1)$ in the class Σ.

Let $F(z) = z^q + a_{q-1}z^{q-1} + \cdots$ be a polynomial of degree $q \ge 2$ and

$$\Omega = \{\zeta : F^{on}(\zeta) \to \infty \text{ as } n \to \infty\}$$

be the basin of attraction of ∞ for F.

Theorem 3. *Let Ω be a simply connected John domain. Then*

$$\sigma^2(0+) = \sigma^2 = \limsup_{r\to 1} \frac{B^2}{\log\bigl(1/(1-r)\bigr)},$$

where $B^2 = \sum |a_k|^2 r^k$; a_k are the coefficients of $\log f'$ and f is the conformal mapping from $D^- = \{|\zeta| > 1\}$ onto Ω.

Proof. Our main idea is an approximation of the function $\log f'$ by lacunary series. Let

$$\psi(\zeta) = \log \frac{F'\bigl(f(\zeta)\bigr)}{q\zeta^{q-1}} = \sum_{k=1}^{q-1} \log \frac{f(\zeta)-\zeta_k}{\zeta} = \sum_{j=0}^{\infty} b_j \zeta^{-j}.$$

It is known [7] that

$$\log f'(\zeta) = -\sum_{k=0}^{\infty} \psi(\zeta^{q^k}) = \varphi(\zeta) + g(\zeta),$$

where $g(\zeta) = \sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty} b_j \zeta^{-jq^k}$ and $\varphi(\zeta) = \sum_{j=0}^{N}\sum_{k=0}^{\infty} b_j\zeta^{-jq^k}$. Fixing $\varepsilon > 0$, we will show that there exists $N = N(\varepsilon)$ such that $|\zeta g'(\zeta)| \le \varepsilon/(|\zeta|^2 - 1)$. From Pommerenke's result [8, p. 100] it follows that $\sum_{j=0}^{\infty} |b_{jk}|^2 j^{1+\alpha} < +\infty$ for John domains, where b_{jk} are the Taylor coefficients of $\log(f(\zeta)-\zeta_k)/\zeta$. This implies that $\sum_{j=0}^{\infty}|b_j|^2 j^{1+\alpha}<+\infty$ for John domains. Therefore, we have

$$\begin{aligned}
|\zeta g'(\zeta)| &= |\sum_{k=0}^{\infty}\sum_{j=N+1}^{\infty} jq^k b_j \zeta^{-jq^k}| \le \sum_{k=0}^{\infty}\sum_{j=N+1}^{\infty} jq^k |b_j| R^{-jq^k} \\
&\le \sum_{k=0}^{\infty} q^k \sqrt{\sum_{j=N+1}^{\infty} j^{1+\alpha}|b_j|^2}\sqrt{\sum_{j=N+1}^{\infty} j^{1-\alpha}R^{-2jq^k}} \\
&\le \varepsilon_{N,\alpha}\sum_{k=0}^{\infty}\frac{q^k R^{-q^k}}{(1-R^{-q^k})^{\alpha/2}} = \varepsilon_{N,\alpha}\sum_{k=0}^{\infty}\frac{q^k r^{q^k}}{(1-r^{q^k})^{\alpha/2}} \\
&\le \varepsilon_{N,\alpha}\sum_{k=0}^{\infty}\frac{q^k r^{q^k}}{(1-r)^{\alpha/2}q^{\alpha k/2}r^{\alpha q^k/2}} \\
&= \frac{\varepsilon_{N,\alpha}}{(1-r)^{\alpha/2}}\sum_{k=0}^{\infty} q^{(1-\alpha/2)k} r^{(1-\alpha/2)q^k} \le C\frac{\varepsilon_{N,\alpha}}{1-r}, \quad \text{where } r=\frac{1}{R}<1.
\end{aligned}$$

So, there exists $N=N(\alpha,\varepsilon)$ such that $|\zeta g'(\zeta)|\le \varepsilon/(|\zeta|^2-1)$. Consider now

$$\int |f'(Re^{i\theta})|^t \,\mathrm{d}\theta = \int |e^{\varphi}|^t|e^{g}|^t \,\mathrm{d}\theta \le \left(\int |e^{\varphi}|^{pt}\,\mathrm{d}\theta\right)^{1/p}\left(\int |e^{g}|^{st}\,\mathrm{d}\theta\right)^{1/s},$$

where $s=1/\varepsilon$, $p=1/(1-\varepsilon)$.

Since $|\zeta g'(\zeta)|\le \varepsilon/(|\zeta|^2-1)$ then it follows from the result of Clunie and Pommerenke [3] that

$$\int |e^{g}|^{st}\,\mathrm{d}\theta \le C\left(\frac{1}{R-1}\right)^{t^2/4}.$$

It is easy to see that φ is a lacunary series with Hadamard gaps. Applying Theorem 2 to this series we see that

$$\int |e^{\varphi}|^{pt}\,\mathrm{d}\theta \asymp \left(\frac{1}{R-1}\right)^{t^2p^2\sigma_{\varphi}^2+O_N(t^{2+1/5})},$$

where

$$\sigma_{\varphi}^2=\frac{1}{2\pi}\limsup_{R\to 1}\frac{\int|\varphi|^2\,\mathrm{d}\theta}{\log\bigl(1/(R-1)\bigr)}$$

Thus,

$$\int |f'(Re^{i\theta})|^t\,\mathrm{d}\theta \le C\left(\frac{1}{R-1}\right)^{t^2(\sigma_{\varphi}^2+O(\varepsilon))+O_N(t^{2+1/5})}.$$

Arguing as above, we obtain

$$\int |f'(Re^{i\theta})|^t\,\mathrm{d}\theta \ge C\left(\frac{1}{R-1}\right)^{t^2({\sigma_{\varphi}}^2+O(\varepsilon))+O_N(t^{2+1/5})}.$$

Obviously, $\sigma_{\varphi}\to\sigma$ as $\varepsilon\to 0$. Hence, $\sigma(0+)^2=\sigma^2$.

Corollary. *Let Ω be a simply connected John domain. Then*

$$\overline{\lim_{R\to 1}} \frac{|\log f'(R\zeta)|}{\sqrt{\log\bigl(1/(R-1)\bigr)\log\log\log\bigl(1/(R-1)\bigr)}} = \sqrt{\sigma^2(0+)}$$

for almost all ζ on $|\zeta| = 1$.

Proof. This formula immediately follows from Theorem 3 and the well-known law of the iterated logarithm for Julia sets [9]:

$$\overline{\lim_{R\to 1}} \frac{|\log f'(R\zeta)|}{\sqrt{\log\bigl(1/(R-1)\bigr)\log\log\log\bigl(1/(R-1)\bigr)}} = \sqrt{\sigma^2}\,.$$

Note also that this equality follows from the classical law of the iterated logarithm for lacunary series [10]. Therefore, we see again that (2) is sharp.

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