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Failure of the condition N below $W^{1,n}$

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Abstract. In this paper we examine the size of the exceptional sets outside which the null sets for the Lebesgue measure are preserved under continuous Sobolev mappings. This persistence is called the condition N. For a $W^{1,1}$-mapping it is true that the condition N is satisfied outside a set of Lebesgue measure zero. On the other hand, for mappings in $W^{1,n}$ this exceptional set can be chosen to have zero Hausdorff dimension. One might expect for some kind of interpolation results between these estimates on the Hausdorff dimension of exceptional sets. We show, however, that this is not the case by constructing examples of homeomorphisms that belong to $\bigcap_{p<n} W^{1,p}$ so that the exceptional set has to be n-dimensional.

1. Introduction

Suppose that f is a continuous mapping from an open bounded subset Ω of $\mathbf{R}^n$, $n \geq 2$, into $\mathbf{R}^n$. We consider the following (Lusin) condition N: if $E \subset \Omega$, $\mathscr{L}^n(E) = 0$, then $\mathscr{L}^n\bigl(f(E)\bigr) = 0$. Physically, this condition requires that there is no creation of matter under the deformation f of the n-dimensional body Ω. This is a natural requirement as the N-property with differentiability a.e. is sufficient for validity of various change-of-variable formulas, including the area formula, and the condition N holds for a homeomorphism f if and only if f maps measurable sets to measurable sets.

If $f \in W^{1,1}(\Omega, \mathbf{R}^n)$, we can ask which integrability conditions on the derivative Df guarantee the condition N. According to a classical result by Marcus and Mizel [11], f satisfies the condition N if $|Df| \in L^p(\Omega)$ for some $p > n$. If $|Df| \in L^p(\Omega)$ with $p \leq n$, the condition N may fail; see e.g. [10] or [15] for examples.

It is then interesting to ask for the size of the exceptional set, i.e. the set outside of which the condition N holds. It is worth pointing out that if f is any mapping in the Sobolev space $W^{1,1}(\Omega, \mathbf{R}^n)$, then f satisfies the condition N outside a set of measure zero (see e.g. [14] for references). Malý and Martio proved in [10] (also see [8]), as a substantial improvement on this fact, that if $f \in W^{1,n}(\Omega, \mathbf{R}^n)$, then the condition N holds for f outside of a set that has Hausdorff dimension zero.

It seems that very little is known about the size of the exceptional set below $W^{1,n}$. It would be natural to expect that the Hausdorff dimension of the

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exceptional set decreases as the exponent p increases to n. In this paper we will prove, however, that if $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ so that (1.1) holds, and thus in particular when $|Df| \in \bigcap_{p<n} L^p(\Omega)$, one cannot, in general, find an exceptional set of Hausdorff dimension smaller than n.

Theorem 1.1. *Let $s \in (0,\infty)$. Then there exists a homeomorphism $f\colon Q_0 \to Q_0$ from the unit cube $Q_0 = [0,1]^n$ onto itself such that the following properties hold*:

(a) *f fixes the boundary ∂Q_0.*
(b) *$f \in W^{1,1}(Q_0, \mathbf{R}^n)$, f is differentiable almost everywhere, and*

$$\sup_{0<\varepsilon\le n-1} \varepsilon \int_{Q_0} |Df(x)|^{n-\varepsilon}\,dx < \infty. \tag{1.1}$$

(c) *The Jacobian determinant $J(x,f)$ is strictly positive for almost every $x \in Q_0$ and $J(\cdot\,,f) \in L^1(Q_0)$.*
(d) *If $E \subset Q_0$ with $\mathscr{H}^h(E) = 0$, where $h(t) = t^n \log^s \log(4+1/t)$ (for small t), then there exists a compact set $F \subset Q_0 \setminus E$ of measure zero such that $\mathscr{L}^n\bigl(f(F)\bigr) > 0$.*

If we put certain additional topological or analytical assumptions on f, then a higher degree of integrability of $|Df|$ can be relaxed for the condition N. Reshetnyak proved in [16] that for a homeomorphism it suffices to assume that $|Df| \in L^n(\Omega)$. There are several generalizations of this result: instead of assuming that f is a homeomorphism it suffices to assume that f is continuous and open [10], or that $J(x,f) > 0$ almost everywhere [2] (see also [17] and [12]). For further results of this type see the survey paper [9]. Recently Kauhanen, Koskela and Malý [6] proved that, for a topologically sense-preserving (and thus in particular for homeomorphical) mapping f, it suffices to assume that

$$\lim_{\varepsilon\to 0+} \varepsilon \int_{\Omega} |Df|^{n-\varepsilon} = 0. \tag{1.2}$$

The construction of f of Theorem 1.1 is a refinement of our construction in [6], where a similar mapping was constructed to show that the condition N may fail under condition (1.1). For the convenience of the reader we will present a complete construction here. The basic idea of the construction comes from Ponomarev [15]; also see [14]. Ponomarev gave an example of a homeomorphism f with $|Df| \in \bigcap_{p<n} L^p(\Omega)$ such that f creates matter. He, however, did not consider the size of the exceptional set and worked only on the L^p-scale.

2. About the function spaces

In this section we briefly review some function spaces that we dealt with in the introduction; for a more detailed discussion see [4]. Let us define $BL^n(\Omega)$ as the collection of all measurable functions u with

$$\|u\|_{n)} = \sup_{0<\varepsilon\le n-1}\left(\varepsilon\int_\Omega |u|^{n-\varepsilon}\right)^{1/(n-\varepsilon)} < \infty.$$

Then $BL^n(\Omega)$ is a Banach space and

$$VL^n(\Omega) = \left\{u \in BL^n(\Omega) : \lim_{\varepsilon\to 0^+}\varepsilon\int_\Omega |u|^{n-\varepsilon} = 0\right\}$$

is a closed subspace. The following inclusions hold (see [3]):

$$L^n(\Omega) \subsetneq L^n\log^{-1}L(\Omega) \subsetneq VL^n(\Omega) \subsetneq BL^n(\Omega) \subsetneq \bigcap_{\alpha<-1} L^n\log^\alpha L(\Omega) \subsetneq \bigcap_{p<n} L^p(\Omega),$$

where the space $L^n\log^\alpha L(\Omega)$ consists of all measurable functions u with

$$\int_\Omega |u|^n\log^\alpha(e+|u|) < \infty.$$

One question that Theorem 1.1 and the other results mentioned in the introduction leave unanswered is the size of the exceptional set for continuous mappings $f \in W^{1,1}(\Omega,\mathbf{R}^n)$ with $|Df|$, for example, in $L^n\log^{-1}L(\Omega)$ or $VL^n(\Omega)$.

3. Proof of Theorem 1.1

Let us begin with some notation. Besides the usual euclidean norm $|x| = (x_1^2+\cdots+x_n^2)^{1/2}$ we will use the cubic norm $\|x\| = \max_i |x_i|$. Using the cubic norm, the x_0-centered closed cube with edge length $2r>0$ and sides parallel to coordinate axes can be represented in the form

$$Q(x_0,r) = \{x \in \mathbf{R}^n : \|x-x_0\| \le r\}.$$

We then call r the radius of Q. $c(a,b,\dots)$ denotes a positive constant depending only on $a,b,\dots$ which might differ from occurrence to occurrence. We write $x \approx c(a,b,\dots)y$ if $x \le c(a,b,\dots)y$ and $y \le c(a,b,\dots)x$.

We will be dealing with radial stretchings that map cubes $Q(0,r)$ onto cubes. The following lemma can be verified by an elementary calculation.

Lemma 3.1. *Let* $\varrho\colon (0,\infty) \to (0,\infty)$ *be a strictly monotone, differentiable function. Then for the mapping*

$$f(x) = \frac{x}{\|x\|}\varrho(\|x\|), \qquad x \neq 0,$$

we have for a.e. x

$$|Df(x)| \approx c(n) \max\left\{\frac{\varrho(\|x\|)}{\|x\|}, \left|\varrho'(\|x\|)\right|\right\} \quad \textit{and} \quad J(x,f) \approx c(n)\frac{\varrho'(\|x\|)\varrho(\|x\|)^{n-1}}{\|x\|^{n-1}}.$$

We will first give two Cantor set constructions in Q_0. Our mapping f will be defined as a limit of a sequence of piecewise continuously differentiable homeomorphisms $f_k\colon Q_0 \to Q_0$, where each f_k maps the k:th step of the first Cantor set construction onto the second one. Then f maps the first Cantor set onto the second one. Choosing the Cantor sets so that the Hausdorff h measure $\mathscr{H}^h$ of the first one is positive and finite and so that the second one has positive measure, we get the property (d). We will explain this argument later.

Let $V \subset \mathbf{R}^n$ be the set of all vertices of the cube $Q(0,1)$. Then the sets $V^k = V \times \cdots \times V$, $k = 1,2,\ldots$, will serve as the sets of indices for our construction (with the exception of the subscript 0). If $w \in V^{k-1}$, we denote

$$V^k[w] = \{v \in V^k : v_j = w_j,\ j = 1,\ldots,k-1\}.$$

Let $z_0 = \left(\frac{1}{2},\ldots,\frac{1}{2}\right)$, $r_0 = \frac{1}{2}$ and denote

$$\psi(k) = \frac{1}{2}\,\frac{\log^{s/n} 2}{\log^{s/n}(k+2)}.$$

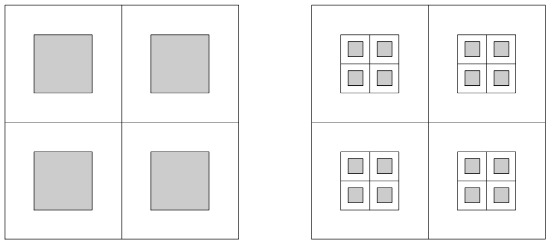
For $v \in V^1 = V$ let $z_v = z_0 + \frac{1}{4}v$, $P_v = Q\left(z_v, \frac{1}{4}\right)$ and $Q_v = Q(z_v, \psi(1)2^{-1})$. If $k = 2,3,4,\ldots$, and $Q_w = Q(z_w, r_{k-1})$, $w \in V^{k-1}$, is a cube from the previous step of construction, then Q_w is divided into 2^n subcubes P_v, $v \in V^k[w]$, with centers z_v and radius $\frac{1}{2}r_{k-1}$ and inside them we pick concentric cubes Q_v, $v \in V^k[w]$, with radius

$$r_k = \psi(k)2^{-k}.$$

Note that $r_k < \frac{1}{2}r_{k-1}$ for all k. Thus, denoting $v = (v_1,\ldots,v_k)$, we have

$$z_v = z_w + \frac{1}{2}r_{k-1}v_k = z_0 + \frac{1}{2}\sum_{j=1}^{k} r_{j-1}v_j,$$

$$P_v = Q\left(z_v, \tfrac{1}{2}r_{k-1}\right), \quad Q_v = Q(z_v, r_k).$$



$k = 1$ $k = 2$

Figure 1. Cubes Q_v, $v \in V^k$.

See Figure 1. The number of cubes in the family $\{Q_v : v \in V^k\}$, $k = 1, 2, 3, \ldots$, is $\# V^k = 2^{nk}$. If W is any subfamily of V_k, we have

$$(3.1)\qquad \begin{aligned} \sum_{w \in W} h(\operatorname{diam} Q_w) &\approx c(n,s) \# W r_k^n \log^s \log(1/r_k) \\ &\approx c(n,s) \# W 2^{-nk} \psi(k)^n \log^s \bigl(\log 2^k + \log \psi(k)^{-1}\bigr) \\ &\approx c(n,s) \# W 2^{-nk} \log^{-s} k \log^s k \approx c(n,s) \# W 2^{-nk}. \end{aligned}$$

This is behind the fact that the Hausdorff h measure of the resulting Cantor set

$$C = \bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q_v$$

is positive and finite. In lack of a convenient reference we give a short proof for this fact.

Lemma 3.2. *We have* $0 < \mathscr{H}^h(C) < \infty$.

Proof. We mimic the proof in [13, Section 4.10]. The estimate $\mathscr{H}^h(C) < \infty$ directly follows from (3.1). Let us justify $\mathscr{H}^h(C) > 0$.

Let $\{U_j\}$ be an open covering of C. It suffices to show that

$$(3.2)\qquad \sum_j h(\operatorname{diam} U_j) \geq c(n,s).$$

Since C is compact, we can assume that $\{U_j\}$ is finite. For each j choose $x_j \in U_j \cap C$ (the intersection can be assumed to be non-empty) and denote $B_j = B(x_j, \operatorname{diam} U_j)$. Then there is k_0 such that for all $k \geq k_0$ every cube Q_v, $v \in V^k$, is contained in some B_j, and because $h(2t) \leq 2^n h(t)$ for $t \geq 0$, we have that

$$\sum_j h(\operatorname{diam} U_j) \geq 2^{-n} \sum_j h(\operatorname{diam} B_j).$$

Therefore it suffices to prove (3.2) for the family $\{B_j\}$ in the place of $\{U_j\}$. We claim that for any open ball B with center in C and any fixed l,

$$\sum_{v\in V^l,\, Q_v\subset B} h(\operatorname{diam} Q_v) \le c(n,s)h(\operatorname{diam} B). \tag{3.3}$$

This gives (3.2), since for $k \ge k_0$, by (3.1),

$$\begin{aligned}\sum_j h(\operatorname{diam} B_j) &\ge c(n,s)\sum_j \sum_{v\in V^k,\ Q_v\subset B_j} h(\operatorname{diam} Q_v)\\ &\ge c(n,s)\sum_{v\in V^k} h(\operatorname{diam} Q_v) \approx c(n,s).\end{aligned}$$

To prove (3.3), suppose that $Q_v \subset B$ for some $v \in V^l$, and let m be the smallest integer for which $Q_w \subset B$ for some $w \in V^m$. Then $m \le l$. Denote

$$W = \{w \in V^m : Q_w \cap B \ne \emptyset\}.$$

By considering the geometry of the construction of C, it is evident that $\#W \le c = c(n,s)$. Thus, using (3.1) twice,

$$\begin{aligned}h(\operatorname{diam} B) &\ge \frac{1}{c}\sum_{w\in W} h(\operatorname{diam} Q_w) \approx c(n,s)\#W2^{-nm} = c(n,s)\#W2^{n(l-m)}2^{-nl}\\ &= c(n,s)\#\{v\in V^l : Q_v \subset Q_w,\ w\in W\}2^{-nl}\\ &\approx c(n,s)\sum_{w\in W}\sum_{v\in V^l,\ Q_v\subset Q_w} h(\operatorname{diam} Q_v)\\ &\ge c(n,s)\sum_{v\in V^l,\, Q_v\subset B} h(\operatorname{diam} Q_v). \quad \square\end{aligned}$$

The second Cantor set construction is similar to the first one except that at this time we denote the centers by z'_v and the cubes by P'_v, Q'_v, $v \in V^k$, with

$$\begin{aligned} z'_v &= z'_w + \frac{1}{2}r'_{k-1}v_k = z_0 + \frac{1}{2}\sum_{j=1}^{k} r'_{j-1}v_j,\\ P'_v &= Q\big(z'_v, \tfrac{1}{2}r'_{k-1}\big), \quad Q'_v = Q(z'_v, r'_k).\end{aligned}$$

We set $r'_0 = \frac{1}{2}$ and

$$r'_k = \varphi(k)2^{-k}$$

for $k = 1, 2, 3, \ldots$, where

$$\varphi(k) = \frac{1}{4}\left(1 + \frac{\log\log 4}{\log\log(k+4)}\right).$$

Note that $r'_k < \frac{1}{2} r'_{k-1}$ for each k. We have

$$\mathscr{L}^n\Big(\bigcap_{k=1}^{\infty} \bigcup_{v\in V^k} Q_v\Big) = \lim_{k\to\infty} \mathscr{L}^n\Big(\bigcup_{v\in V^k} Q_v\Big) = \lim_{k\to\infty} 2^{nk}(2r'_k)^n = 2^{-n} > 0.$$

We are now ready to define the mappings f_k. Define $f_0 = \mathrm{id}$. We will give a mapping f_1 that stretches each cube Q_v, $v \in V^1$, homogeneously so that $f_1(Q_v)$ equals Q'_v. On the annulus $P_v \setminus Q_v$, f_1 is defined to be an appropriate radial map with respect to z_v and z'_v in the image in order to make f_1 a homeomorphism. The general step is the following: If $k > 1$, f_k is defined as f_{k-1} outside the union of all cubes Q_w, $w \in V^{k-1}$. Further, f_k remains equal to f_{k-1} at the centers of cubes Q_v, $v \in V^k$. Then f_k stretches each cube Q_v, $v \in V^k$, homogeneously so that $f(Q_v)$ equals Q'_v. On the annulus $P_v \setminus Q_v$, f is defined to be an appropriate radial map with respect to z_v in preimage and z'_v in image to make f_k a homeomorphism (see Figure 2). Notice that the Jacobian determinant $J(x, f_k)$ will be strictly positive almost everywhere in Q_0.

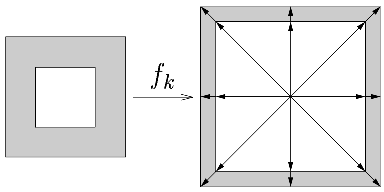


Figure 2. The mapping f_k acting on P_v, $v \in V^k$.

To be precise, let $f_0 = \mathrm{id} \mid Q_0$ and for $k = 1, 2, 3, \ldots$ define

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{if } x \notin \bigcup_{v\in V^k} P_v, \\ f_{k-1}(z_v) + a_k(x - z_v) + b_k \dfrac{x - z_v}{\|x - z_v\|} & \text{if } x \in P_v \setminus Q_v,\ v \in V^k, \\ f_{k-1}(z_v) + c_k(x - z_v) & \text{if } x \in Q_v,\ v \in V^k, \end{cases}$$

where a_k, b_k and c_k are chosen so that f_k maps each Q_v onto Q'_v, is continuous and fixes the boundary ∂Q_0:

$$a_k r_k + b_k = r'_k, \qquad \tfrac{1}{2} a_k r_{k-1} + b_k = \tfrac{1}{2} r'_{k-1}, \qquad c_k r_k \tag{3.4}$$

Clearly the limit $f = \lim_{k\to\infty} f_k$ is differentiable almost everywhere, its Jacobian determinant is strictly positive almost everywhere, and f is absolutely continuous on almost all lines parallel to coordinate axes. Continuity of f follows from the uniform convergence of the sequence (f_k): for any $x \in Q_0$ and $l \ge j \ge 1$ we have

$$|f_l(x) - f_j(x)| \le c(n) r'_j \to 0$$

as $j\to\infty$.

It is easily seen that f is a one-to-one mapping of Q_0 onto Q_0. Since f is continuous and Q_0 is compact, it follows that f is a homeomorphism.

To finish the proof of the properties (b) and (c), we next estimate $|Df(x)|$ and $J(x,f)$ at x in the interior of the annulus $P_v\setminus Q_v$, $v\in V^k$, $k=1,2,3,\dots$. Denote $r=\|x-z_v\|\approx c(n,s)r_k$. In the annulus

$$f(x)=f_{k-1}(z_v)+\bigl(a_k\|x-z_v\|+b_k\bigr)\frac{x-z_v}{\|x-z_v\|}$$

whence denoting $\varrho(r)=a_kr+b_k$ we have by Lemma 3.1 (it is easy to check that $b_k>0$ for large k)

$$|Df(x)|\approx c(n,s)(a_k+b_k/r_k)$$

and

$$J(x,f)\approx c(n,s)a_k(a_k+b_k/r_k)^{n-1}.$$

From the equations (3.4) it follows that

$$a_k=\frac{\frac12 r'_{k-1}-r'_k}{\frac12 r_{k-1}-r_k}=\frac{\varphi(k-1)-\varphi(k)}{\psi(k-1)-\psi(k)}\approx c(n,s)\frac{\varphi'(k)}{\psi'(k)}\approx c(n,s)\frac{\log^{s/n}k}{\log^2\log k}$$

and

$$a_k+b_k/r_k=r'_k/r_k=\frac{\varphi(k)}{\psi(k)}\approx c(n)\frac{1}{\psi(k)}\approx c(n,s)\log^{s/n}k.$$

Therefore

$$|Df(x)|\approx c(n,s)\log^{s/n}k\qquad\text{and}\qquad J(x,f)\approx c(n,s)\frac{\log^s k}{\log^2\log k}.$$

The measure of $\bigcup\limits_{v\in V^k}(P_v\setminus Q_v)$ is

$$\begin{aligned}2^{nk}\bigl(r_{k-1}^n-(2r_k)^n\bigr)&\approx c(n)\bigl(\psi(k-1)^n-\psi(k)^n\bigr)\\&\approx c(n,s)\left(\frac{1}{\log^s(k-1)}-\frac{1}{\log^s k}\right)\\&\approx -c(n,s)\frac{d}{dk}\left(\frac{1}{\log^s k}\right)\approx c(n,s)\frac{1}{k\log^{1+s}k}\end{aligned}$$

and so for $0<\varepsilon\le n-1$

$$\begin{aligned}\varepsilon\int_{Q_0}|Df(x)|^{n-\varepsilon}\,dx&\approx c(n,s)\varepsilon\sum_{k=2}^{\infty}\frac{1}{k\log^{1+s}k}\log^{(s/n)(n-\varepsilon)}k\\&=c(n,s)\varepsilon\sum_{k=2}^{\infty}\frac{1}{k\log^{1+(s/n)\varepsilon}k}\\&\approx c(n,s)\varepsilon\int_2^\infty\frac{dt}{t\log^{1+(s/n)\varepsilon}t}\approx c(n,s).\end{aligned}\tag{3.5}$$

This proves (1.1), and it follows that $f \in W^{1,1}(Q_0, \mathbf{R}^n)$. Similarly we prove the property (c):

$$\begin{aligned}\int_{Q_0} J(x,f)\,dx &\approx c(n,s)\varepsilon \sum_{k=4}^{\infty} \frac{1}{k\log^{1+s} k}\,\frac{\log^s k}{\log^2 \log k} \\ &\approx c(n,s)\int_4^\infty \frac{dt}{t\log t \log^2 \log t} < \infty.\end{aligned} \tag{3.6}$$

Let us now prove the property (d). Since $\mathscr{H}^h$ is Borel regular, we find a Borel set $E' \supset E \cap C$ such that $\mathscr{H}^h(E') = 0$. Now $C \setminus E'$ is a Borel set, whence there is a compact set $F \subset C \setminus E'$ such that $\mathscr{H}^h(F) > 0$. Denote

$$W^k = \{v \in V^k : Q_v \cap F \neq \emptyset\}.$$

Then there exists a constant $c > 0$ such that

$$\frac{\# W^k}{\# V^k} \geq c \tag{3.7}$$

for all k, since otherwice we would have by (3.1) that

$$\sum_{v \in W^k} h(\operatorname{diam} Q_v) \approx c(n,s)\# W^k\, 2^{-nk} = c(n,s)\frac{\# W^k}{\# V^k} \to 0$$

as $k \to \infty$ for k in some subsequence $(k) \subset \mathbf{N}$, whence we would have $\mathscr{H}^h(F) = 0$, a contradiction. Since F is compact, we have that

$$F = \bigcap_{k=1}^{\infty} \bigcup_{v \in W^k} Q_v,$$

whence (3.7) yields

$$\begin{aligned}\mathscr{L}^n\big(f(F)\big) &= \mathscr{L}^n\left(f\left(\bigcap_{k=1}^{\infty}\bigcup_{v\in W^k} Q_v\right)\right) = \mathscr{L}^n\left(\bigcap_{k=1}^{\infty} f\left(\bigcup_{v\in W^k} Q_v\right)\right) \\ &= \lim_{k\to\infty} \mathscr{L}^n\left(\bigcup_{v\in W^k} Q'_v\right) \approx c(n) \lim_{k\to\infty} \# W^k \big(\varphi(k)2^{-k}\big)^n \\ &\geq c(n)\lim_{k\to\infty} \# W^k\, 2^{-nk} = c(n)\lim_{k\to\infty}\frac{\# W^k}{\# V^k} \geq c(n)c > 0.\end{aligned}$$

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References

[1] Falconer, K.: Fractal Geometry. Mathematical Foundations and Applications. - John Wiley & Sons, 1990.

[2] Gol'dstein, V., and S. Vodopyanov: Quasiconformal mappings, and spaces of functions with first generalized derivatives. - Sibirsk. Mat. Zh. 17, 1976, 515–531 (Russian).

[3] Greco, L.: A remark on the equality $\det Df = \operatorname{Det} Df$. - Differential Integral Equations 6, 1993, 1089–1100.

[4] Iwaniec, T., P. Koskela, and J. Onninen: Mappings of finite distortion: Monotonicity and continuity. - Invent. Math. (to appear).

[5] Kauhanen, J., P. Koskela, and J. Malý: On functions with derivatives in a Lorentz space. - Manuscripta Math. 100, 1999, 87–101.

[6] Kauhanen, J., P. Koskela, and J. Malý: Mappings of finite distortion: Condition N. - Michigan Math. J. 49, 2001, 169–181.

[7] Kauhanen, J., P. Koskela, and J. Malý: Mappings of finite distortion: Discreteness and openness. - Arch. Ration. Mech. Anal. 160, 2001, 135–151.

[8] Malý, J.: The area formula for $W^{1,n}$-mappings. - Comment. Math. Univ. Carolin. 35, 1994, 291–298.

[9] Malý, J.: Sufficient Conditions for Change of Variables in Integral. - Proceedings of the Conference Analysis & Geometry (to appear).

[10] Malý, J., and O. Martio: Lusin's condition (N) and mappings of the class $W^{1,n}$. - J. Reine Angew. Math. 458, 1995, 19–36.

[11] Marcus, M., and V. Mizel: Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. - Bull. Amer. Math. Soc. 79, 1973, 790–795.

[12] Martio, O., and W. Ziemer: Lusin's condition (N) and mappings with non-negative Jacobians. - Michigan Math. J. 39, 1992, 495–508.

[13] Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability. - Cambridge Stud. Adv. Math. 44, Cambridge University Press, 1995.

[14] Müller, S., and S. Spector: An existence theory for nonlinear elasticity that allows for cavitation. - Arch. Rational Mech. Anal. 131, 1995, 1–66.

[15] Ponomarev, S.: Property N of homeomorphisms of the class W_p^1. - Sibirsk. Mat. Zh. 28, 1987, 140–148 (Russian).

[16] Reshetnyak, Yu.: Some geometrical properties of functions and mappings with generalized derivatives. - Sibirsk. Mat. Zh. 7, 1966, 886–919 (Russian).

[17] Šverák, V.: Regularity properties of deformations with finite energy. - Arch. Rational Mech. Anal. 100, 1988, 105–127.

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