Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 27, 2002, 57–67

ABSOLUTELY MINIMIZING LIPSCHITZ EXTENSIONS ON A METRIC SPACE

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Abstract. In this note, we consider the problem of finding an absolutely minimizing Lipschitz extension of a given function defined in a subset of an arbitrary metric space. Using a version of Perron's method due to Aronsson, we prove the existence under the assumption that the space is a separable length space.

1. Introduction

Let (X,d) be an arbitrary metric space and let A be any nonempty subset of X. If we are given a Lipschitz function $f\colon A\to\mathbf{R}$, then it is well known that there exists a *minimal Lipschitz extension* (an MLE for short) of f to X, that is, a function $h\colon X\to\mathbf{R}$ such that $h|_A=f$ and the Lipschitz constant of h in X equals to the Lipschitz constant of f in A. Such extensions were explicitly constructed already by McShane [10] and Whitney [13], and can be written as

$$\overline{f}(x)=\inf_{y\in A}\{f(y)+L(f,A)\,d(x,y)\},\qquad \underline{f}(x)=\sup_{y\in A}\{f(y)-L(f,A)\,d(x,y)\}.$$

Here, and in what follows, $L(g,E)$ denotes the (smallest) Lipschitz constant of a function $g\colon E\to\mathbf{R}$. It is easy to verify that both the lower and the upper extension $\underline{f}$ and $\overline{f}$ really are minimal Lipschitz extensions. Moreover, if u is any minimal Lipschitz extension of f to X then $\underline{f}\le u\le\overline{f}$ pointwise in X.

The notion of a minimal Lipschitz extension is not completely satisfactory. It involves only the global Lipschitz constant of the extension and ignores what may happen locally. To illustrate this point, let us consider the following simple example. Let $X=[-2,2]$, equipped with the usual metric of $\mathbf{R}$, and let $A=\{-2,-1,1,2\}$. If we define the function f by setting $f(\pm1)=1$, $f(\pm2)=2$, then $h\colon[-2,2]\to\mathbf{R}$ is an MLE of f if and only if it can be written as

$$h(x)=\begin{cases}|x|, & \text{if } 1<|x|\le 2,\\ \hat{h}(x), & \text{if } |x|\le 1,\end{cases}$$

2000 Mathematics Subject Classification: Primary 54C20; Secondary 49J99.

This note was written at the Mittag-Leffler Institute during the program "Potential Theory and Nonlinear Partial Differential Equations". The author is supported by the Academy of Finland (Project 41964).

where $\hat{h}: [-1,1] \to \mathbf{R}$ is any function such that $\hat{h}(\pm 1) = 1$ and $L(\hat{h}, [-1,1]) \le 1$. Yet the most natural, and in a sense the best extension would be $\max\{1, |x|\}$, which is the only MLE that has also locally the smallest possible Lipschitz constant.

With these remarks in mind, it is natural to ask whether it is always possible to find an MLE which has also locally the optimal Lipschitz constant, and if such an extension could be unique. These questions have been thoroughly studied in $\mathbf{R}^n$, starting from the pioneering work of Aronsson [3]. He introduced the concept of absolute minimizers (also known as "absolutely minimizing Lipschitz extensions", see Definition 2.1 below) and gave an affirmative answer to the existence by using a nice variant of Perron's method. Aronsson also proposed another approach to the problem, based on approximation by p-harmonic functions as $p \to \infty$. This method provides an Euler equation for the problem, called the infinity Laplace equation, and it has turned out to be crucial in the proof of uniqueness; see the celebrated paper of Jensen [7]. Nowadays there is a fast growing literature on the various aspects of the problem in $\mathbf{R}^n$, in particular, on the infinity Laplace equation and its generalizations. See [4], [7], [8], [9], and the references therein.

The objective of this paper is to study the problem in the setting of an arbitrary metric space. Since, in general, there is no theory of PDE's available, we try to follow in Aronsson's footsteps and show the existence of an "absolute minimizer" by using Perron's method. Initially, (X,d) can be any metric space, but later on we have to make some restrictions. Our main result, Theorem 4.3, states that if the space is a separable length space, then there exists an absolutely minimizing Lipschitz extension of any given Lipschitz continuous function.

This paper is organized as follows. In Section 2, we give the precise definition of an absolutely minimizing Lipschitz extension, which is not entirely trivial in this generality. The classes of subsolutions and supersolutions of the extension problem, needed for the Perron's method, are introduced in Section 3. The existence result is then proved in Section 4, and we conclude with a short appendix on subsolutions and supersolutions in $\mathbf{R}^n$.

2. Definitions

The first thing we need to do is to precisely formulate what it means for an MLE to have the smallest possible Lipschitz constant locally also.

2.1. Definition. Let A be any nonempty subset of X and let $f: A \to \mathbf{R}$ be Lipschitz. A function $h: X \to \mathbf{R}$ is an *absolutely minimizing Lipschitz extension* (an AMLE for short) of f to X if

(1) h is an MLE of f to X;

(2) whenever $B \subset X$ and $g: X \to \mathbf{R}$ is an MLE of f to X such that $h = g$ in $X \setminus B$, then
$$L(h,B) \le L(g,B).$$

The above definition is quite natural as it requires that no admissible variation can decrease the local Lipschitz constant. Aronsson's original (and essentially equivalent) definition in $\mathbf{R}^n$ was formulated in a slightly different way. He assumed that A is a compact set and required that $L(h,D) = L(h,\partial D)$ for every bounded open set D in $\mathbf{R}^n \setminus A$. In our situation, this kind of definition would be somewhat ambiguous because the boundary of an open subset of a metric space may very well be empty.

It is easy to find pathological examples where a given Lipschitz function does not have an absolutely minimizing extension.

2.2. Example. Let $X = \{x, y, z\}$ and define a metric d on X as follows:

$$\begin{cases} d(x,y) = \frac{3}{2}, \\ d(x,z) = d(y,z) = 1. \end{cases}$$

Let $A = \{x, y\}$ and set $f(x) = 0$, $f(y) = 1$. If $h\colon X \to \mathbf{R}$ is an AMLE of f to A then $h(x) = 0$, $h(y) = 1$, and $\frac{1}{3} \le h(z) \le \frac{2}{3}$, because $L(f,A) = \frac{2}{3}$. Choose $B = \{x, z\}$ and let g be the MLE of f for which $g(z) = \frac{1}{3}$. Then $L(g,B) = \frac{1}{3}$, which implies that $L(h,B) = \frac{1}{3}$, and consequently $h(z) = \frac{1}{3}$. By a similar argument, this time using the set $\{y, z\}$, we see that $h(z) = \frac{2}{3}$, which is not possible. Hence there does not exist an AMLE of f to X.

One could now argue that Definition 2.1 is not the correct one because we allow testing with sets B that intersect A. If, however, we weaken the definition so that B is required to be a subset of $X \setminus A$, then in the above example every MLE is an AMLE. Heuristically, the issue here is to find the right way to interpret the "boundary condition". We will not pursue this discussion any further because the choice between these two versions plays no role in the rest of the paper. Furthermore, it is not hard to see that in any length space the two definitions are equivalent.

2.3. Remark. In a sufficiently well-behaved space it also makes sense to talk about local AMLEs without any reference to a function f that is to be extended. This more intrinsic version of the definition has been favored in most of the recent papers since it fits better with the PDE theory. In this note, we will not need the local concept.

3. Subsolutions and supersolutions

Since our proof of existence is based on Perron's method, we need to introduce the classes of subsolutions and supersolutions of the extension problem. As in [1], these are defined via comparison with respect to functions that are, in some sense, extremal solutions to a local extension problem.

Let us assume that (X,d) is separable and perfect. We fix f and $A\subset X$, and from now on refer to MLEs of f to X simply as MLEs. For any MLE $u\colon X\to \mathbf{R}$ and any $B\subset X$ let

$$\mathscr{L}(u,B)=\inf\{L(g,B)\colon g \text{ is an MLE, } g=u \text{ in } X\setminus B\}.$$

Using this notation, the condition (2) in Definition 2.1 can be reformulated as
(2′) $L(u,B)=\mathscr{L}(u,B)$ for all $B\subset X$,
which is closer to the original definition of Aronsson [3].

3.1. Lemma. *If u is an MLE and $B\subset X$, then there exists an MLE g_0 such that $g_0=u$ in $X\setminus B$ and $L(g_0,B)=\mathscr{L}(u,B)$.*

Proof. Let g_j be a sequence of MLEs such that $g_j=u$ in $X\setminus B$ and $L(g_j,B)\le \mathscr{L}(u,B)+1/j$. Since the family $\{g_j\}$ is bounded pointwise by the McShane–Whitney extensions $\underline{f}$ and $\overline{f}$, and (X,d) is separable, Ascoli's theorem, see for example [12], provides us with a subsequence g_{j_k} converging uniformly on compact subsets to some function g_0. It is elementary to check that g_0 satisfies the properties stated above. □

Next set

$$\Lambda_{u,B}(x)=\inf\{g(x): g \text{ is an MLE, } g=u \text{ in } X\setminus B, \text{ and } L(g,B)=\mathscr{L}(u,B)\},$$
$$\Upsilon_{u,B}(x)=\sup\{g(x): g \text{ is an MLE, } g=u \text{ in } X\setminus B, \text{ and } L(g,B)=\mathscr{L}(u,B)\}$$

These functions are clearly well-defined. Moreover, it is easy to check that $\Lambda_{u,B}$ and $\Upsilon_{u,B}$ are MLEs, $u=\Lambda_{u,B}=\Upsilon_{u,B}$ in $X\setminus B$, and

$$L(\Lambda_{u,B},B)=L(\Upsilon_{u,B},B)=\mathscr{L}(u,B).$$

Following Aronsson's idea, we now define sub- and supersolutions using the local extremal extensions $\Lambda_{u,B}$ and $\Upsilon_{u,B}$.

3.2. Definition. Let u be an MLE of f to X. We say that u is a *subsolution* of the extension problem if $u\le \Upsilon_{u,B}$ in X for every $B\subset X$. Similarly, we say that u is a *supersolution* of the extension problem if $u\ge\Lambda_{u,B}$ in X for every $B\subset X$.

Note that the set of supersolutions is never empty because the upper McShane–Whitney extension is a supersolution. This follows immediately from the fact that $\overline{f}$ is the largest MLE of f to X. An analogous remark holds for the subsolutions.

3.3. Example. Let $(X,d)=(\mathbf{R}^n,|\cdot|)$. Then $\mathscr{L}(u,B)=L(u,\partial B)$ and

$$\Upsilon_{u,B}(x)=\inf_{y\in\partial B}\{u(y)+L(u,\partial B)|x-y|\}$$

for all $x\in B$ and for all open $B\subset(X\setminus A)$.

The terminology used in the above definition is justified by the following proposition.

3.4. Proposition. *A function $u\colon X \to \mathbf{R}$ is an AMLE if and only if u is both a subsolution and a supersolution.*

Proof. Assume first that u is an AMLE. Then $L(u,B) = \mathscr{L}(u,B)$ for each $B \subset X$, and hence $\Lambda_{u,B} \le u \le \Upsilon_{u,B}$ by the definitions. Consequently, u is both a subsolution and a supersolution.

For the opposite direction, we argue by contradiction and assume that u fails to be an AMLE. This means that there exist a subset B of X and points $x_0, y_0 \in B$ such that

$$M = \frac{u(x_0) - u(y_0)}{d(x_0,y_0)} > \mathscr{L}(u,B).$$

We may assume, without loss of generality, that both x_0 and y_0 are interior points of B. Consider the set $\widetilde{B} = B \setminus \{y_0\}$. We show first that $\mathscr{L}(u,B) = \mathscr{L}(u,\widetilde{B})$. In order to do this, let

$$g(x) = \min\{\Upsilon_{u,B}(x), \Lambda_{u,B}(x) + c\},$$

where $c = u(y_0) - \Lambda_{u,B}(y_0)$. Then clearly $L(g,B) = \mathscr{L}(u,B)$. Because u is a subsolution and a supersolution, we have

$$\Lambda_{u,B}(y_0) \le u(y_0) \le \Upsilon_{u,B}(y_0).$$

Hence $g(y_0) = u(y_0)$. Moreover, since $\Lambda_{u,B}$ and $\Upsilon_{u,B}$ agree with u in the complement of B, then so does g. Thus g is an MLE such that $g|_{X\setminus\widetilde{B}} = u|_{X\setminus\widetilde{B}}$, and therefore

$$\mathscr{L}(u,\widetilde{B}) \le L(g,B) = \mathscr{L}(u,B).$$

Since the opposite inequality is trivial, we obtain that $\mathscr{L}(u,\widetilde{B}) = \mathscr{L}(u,B)$.

The rest is now easy. Using the facts that u is a subsolution and (X,d) is perfect, we have

$$u(x_0) \le \Upsilon_{u,\widetilde{B}}(x_0) \le u(y_0) + \mathscr{L}(u,B)d(x_0,y_0) < u(y_0) + Md(x_0,y_0) = u(x_0),$$

which is clearly a contradiction. Hence u is an AMLE, and the proposition is proved. □

3.5. Corollary. *Suppose that u is a unique MLE of f to X. Then u is an AMLE.*

Proof. By the uniqueness, $u = \overline{f} = \underline{f}$. In particular, u is both a supersolution and a subsolution, and hence by the above proposition an AMLE. □

4. Existence of an AMLE

In this section, we prove the existence of an AMLE under the assumption that (X, d) is a separable length space. Recall that a metric space is a length space if for all $x, y \in X$

$$d(x,y) = \inf\{l(\gamma) : \gamma\colon [0,1] \to X \text{ is continuous, } \gamma(0) = x, \text{ and } \gamma(1) = y\}.$$

Here $l(\gamma)$ denotes the length of the curve γ.

As always with Perron's method, the idea is to show that the infimum of all supersolutions is a solution. We have divided the proof into two lemmas.

4.1. Lemma. *Let $\mathscr{F}$ be a nonempty collection of supersolutions and define*

$$h(x) = \inf_{u \in \mathscr{F}} u(x).$$

Then h is a supersolution.

Proof. It is clearly enough to show that $u \geq \Lambda_{h,B}$ for every $u \in \mathscr{F}$ and every $B \subset X$. Let us therefore fix B and u, and consider the set

$$D = \{x \in X : u(x) < \Lambda_{h,B}(x)\}.$$

Observe that $D \subset B$ because $\Lambda_{h,B} = h \leq u$ in $X \setminus B$. We define

$$\Lambda(x) = \min\{\Lambda_{h,B}(x), \Lambda_{u,D}(x)\} = \begin{cases} h(x) & \text{in } X \setminus B, \\ \Lambda_{h,B}(x) & \text{in } B \setminus D, \\ \Lambda_{u,D}(x) & \text{in } D. \end{cases}$$

If we can show that $L(\Lambda, B) = \mathscr{L}(h, B)$, then D must be empty. Indeed, otherwise we would contradict the definition of $\Lambda_{h,B}$ since $\Lambda = \Lambda_{u,D} \leq u < \Lambda_{h,B}$ in D.

To this end, consider first the function $g = \max\{u, \Lambda_{h,B}\}$. Clearly g is an MLE and $g = u$ outside D. Hence

$$\mathscr{L}(u, D) \leq L(g, D) = L(\Lambda_{h,B}, D) \leq L(\Lambda_{h,B}, B) = \mathscr{L}(h, B)$$

by the definitions. In particular,

$$L(\Lambda, D) = L(\Lambda_{u,D}, D) = \mathscr{L}(u, D) \leq \mathscr{L}(h, B).$$

On the other hand,

$$L(\Lambda, B \setminus D) = L(\Lambda_{h,B}, B \setminus D) \leq \mathscr{L}(h, B).$$

In order to conclude that $L(\Lambda, B) = \mathscr{L}(h, B)$, we invoke the assumption that (X, d) is a length space. Let $x \in D$ and $y \in B \setminus D$, and let $\gamma_j \colon [0,1] \to X$ be a path joining x to y such that $l(\gamma_j) \le d(x,y) + 1/j$. Then there is $t_j \in [0,1]$ such that the point $z_j = \gamma_j(t_j)$ is at the boundary of D. Hence

$$\begin{aligned}|\Lambda(x) - \Lambda(y)| &\le |\Lambda_{u,D}(x) - \Lambda_{u,D}(z_j)| + |\Lambda_{h,B}(z_j) - \Lambda_{h,B}(y)| \\ &\le \mathscr{L}(h,B)\big(d(x,z_j) + d(z_j,y)\big) \le \mathscr{L}(h,B)\big(l(\gamma_j|_{[0,t_j]}) + l(\gamma_j|_{[t_j,1]})\big) \\ &= \mathscr{L}(h,B)\big(d(x,y) + (1/j)\big)\end{aligned}$$

for each $j \in \mathbf{N}$. This shows that $L(\Lambda, B) = \mathscr{L}(h, B)$. □

The second lemma provides us with a "weak Poisson modification" of a supersolution.

4.2. Lemma. *Let u be a supersolution that is not a subsolution. Then there exists a supersolution $\hat{u}$ such that*

(1) $u \ge \hat{u}$ *in* X,
(2) $\sup_{x \in X}\big(u(x) - \hat{u}(x)\big) > 0$.

Proof. Let $E \subset X$ be such that

$$E' = \{x \in X : u(x) > \Upsilon_{u,E}(x)\} \subset E$$

is not empty. Since (X, d) is a length space,

$$\mathscr{L}(u, E') = \mathscr{L}(\Upsilon_{u,E}, E') \le L(\Upsilon_{u,E}, E') \le \mathscr{L}(u, E),$$

and consequently $\Upsilon_{u,E'} \le \Upsilon_{u,E} < u$ in E' by the definition of $\Upsilon_{u,E}$. Let us now define

$$\hat{u}(x) = \Upsilon_{u,E'}(x).$$

It is clear that this function satisfies (1) and (2), and thus we only need to prove that $\hat{u}$ is a supersolution. To this end, we argue by contradiction and assume that there is $B \subset X$ such that the set $B' = \{\hat{u} < \Lambda_{\hat{u},B}\}$ is not empty. By using a similar reasoning as above, we may in fact assume that $B' = B$. Furthermore, $B \cap E' \ne \emptyset$, because otherwise we would contradict the fact that u is a supersolution.

To conclude the proof, we consider the set $D = \{\Lambda_{\hat{u},B} > u\} \subset B$. By the various definitions,

$$\mathscr{L}(u, D) = \mathscr{L}(\Lambda_{\hat{u},B}, D) \le L(\Lambda_{\hat{u},B}, D) \le \mathscr{L}(\hat{u}, B),$$

and we obtain that $\Lambda_{u,D} \ge \Lambda_{\hat{u},B}$ in D. Since u is a supersolution, this implies that $u \ge \Lambda_{\hat{u},B}$ in X, and consequently $D = \emptyset$. Now we can easily deduce that $B \subset E'$. Indeed, if $x \in X \setminus E'$, then $\hat{u}(x) = u(x) \ge \Lambda_{\hat{u},B}(x)$, which means that $X \setminus E' \subset X \setminus B$. Using once again the assumption that (X, d) is a length space, we see that $L(\max\{\hat{u}, \Lambda_{\hat{u},B}\}, E') = \mathscr{L}(u, E')$. Hence $\hat{u} \ge \Lambda_{\hat{u},B}$ by the definition of $\hat{u}$, and we finally conclude that $B = \emptyset$. □

The existence of an AMLE is a direct consequence of Lemmas 4.1 and 4.2.

4.3. Theorem. *Let (X,d) be a separable length space, and suppose $f\colon A\to\mathbf{R}$ is Lipschitz. Then there exists an absolutely minimizing Lipschitz extension of f to X.*

Proof. Define

$$h(x)=\inf\{u(x): u \text{ is a supersolution}\}.$$

By Lemma 4.1, h is a supersolution. If it was not a subsolution, we could use Lemma 4.2 to construct a supersolution $\hat{h}\le h$ such that $\hat{h}(x)<h(x)$ for some $x\in X$, which obviously contradicts the definition of h. Hence h is both a subsolution and a supersolution, and it then follows from Proposition 3.4 that it is an AMLE of f to X. □

We do not know how optimal our assumptions on the space (X,d) are. The separability was needed to guarantee the validity of Ascoli's theorem, and it was not used anywhere else. The second assumption, that (X,d) is a length space, in turn ensures that we can "glue" together two Lipschitz functions without increasing the Lipschitz constant, see the proof of Lemma 4.1. The following somewhat artificial example shows that this need not be true even if the space is close to being a length space.

4.4. Example. For $\alpha\in(0,\frac{1}{2}\pi)$ let

$$X=\bigl\{(x,y)\in\mathbf{R}^2 : x\in\mathbf{R},\ y=|x|\tan\alpha\bigr\}.$$

Then the space X equipped with the usual Euclidean distance in $\mathbf{R}^2$ is quasiconvex with constant $\sqrt{1+\tan^2\alpha}$, i.e., for all $z_1,z_2\in X$ there exists a curve γ in X connecting z_1 to z_2 such that

$$l(\gamma)\le\sqrt{1+\tan^2\alpha}\,|z_1-z_2|.$$

Moreover, the constant $\sqrt{1+\tan^2\alpha}$ is sharp. Let

$$u(z)=\begin{cases}|z|, & \text{if } z\in X^+=X\cap\{x>0\},\\ -|z|, & \text{if } z\in X\setminus X^+.\end{cases}$$

It is elementary to check that $L(u,X)=\sqrt{1+\tan^2\alpha}$, although $L(u,X^+)=L(u,X\setminus X^+)=1$.

4.5. Remarks. (i) An open question in this general setting is the uniqueness of AMLEs. In the case of $\mathbf{R}^n$, it is known that if $\mathbf{R}^n\setminus A$ is bounded, then for each Lipschitz function f defined in A there exists a unique AMLE. The proof of this result uses heavily the Euler equation of the problem, see [7], and the same method cannot be directly applied here.

(ii) As mentioned in the introduction, an alternative way to prove the existence of an AMLE in $\mathbf{R}^n$ is via approximation by p-harmonic functions as $p \to \infty$. Thanks to the development of the theory of Sobolev spaces on metric measure spaces (see [6] and the references therein), it is nowadays possible to consider p-harmonic functions in a quite abstract setting, cf. [11], and hence one might be able to base a general existence result on the same idea. However, in such an approach a doubling measure and a suitable Poincaré inequality are needed, whereas the extension problem itself contains no reference to a measure of any kind.

5. Appendix: Subsolutions and supersolutions in $\mathbf{R}^n$

In $\mathbf{R}^n$ (equipped with the usual metric), a theorem of Jensen [7] states that a continuous function is a (local) AMLE if and only if it is a viscosity solution of the *infinity Laplace equation*

$$-\Delta_\infty u(x) = 0, \tag{5.1}$$

where

$$\Delta_\infty u = \tfrac{1}{2}\nabla\left(|\nabla u|^2\right)\cdot \nabla u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}.$$

By definition, being a viscosity solution is equivalent to being both a viscosity subsolution and a viscosity supersolution (see [5], [7]). Thus, in $\mathbf{R}^n$, there are two separate concepts of a subsolution (respectively supersolution) that, a priori, could give two different classes of functions. We show below that this is not the case. The proof uses the following characterization of the viscosity subsolutions of the infinity Laplacian due to Crandall, Evans and Gariepy [4].

5.2. Theorem. *A continuous function u is a viscosity subsolution of* (5.1) *in $\Omega \subset \mathbf{R}^n$ if and only if it enjoys the following comparison with cones from above*: *whenever $D \Subset \Omega$ is open, $x_0 \in \mathbf{R}^n$, and*

$$C(x) = a + b|x - x_0|, \qquad a, b \in \mathbf{R}$$

is such that $u(x) \le C(x)$ for every $x \in \partial(D \setminus \{x_0\})$, one has $u(x) \le C(x)$ for all $x \in D$.

Recall that we may now use the equivalent definition of subsolutions given by Aronsson. Thus u is a local subsolution of the extension problem in $\Omega \subset \mathbf{R}^n$ if and only if

$$u(x) \le \inf_{y\in\partial D} \{u(y) + L(u, \partial D)|x - y|\}$$

for every $x \in D$ and for every open $D \Subset \Omega$.

5.3. Proposition. *A function* $u\colon \Omega \to \mathbf{R}$ *is a local subsolution of the extension problem if and only if it is a viscosity subsolution of* (5.1).

Proof. Assume first that u is a viscosity subsolution of the infinity Laplace equation. Then u is locally Lipschitz continuous, see [4], [9]. Next notice that, for each fixed $y \in \partial D$, $D \Subset \Omega$ open, the function

$$x \mapsto u(y) + L(u, \partial D)|x - y|$$

is a classical solution of (5.1) in D. Hence, by the standard theory of viscosity solutions, see [5], [8],

$$\Upsilon_{u,D}(x) = \inf_{y \in \partial D} \{u(y) + L(u, \partial D)|x - y|\}$$

is a viscosity supersolution. Since $\Upsilon_{u,D} = u$ on ∂D, the comparison principle, cf. [7], yields

$$u \le \Upsilon_{u,D} \qquad \text{in } D,$$

as required.

For the converse implication, we use Theorem 5.2 above. Suppose that the assertion does not hold. Then there is a subset $D \Subset \Omega$ and a cone $C(x) = a + b|x - x_0|$ such that

$$C(y) \ge u(y) \qquad \text{for all } y \in \partial(D \setminus \{x_0\})$$

and

$$C(z) < u(z) \qquad \text{for some } z \in D.$$

Let D_1 be the connected component of

$$\{x \in D : C(x) < u(x)\}$$

containing z. This means that $C(x) = u(x)$ on ∂D_1, and, in particular, that $L(u, \partial D_1) = L(C, \partial D_1) = |b|$. Moreover, the vertex x_0 cannot be in D_1.

To conclude the proof, we consider separately the cases $b \ge 0$ and $b < 0$. First, if $b \ge 0$, let $y^* \in \partial D_1$ be a point on the line segment between z and x_0. Then

$$\begin{aligned} u(z) &\le \inf_{y \in \partial D_1} \{u(y) + L(u, \partial D_1)|z - y|\} \le u(y^*) + L(u, \partial D_1)|z - y^*| \\ &= C(y^*) + b|z - y^*| = a + b|x_0 - y^*| + b|z - y^*| = a + b|x_0 - z| = C(z). \end{aligned}$$

This obviously contradicts the choice of z. On the other hand, if $b < 0$, we choose $y^* \in \partial D_1$ so that z lies on the line segment between x_0 and y^*. Then again

$$\begin{aligned} u(z) &\le u(y^*) + L(u, \partial D_1)|z - y^*| = C(y^*) - b|z - y^*| \\ &= a + b|x_0 - y^*| - b|z - y^*| = a + b|x_0 - z| = C(z), \end{aligned}$$

and we are done. □

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Received 14 June 2000