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A FREE BOUNDARY PROBLEM RELATED TO SINGLE-LAYER POTENTIALS

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Abstract. Let Ω be a bounded domain in $\mathbf{R}^n$ with sufficiently regular boundary Γ, and let J be the operator which maps a function f on Γ to the restriction Jf to Γ of its single layer potential. In the present paper, we consider the problem of characterizing those domains Ω for which the constant function $f\equiv 1$ is an eigenfunction for J and some related inverse problems. One of our main results concerning the former problem in $\mathbf{R}^2\cong\mathbf{C}$ is the following: If Γ is a rectifiable curve such that $\mathbf{C}\setminus\overline{\Omega}$ is a Smirnov domain and $f\equiv 1$ is an eigenfunction for J, then Γ is a circle.

1. Introduction

Let Ω be a bounded open set in $\mathbf{R}^n$, with sufficiently smooth boundary Γ, such that $\overline{\Omega}$ is homeomorphic to a closed ball. If dS denotes "area" (i.e. $(n-1)$-dimensional Hausdorff measure) on Γ, we may associate to each $f\in C(\Gamma)$ (the space of real-valued continuous functions on Γ) its *single-layer potential*

$$u(x)=\int_\Gamma k_n(x-y)f(y)\,dS(y), \tag{1.1}$$

where k_n denotes the kernel of Newtonian potential theory (so it satisfies, in the sense of distributions on $\mathbf{R}^n$, the equation

$$\Delta k_n=-\delta$$

where δ is, as usual, a unit mass at $x=0$), i.e.

$$k_n(x)=\begin{cases}-(1/2\pi)\log|x|, & n=2,\\ [1/(n-2)\sigma_n]|x|^{2-n}, & n\geq 3,\end{cases}$$

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and σ_n denotes the measure of the unit sphere in $\mathbf{R}^n$. It is well known that u, which obviously is harmonic on $\mathbf{R}^n \setminus \Gamma$, extends continuously to all of $\mathbf{R}^n$. In fact, this is true under weaker conditions e.g. for $n \geq 3$ it suffices that $f \in L^p(\Gamma) := L^p(\Gamma; dS)$ with $p > (n-1)/(n-2)$. Moreover, the map of $C(\Gamma)$ taking f to $u|_\Gamma$ is compact.

It is also known that for any p, $1 \leq p < \infty$, if $f \in L^p(\Gamma)$, then u belongs to a Sobolev space on $\mathbf{R}^n$ which admits L^p "traces" on $(n-1)$-dimensional smooth hypersurfaces, so that $f \to u|_\Gamma$ is a well defined operator on $L^p(\Gamma)$ and this operator is moreover compact. (There is a very extensive literature on these matters, see [DL], [Fo], [V]). Here we will only consider $p = 2$ and denote by J the corresponding map $f \to u|_\Gamma$ which is thus a compact linear operator in $L^2(\Gamma; dS)$. J is self-adjoint, and it is also well known that it is injective when $n \geq 3$. Thus $L^2(\Gamma)$ is, for $n \geq 3$, spanned by mutually orthogonal eigenfunctions of J, i.e. solutions φ to

$$J\varphi = \lambda\varphi \tag{1.2}$$

with $\lambda > 0$. (A slight exception occurs for $n = 2$, see below.)

The goal of this paper is to survey some "inverse" problems (also called free boundary problems) naturally arising in this framework; i.e. find those domains with certain spectral features, such as admitting a given number λ as eigenvalue, or a given function (especially, $\varphi \equiv 1$) as eigenfunction, of the associated single-layer potential operator J. This will also lead us in to some deeper aspects of classical theorems of Newton, Dive and others concerning gravitational attraction.

As a starting point, let us compute the spectrum and eigenfunctions when Ω is the unit ball in $\mathbf{R}^n$. (Presumably, these results are known in the literature, but since we could not find the sources we present the simple derivations.) Our argument is based on the following, in principle well known, calculus lemma, whose proof (a simple application of Green's identity) is left to the reader.

Lemma 1.3. *Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary Γ is a smooth hypersurface. Let u, v be harmonic in Ω, Ω_e respectively (where $\Omega_e = \mathbf{R}^n \setminus \overline{\Omega}$). Suppose u and v extend, as well as their first order partial derivatives, continuously to each point of Γ and that $u(y) = v(y)$ for all y in Γ. Let F denote the distribution (in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$) which equals u in Ω and v in Ω_e. Then ΔF is a bounded measure μ supported on Γ, absolutely continuous with respect to hypersurface measure dS, and*

$$\frac{d\mu}{dS} = \frac{\partial v}{\partial N} - \frac{\partial u}{\partial N},$$

at each point of Γ, where N is an outer-directed unit normal vector. If $n \geq 3$ and $v(\infty) = 0$, then F is the single layer potential of ΔF.

As an application, let H_m denote a homogeneous harmonic polynomial of degree m in $x = (x_1, \dots, x_n)$. Then, as is well known (and easy to check), the function $H_m(x)/|x|^{2m+n-2}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ (indeed, it is the Kelvin transform of H_m) so we may apply the lemma to the situation where Ω is the unit ball of $\mathbf{R}^n$, $u(x) = H_m(x)$ and $v(x) = H_m(x)/|x|^{2m+n-2}$. At a point $y \in \partial\Omega$ we have

$$\frac{\partial u}{\partial N}(y) = \frac{\partial H_m}{\partial N}(y) = \langle (\nabla H_m)(y), y \rangle = mH_m(y),$$

where ∇ denotes the gradient operator and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbf{R}^n$, and, by a short calculation,

$$\frac{\partial v}{\partial N}(y) = (-m - n + 2)H_m(y).$$

Hence ΔF is a measure on the unit sphere Γ, whose Radon–Nikodym derivative with respect to dS is equal, at y, to $(-2m-n+2)H_m(y)$. On the other hand, the single-layer potential U of this measure satisfies $\Delta U = -\Delta F$, so $U + F$ is harmonic on $\mathbf{R}^n$ and, since it vanishes at ∞, it is identically zero. Consequently, *F is the single-layer potential of the measure $(2m + n - 2)H_m dS$ on Γ.* This shows that

$$JH_m = [1/(2m + n - 2)]H_m.$$

Thus, all homogeneous harmonics ("spherical harmonics") are eigenfunctions of J and, since these span $C(\Gamma)$ they yield all the eigenfunctions. (Of course, $1/(2m + n - 2)$ is an eigenvalue whose multiplicity is the dimension of the space of spherical harmonics of degree m, i.e. (for $m \geq 2$),

$$A(n, m) - A(n, m - 2),$$

where

$$A(n, m) = \frac{n(n + 1)(n + 2)\dots(n + m - 1)}{m!}$$

is the dimension of the vector space of all homogeneous polynomials of degree m in n variables. The choice of a basis within the space of eigenfunctions corresponding to this eigenvalue is of course arbitrary.)

Note that a slight modification is needed in the case $n = 2$, $m = 0$ where the argument presented breaks down. It is easy to see that for $n = 2$, the constant function 1 is in $\ker J$, i.e. is associated to the eigenvalue 0. If the disk had radius R, then 1 would be an eigenfunction with the associated eigenvalue $\log R$.

One of the first things we observe from this example is that, in all dimensions, 1 is an eigenfunction for the sphere, and it is natural to study the converse assertion:

Conjecture 1.4. *If the function $\phi \equiv 1$ is an eigenfunction for J, then Γ is a sphere.*

This can be formulated in another, equivalent way. In view of Lemma 1.3, 1 is an eigenfunction if and only if there is a harmonic function v on Ω_e vanishing at infinity and equal to 1 on $\partial\Omega$, whose normal derivative on $\partial\Omega$ is constant. This v is thus (a constant multiple of) what is called in electrostatics (when $n = 3$) the *conductor potential* associated to the compact set $\partial\Omega$, and its normal derivative equals the charge density (with respect to surface area) arising from a unit charge on the "conductor" $\partial\Omega$ distributing itself into electrostatic equilibrium (for the physics of this, see e.g. [Je], [Max]; for the mathematical aspects [Ke], [W]). Hence, an alternative formulation of Conjecture 1.4 is:

Conjecture 1.4$'$. *If the equilibrium measure on $\partial\Omega$ is proportional to surface measure (i.e., the equilibrium charge distribution has constant density with respect to surface measure), Ω is a ball.*

In this form, the conjecture was apparently first enunciated by P. Gruber, and proved by W. Reichel [R] as a consequence of much more general results concerning elliptic boundary value problems. The two-dimensional version had been proved earlier by Martensen [Mar]; see [R] for further references. Reichel's paper has inspired further generalizations ([AB], [GS]). Reichel's work is based on the "moving hyperplane" method of Alexandroff and Serrin. (We are indebted to Henrik Shahgholian for directing us to all this literature.)

Of course, Conjecture 1.4$'$ only becomes precise after we enunciate the regularity hypotheses imposed on Ω. Reichel assumed the boundary has $C^{2,\alpha}$ regularity for some $\alpha > 0$ ([R, Theorem 2]). We shall deduce below, as part of a more general result, that for $n = 2$ weaker regularity suffices (the so-called Smirnov condition, cf. [Du]). The referee pointed out that, in the recent paper [MR], Conjecture 1.4$'$ was proved for Lipschitz domains in $\mathbf{R}^2$ and convex bodies in $\mathbf{R}^n$, $n \geq 3$.

From the above calculations, we also infer:

Proposition 1.5. *If Γ is the unit sphere of $\mathbf{R}^n$, then the associated single-layer potential operator J on $L^2(\Gamma; dS)$ is compact, and in the Schatten class $\mathscr{S}_p$ if and only if $p > n-1$. (In particular, it is Hilbert–Schmidt only for $n = 2$, and never trace class.)*

For definition of $\mathscr{S}_p$ see [GK]. Presumably these results hold for every sufficiently regular closed surface, but we are not aware of any studies in this direction.

2. The two-dimensional case

We keep the notation introduced in the previous section, and start out with an observation which is valid in all dimensions. Let u be defined by (1.1) with $f \equiv 1$, and assume that $f \equiv 1$ is an eigenfunction for the operator J with eigenvalue c. Then, as discussed in Section 1, $u \equiv c$ in the domain Ω, while in the exterior domain Ω_e, u is a harmonic function vanishing at ∞ for $n \geq 3$ or being $O(\log|z|)$ for $n=2$, and constant on Γ. Thus, up to a scalar multiple, u is the conductor potential of Γ. Moreover, if Γ is sufficiently smooth, say C^2, then

$$\left(\frac{\partial u}{\partial N}\right)_e - \left(\frac{\partial u}{\partial N}\right)_i = f \equiv 1, \tag{2.1}$$

where we have used the notation v_i, v_e for the limits of a function $v(x)$ harmonic in $\mathbf{R}^n \setminus \Gamma$ as $x \to \Gamma$ in Ω, Ω_e respectively. We conclude, by the assumption that $u \equiv c$ in Ω, that

$$\left(\frac{\partial u}{\partial N}\right)_e \equiv 1; \tag{2.2}$$

i.e. assuming that $f \equiv 1$ is an eigenfunction for J, we deduce that the equilibrium potential u has also constant normal derivative on Γ in Ω_e. (For a merely rectifiable surface Γ (2.1), and hence (2.2), holds almost everywhere on Γ.)

We will need the theory of Hardy and Smirnov classes of analytic functions. The reader is referred to [Du], [Go], [Pr], [Kh1], and references therein for a basic account of the subject. Here is a synopsis.

Recall that a Jordan domain $\Omega \subset \mathbf{C}$ bounded by a rectifiable curve Γ is called *Smirnov* if the derivative φ' of the Riemann mapping function $\varphi\colon \mathbf{D} \to \Omega$, $\mathbf{D} = \{z : |z| < 1\}$, (which necessarily belongs to the Hardy class $H^1(\mathbf{D})$), does not have a singular inner factor (φ is allowed to have a simple pole if $\infty \in \Omega$). Note in passing that in a recent preprint [JS] it was shown that, surprisingly, in a great many cases, even if one of the domains Ω, Ω_e bounded by Γ is non-Smirnov, the other nevertheless is Smirnov—the solution of a long standing problem. If a curve Γ is sufficiently smooth, e.g., a C^1-curve, it is well known that both domains Ω, Ω_e are Smirnov.

An analytic function f is said to belong to the Smirnov class $E_p(\Omega)$ if there exists a sequence of rectifiable curves $\{\Gamma_i\}$ in Ω, $\Gamma_i \to \Gamma := \partial\Omega$, so that

$$\sup_i \left\{ \int_{\Gamma_i} |f|^p \, ds \right\} < +\infty.$$

We define $E_\infty(\Omega)$ to be $H^\infty(\Omega)$, the space of bounded analytic functions in Ω. $E_1(\Omega)$ can be viewed as the space of analytic functions representable by the Cauchy integral of their boundary values on Γ, the latter being moreover integrable on Γ. In Smirnov domains (and only in Smirnov domains—cf. [Kh2]), if $f \in E_{p_1}$, $f|_\Gamma \in L^{p_2}$, $p_2 > p_1$, then $f \in E_{p_2}$. An analytic function f in Ω is said to belong to the Smirnov class $N^+(\Omega)$ if $\log^+|g \circ \varphi|$, $\varphi\colon \mathbf{D} \to \Omega$ being a Riemann mapping, is uniformly integrable on concentric circles $\{|z| = r,\ 0 < r < 1\}$. If Ω is Smirnov, $f \in N^+(\Omega)$, and $f|_\Gamma \in L^p$ (Γ, ds), $0 < p \le \infty$, then $f \in E_p(\Omega)$.

Theorem 2.3. *Let Γ be a rectifiable curve such that Ω_e is a Smirnov domain and assume that the function $\phi \equiv 1$ on Γ is an eigenfunction of the operator J. Then, Γ is a circle.*

Proof. Let u as above denote the equilibrium potential of Γ. The function

$$h(z) := -\frac{1}{4\pi}\int_\Gamma \frac{ds(\zeta)}{\zeta - z} = \frac{\partial u}{\partial z}, \tag{2.4}$$

where as usual

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

is analytic off Γ and vanishes in Ω. Thus, $h \in E_1(\Omega_e)$ and since h is representable in Ω_e by a Cauchy integral with density $ds/d\zeta \in L^\infty(\Gamma)$, has boundary values $(i/2)ds/d\zeta = h(\zeta)$ a.e. on Γ, and Ω_e is Smirnov, we conclude that $h \in H^\infty(\Omega_e)$. From (2.2) it also follows that

$$\left(\frac{\partial u}{\partial \bar z}\right)_e = \left(\frac{1}{2}\,\nabla u\right)_e = \frac{1}{2}\left(\frac{\partial u}{\partial N}\right)_e N = -\frac{1}{2}\,iT,$$

where $T = d\zeta/ds$ is the unit tangent vector on Γ encoded (as is the unit normal vector N) as a complex number, and therefore a.e. on Γ we have

$$-\frac{1}{2}\,\overline{T} = ih(z) =: H(z), \tag{2.5}$$

where $H \in H^\infty(\Omega_e)$, $H(\infty) = 0$. Let $\varphi\colon \mathbf{D} \to \Omega_e$ be the Riemann mapping function onto Ω_e normalized by $\varphi(0) = \infty$. Thus,

$$\varphi(z) = \frac{a}{z} + k(z), \tag{2.6}$$

where a is a constant and k is analytic in $\mathbf{D}$. Since Γ is rectifiable $k' \in H^1(\mathbf{D})$. Set $G(z) = H\big(\varphi(z)\big)$. Then, G is analytic in $\mathbf{D}$, $G(0) = 0$ and $G \in H^\infty(\mathbf{D})$.

Now, transfer (2.5) to $\mathbf{D}$. We obtain

$$G(z) = -\frac{1}{2}\,\frac{\overline{\varphi'}\,d\bar z}{|\varphi'|\,|dz|} \tag{2.7}$$

a.e. on $\mathbf{T}$, the unit circle. Squaring both sides of (2.7) and observing that on $\mathbf{T}$

$$\frac{d\bar z}{|dz|} = -i\,\bar z = -\frac{i}{z},$$

we obtain from (2.7) that

$$z^2 G^2 \varphi' = \frac{1}{2}\,\overline{\varphi'} = \frac{1}{2}\,\overline{\left(-\frac{a}{z^2} + k'(z)\right)} = \frac{1}{2}\left(-\bar{a}z^2 + \overline{k'(z)}\,\right) \tag{2.8}$$

holds almost everywhere on $\mathbf{T}$, or, equivalently,

$$z^2\left(G^2\varphi' + \tfrac{1}{2}\,\bar{a}\right) = \tfrac{1}{2}\,\overline{k'(z)} \tag{2.9}$$

a.e. on $\mathbf{T}$. Now, the left-hand side of (2.9) is the boundary value of an $H^1(\mathbf{D})$-function ($G^2\varphi' \in H^1(\mathbf{D})$!), while the right-hand side is the conjugate of an H^1-function.

Since the only real H^1-functions are constants, both sides are constants. But the left-hand side vanishes at 0, hence identically. Therefore, $k' = 0$ and k is a constant, i.e., $\phi(z)$ is a Möbius transformation and Γ must be a circle. ▫

Remark. Another proof of Theorem 2.3 is contained in that of Theorem 3.26 below.

Observe, that the major step in the proof of Theorem 2.3 is the following proposition (cf. (2.5) ff).

Proposition 2.10. *Let Γ be a closed, rectifiable Jordan curve and $T(z)$ the unit tangent vector to Γ defined a.e. on Γ. Suppose that*

$$\overline{T(z)} = H(z), \qquad \text{a.e. on } \Gamma, \tag{2.11}$$

where $H(z)$ stands for nontangential boundary values of a bounded analytic function H in the exterior Ω_e of Γ with $H(\infty) = 0$. Then, Γ must be a circle.

Since, in view of a theorem of Keldysh and Lavrentiev (see e.g. [Du]) a holomorphic function F is in $E_p(\Omega)$ for a bounded domain Ω if and only if $(F\circ\varphi)(\varphi')^{1/p}$ belongs to $H^p(\mathbf{D})$ (φ being a Riemann map of $\mathbf{D}$ onto Ω), the conclusion remains valid if instead of $H \in H^\infty(\Omega)$, one takes $H \in E_2(\Omega)$, normalized by $H(z^0) = 0$ for some $z^0 \in \Omega$. This suggests the natural question whether Proposition 2.10 remains true if we replace $\overline{T}$ by T. The following example shows that without any additional hypothesis the answer is “No”.

Example 2.12. Let Ω be a non-Smirnov “pseudocircle” (cf. [Du], [Pr], [Shl]), “centered” at the origin, i.e., the conformal mapping $\varphi\colon \mathbf{D} \to \Omega$ with $\varphi(0) = 0$ satisfies

$$\varphi'(z) = \exp\!\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right), \tag{2.13}$$

where $\mu \geq 0$ and is singular with respect to Lebesgue measure. (From [JS] it follows, however, that Ω_e is Smirnov.) Then, on $\Gamma := \varphi(\mathbf{T})$ we have ($|\varphi'| = 1$ a.e.):

$$T\big(\varphi(z)\big) = \frac{d\varphi}{ds} = i\,z\varphi',$$

so $T \circ \varphi$ is equal to a bounded analytic function on $\mathbf{T}$, but Γ is not a circle.

However, assuming some additional regularity for the curve Γ, one can prove an analogue of Proposition 2.10.

Theorem 2.14. *Let Γ be a C^1-curve, i.e. $T(z)$ is continuous on Γ. If*

$$T(z) = g(z) \qquad \text{a.e. on } \Gamma, \tag{2.15}$$

where g is a function in the Smirnov class N^+ in either Ω, or Ω_e, then Γ is a circle.

Remark. Since Γ is assumed to be C^1, both domains Ω, Ω_e are Smirnov and accordingly $E_p(\Omega) \subset N^+(\Omega)$, $E_p(\Omega_e) \subset N^+(\Omega_e)$ for all $p > 0$ (cf. [Kh1]).

Proof. Without loss of generality assume $g \in N^+(\Omega)$. Since $g(z) = T(z)$ is bounded on Γ and $g \in N^+(\Omega)$, g is in $H^\infty(\Omega)$. As above, let φ denote a Riemann mapping φ: $\mathbf{D} \to \Omega$ and similarly to (2.8) rewrite (2.15) in the form

$$-\frac{z^2\varphi'}{G^2} = \overline{\varphi'} \qquad \text{a.e. on } \mathbf{T}, \tag{2.16}$$

where $G := g \circ \varphi \in H^\infty(\mathbf{D})$. Since $G = T \circ \varphi$ a.e. on $\mathbf{T}$ and $T \circ \varphi \in C(\mathbf{T})$, G is continuous in $\overline{\mathbf{D}}$. Hence, taking into account that G is an inner function, we conclude that G must be a finite Blaschke product. But the change in $\arg G$ going once around the circle $\mathbf{T}$ equals the corresponding change in $\arg T \circ \varphi$, which is 2π, so G must be a Möbius automorphism of $\mathbf{D}$, say $G = e^{i\alpha}(z - a)/(1 - \bar{a}z)$ for some $|a| < 1$ and real constant α. Then, from (2.16) it follows that

$$-e^{-2i\alpha}(1 - \bar{a}z)^2\varphi' = \overline{(1 - \bar{a}z)^2\varphi'} \qquad \text{a.e. on } \Gamma,$$

and, as before, both sides are constants and so $\varphi' = c(1 - \bar{a}z)^{-2}$ which implies $\varphi = c_1(1 - \bar{a}z)^{-1} + c_2$. Thus, Γ is indeed a circle. ▫

3. A "quadrature formula" interpretation of the equilibrium distribution

With $\Omega \subset \mathbf{R}^n$ as in Section 1, there is (as already discussed, cf. [W]) a unique probability measure on its boundary Γ whose potential is constant in Ω (the "equilibrium distribution"). It has minimal energy among all probability measures on Γ, and represents physically the distribution of charge at equilibrium into which a unit charge placed on the conductor Γ resolves itself. We recall that the equilibrium distribution is of the form $f\,dS$ for some smooth function f, the equilibrium charge density on Γ, which apart from a constant factor of normalization equals $-\partial U/\partial N$, where U (the "conductor potential" of Γ) is the unique harmonic function in Ω_e equal to 1 on Γ and 0 at infinity. (Here, and henceforth in this section, we assume that the number n of dimensions is ≥ 3.)

For a function u, which is harmonic and $O(|x|^{2-n})$ on a neighborhood of ∞ in $\mathbf{R}^n$, we have a convergent representation

$$u(x) = \sum_{m=0}^{\infty} \frac{h_m(x)}{|x|^{2m+n-2}} \tag{3.1}$$

valid on $\{|x| > R\}$ for some R, where h_m is a uniquely determined homogeneous harmonic polynomial of degree m (or zero). Indeed, the $O(|x|^{2-n})$ assumption is equivalent to the boundedness near 0 of the Kelvin transform $u^{\#}$ of u (so, 0 is a removable singularity for $u^{\#}$), and then (3.1) is just the result of applying Kelvin transformation to the Taylor expansion of $u^{\#}$ at 0 (see below for more discussion of Kelvin transforms). In particular, $h_0(x)$ in (3.1) is a constant important for our discussion and we will sometimes denote it by $h_0[u]$:

$$h_0[u] = \lim_{x\to\infty} |x|^{n-2}u(x) = u^{\#}(0). \tag{3.2}$$

Proposition 3.3. *Let $f\,dS$ be the equilibrium distribution for Γ. Then, there exists a constant C such that, for every u harmonic on $\overline{\Omega}_e$ and $O(|x|^{2-n})$ at ∞, we have the "quadrature formula"*

$$h_0[u] = C\int_\Gamma uf\,dS. \tag{3.4}$$

(If $0 \in \Omega$, C can be determined by setting $u = k_n$ in (3.4)). Conversely, a formula $h_0[u] = \int_\Gamma u\,d\mu$ with μ a bounded measure on Γ, valid for all u in the given class, implies that μ is a constant multiple of the equilibrium distribution.

Proof. For the equilibrium distribution $f\,dS$ we have

$$\int_\Gamma |x-y|^{2-n} f(x)\,dS(x) = A$$

for all $y \in \Omega$, where A is a constant (independent of y). Thus, for $u(x) = |x-y|^{2-n}$ (note that $h_0[u] = 1$) (3.4) holds, if we choose $C = A^{-1}$.

To complete the proof of formula (3.4), we need only show that finite linear combinations

$$\sum_{j=1}^{m} c_j|x-y^j|^{2-n} \qquad (y^j \in \Omega)$$

approximate uniformly, on $\overline{\Omega}_e$, all harmonic functions u of the stated class. In view of (3.1) it is enough to show that functions of the form $h_n(x)/|x|^{2m+n-2}$ can be approximated. This is quite elementary, and left to the reader.

As to the converse: it is enough to show that $\int_\Gamma u\,d\mu = 0$ for all u of the given class, implies the measure μ is 0, i.e. that the restrictions $u|_\Gamma$ are dense in $C(\Gamma)$. But, it is well known that for every smooth function g on Γ, the exterior Dirichlet problem $\Delta u = 0$ in Ω_e, $u = g$ on Γ has a unique solution with $u(\infty) = 0$. Moreover, such u is necessarily $O(|x|^{2-n})$ since $u^{\#}(y) = o(|y|^{2-n})$ at $y = 0$, implying that $y = 0$ is a removable singularity for $u^{\#}$. The proposition is proved. □

Our next aim is to transform Proposition 3.3, via Kelvin transformation, to *bounded* domains. For this, we shall require some well-known properties of inversion and Kelvin transforms on $\mathbf{R}^n$. Since it is hard to find suitable references, we outline the proofs.

We recall the definition of *inversion* with respect to the unit sphere Σ of $\mathbf{R}^n$: The inverse point x' to x is

$$x' = I(x) := |x|^{-2}x \tag{3.5}$$

with the convention that $I(0) = \infty$, $I(\infty) = 0$. Clearly I is an involutive, sense reversing diffeomorphism of (the one-point compactification of) $\mathbf{R}^n$ on itself, which leaves points of Σ fixed. Moreover, it commutes with linear isometric transformations of $\mathbf{R}^n$ that leave 0 fixed.

Proposition 3.6. *For $\xi \in \mathbf{R}^n \setminus \{0\}$, and $x \in \mathbf{R}^n$, $|x|$ small:*

$$I(\xi + x) = |\xi|^{-2}(\xi + R_\xi x) + O(|x|^2) = I(\xi) + |\xi|^{-2}R_\xi x + O(|x|^2) \tag{3.7}$$

where R_ξ denotes reflection in the hyperplane H_ξ through 0 and orthogonal to ξ.

Proof. Since I commutes with rotations about axes through 0, it is enough to check (3.7) for $\xi = (a, 0, \dots, 0)$ with $a > 0$ (and then, R_ξ is reflection in $\{x_1 = 0\}$). The simple computation is left to the reader. □

It follows easily that, if Γ is a smooth hypersurface and Γ' its image under inversion and ξ, ξ' are corresponding points on Γ, Γ' then the unit normal vectors at those points transform into one another by the reflection R_ξ (note that $R_\xi = R_{\xi'}$).

If u is a smooth function on $\mathbf{R}^n$, and $u^{\#}$ its *Kelvin transform*:

$$u^{\#}(x) = u\big(I(x)\big)|x|^{2-n}, \tag{3.8}$$

a fairly straightforward computation shows that

$$\nabla u^{\#}(x') = |x'|^{-n}\big[(2-n)u(x)x' + R_x\big((\nabla u)(x)\big)\big] \tag{3.9}$$

relates the gradients of u, $u^{\#}$ at a pair of inverse points x, x'. Using this, it is not hard to check that for a pair Γ, Γ' of mutually inverse hypersurfaces the normal derivative of $u^{\#}$ at a point $x' \in \Gamma'$ and that of u at the corresponding point $x \in \Gamma$ relate by

$$\frac{\partial u^{\#}}{\partial N'}(x') = (n-2)|x|^{n-2}\langle x, N\rangle u(x) + |x|^n \frac{\partial u}{\partial N}(x) \tag{3.10}$$

(again N, N' denote corresponding unit normal vectors at x, x').

Let us now return to the single-layer potential of $f\,dS$. In view of Lemma 1.3 it is equivalent to consider a pair of harmonic functions u, v in Ω, Ω_e respectively with $v(\infty) = 0$, and $u = v$ on Γ. The jump $(\partial v/\partial N) - (\partial u/\partial N)$ across a point x of Γ then gives $-f$.

It is clear that if we subject this whole picture to inversion in Σ, and Kelvin-transform u and v, we get a pair of harmonic functions $v^{\#}$ in Ω'_e, and $u^{\#}$ in Ω' which again have the properties characterizing a single-layer potential.

In view of (3.10) the jump across Γ' of the normal derivative $\partial u^{\#}/\partial N' - \partial v^{\#}/\partial N'$ at x' equals $|x|^n$ times the corresponding jump $(\partial v/\partial N) - (\partial u/\partial N)$.

We have the following consequence, in view of Proposition 3.3:

Proposition 3.11. *With Ω, Γ having their usual meaning and $0 \in \Omega$, there is a "quadrature identity"*

$$u(0) = \int_\Gamma |x|^{-n} g(x) u(x)\, dS(x) \tag{3.12}$$

valid for all harmonic functions u in Ω. Here, $g(x)$ is some constant multiple of $f\big(I(x)\big)$, the equilibrium distribution density on Γ', the inversion of Γ with respect to the unit sphere.

A case of special interest is that where, in (3.4), f is constant, i.e. 1 is an eigenfunction of the operator J introduced in Section 1. Then also g in (3.12) is constant, and we can assert:

Proposition 3.13. *The operator J admits the constant function 1 as eigenfunction, for a domain Ω' containing 0, if and only if the domain Ω obtained by inverting $\mathbf{R}^n \setminus \overline{\Omega'}$ in the unit sphere admits the quadrature identity*

$$u(0) = c\int_\Gamma |x|^{-n} u(x)\, dS \tag{3.14}$$

for functions $u \in C(\overline{\Omega})$ that are harmonic on Ω. Here Γ denotes $\partial\Omega$, and c denotes the constant $(\int_\Gamma |x|^{-n}\, dS)^{-1}$.

Thus, Conjecture 1.4 is equivalent to (always, modulo our standing assumptions that our domains have C^1 boundaries and connected complements):

Conjecture 3.15. *A domain in* $\mathbf{R}^n$ *admitting the quadrature identity* (3.14) *is a ball.*

The affirmative answer follows from Reichel's theorem [R] mentioned in the introduction for domains with $C^{2,\alpha}$ boundaries. A more ambitious conjecture would be (we will see later that it is not true for all α):

Conjecture 3.16. *A domain in* $\mathbf{R}^n$ *admitting the quadrature identity*

$$u(0) = c \int_\Gamma |x|^\alpha u(x)\, dS(x) \tag{3.17}$$

where $c = (\int_\Gamma |x|^\alpha\, dS)^{-1}$ *and* $\alpha \in \mathbf{R}$, *is a ball.*

Remark. Since obviously Conjecture 1.4 is invariant with respect to translation, it follows from the above that, in particular, (3.14) holds for every *ball* Ω containing 0 (and not necessarily centered at 0, which at first sight may seem surprising). This is easy to check directly: it is equivalent to the assertion that for a fixed point y in a ball $B \subset \mathbf{R}^n$, the Poisson kernel as a function of $\xi \in \partial B$ has the form $C(y,n,R)\ |y-\xi|^{-n}$, where C depends only on y, n and the radius R, and this is true. Indeed, taking B to be centered at 0, and $y = (\rho, 0, \ldots, 0)$ for some $0 \le \rho < R$ (which involves no loss of generality), $|y-\xi|^2 = (\rho-\xi_1)^2 + \xi_2^2 + \cdots + \xi_n^2 = \rho^2 - 2\rho\xi_1 + R^2$, and inspection of the Poisson kernel (see [CH, p. 265] confirms our assertion.

Now, it is interesting that Conjecture 3.16 is known to be true for $\alpha \ge 1-n$, and even in the strengthened form that the ball must be centered at 0! This is implicitly contained in an argument of Shahgholian [Shah], who showed:

Theorem 3.18 (Shahgholian). *If* Ω *is a bounded domain with smooth boundary in* $\mathbf{R}^n$, *and for some continuous increasing function* F *on* $\mathbf{R}^+$ *we have*

$$\int_\Gamma \frac{F(|x|)\, dS(x)}{|x-y|^{n-2}} = \frac{c_1}{|y|^{n-2}} \tag{3.19}$$

for all $y \in \mathbf{R}^n \setminus \overline{\Omega}$, *and appropriate constant* c_1, *then* Ω *is a ball centered at* 0.

By a simple approximation argument (i.e. approximating harmonic functions on Ω by potentials of discrete measures on $\mathbf{R}^n \setminus \overline{\Omega}$) it follows that (3.19) is equivalent to the quadrature identity

$$u(0) = c_1 \int_\Gamma u(x) F(|x|)\, dS(x), \tag{3.20}$$

for harmonic functions u, and comparing this with (3.17) we see: *Shahgholian's theorem implies the truth of Conjecture* 3.16 *when* $\alpha \ge 0$. Actually, Shahgholian's argument yields a stronger conclusion than 3.18, as we show below: *The conclusion of Theorem* 3.18 *holds even under the weaker assumption that* $t^{n-1}F(t)$ *is nondecreasing*, so in particular Conjecture 3.16 is true for all $\alpha \geq 1-n$. But, the case of main interest, $\alpha = -n$, eludes this analysis. We are able, however, to completely settle Conjecture 3.16 (for all α) when $n = 2$. We shall return to this point.

We emphasize once more that regularity hypotheses are essential for Shahgholian's result: already the case $F =$ const. in (3.20) is known, due to a classical construction by Keldysh and Lavrentieff in dimension 2, to allow solutions which are Jordan domains with rectifiable (but not C^1) boundaries (see [Pr]), and this was extended to higher dimensions by Lewis and Vogel [LV].

To get a better perspective on Conjecture 3.16, let us assemble some equivalent formulations. The following is a mosaic of well-known results (for simplicity we tacitly assume $n \geq 3$):

Proposition 3.21. *For a bounded domain* Ω *containing* 0, *and having connected complement and* C^1 *boundary* Γ, *the following are equivalent properties of a measure* μ *on* Γ:

(a) *The Newtonian potential of* μ *equals a constant multiple of* $|x|^{2-n}$ *for sufficiently large* $|x|$.

(b) *The quadrature identity*

$$u(0) = c \int_\Gamma u \, d\mu \tag{3.22}$$

holds for all $u \in C(\overline{\Omega})$ *that are harmonic on* Ω.

(c) *Harmonic measure on* Γ, *with respect to* 0, *is a constant multiple of* μ.

(d) *The balayage onto* Γ *of a point mass at* 0 *is a constant multiple of* μ.

(e) *For a suitable constant* c, *the distributional partial differential equation*

$$\Delta v = c\mu - \delta$$

has a solution v *on* $\mathbf{R}^n$ *whose support is in* $\overline{\Omega}$.

Proof. We merely sketch the steps. Assumption (a) can be phrased thus, that (3.22) holds when u is the potential of a point mass at an arbitrary point $y \in \mathbf{R}^n \setminus \overline{\Omega}$. Then (3.22) for general u follows by linearity and approximation.

Now, (3.22) says $c\mu$ is a representing measure for evaluating harmonic functions at 0. As is well known, this measure is unique and is by definition the harmonic measure on Γ with respect to 0.

That (b) and (d) are equivalent is just the definition of balayage, see [L, p. 205]. Next, if (b) holds, the measure $c\mu - \delta$ annihilates harmonic functions and by a well-known general theorem [H, vol. 2] the distributional p.d.e.

$$\Delta v = c\mu - \delta \tag{3.23}$$

has a solution v on $\mathbf{R}^n$ of compact support (we can also construct this solution directly as the convolution of $c\mu - \delta$ with k_n). Since v is harmonic for large $|x|$, it vanishes off $\overline{\Omega}$, which establishes (e). Finally, assume (3.23) holds, and let u denote the single-layer potential of μ (normalized so that $\Delta u = \mu$). Then, $\Delta v = c\Delta u + \Delta k_n$ (where k_n is as in Section 1), so $v - cu - k_n =: h$ is a harmonic function on $\mathbf{R}^n$. Since $h(x) \to 0$ as $|x| \to \infty$, h vanishes identically so

$$cu + k_n = v$$

and, since v vanishes off $\overline{\Omega}$, u coincides there with a constant multiple of k_n, and hence of $|x|^{2-n}$, which proves (a). ▫

Thus, an equivalent form of Conjecture 3.16 in its strengthened form (in view of Proposition 3.21) is

Conjecture 3.24. *With Ω, Γ and dS as above, if harmonic measure on Γ with respect to 0 is proportional to $|x|^\alpha \, dS$, then Ω is a ball centered at 0.*

Remark. For $\alpha = -n$, the conjecture reads, however, merely that Ω is a ball; as we have seen, all balls containing 0 really are solutions in that case.

As remarked above, Shahgholian's theorem implies the correctness of this conjecture for $\alpha \geq 0$ but his argument really shows more:

Theorem 3.25. *With Ω, Γ and dS as above, if harmonic measure on Γ with respect to 0 is proportional to $|x|^\alpha dS$ for some real α, $\alpha \geq 1 - n$ then Ω is a ball centered at 0.*

Proof. The following argument is inspired by that of Shahgholian [Shah]. It could as well be carried out in his framework of single-layer potentials, but we shall transpose it to the framework of harmonic measure. We will prove slightly more, that if the harmonic measure equals $F(|x|)\, dS$ where F *is continuous on* $\mathbf{R}^+$ *with* $t^{n-1}F(t)$ *non-decreasing*, then Ω is a ball centered at 0.

The proof is by contradiction: suppose $F(|x|)\, dS$ is the harmonic measure, with F as above, and Ω is not a ball centered at 0. Let r_1, r_2 (where $0 < r_1 < r_2$) denote $\min|x|$ and $\max|x|$, respectively, for $x \in \Gamma$, and B_i be the open ball $\{|x| < r_i\}$ for $i = 1, 2$. Let G_i $(i = 1, 2)$ be the Green function for B_i with pole at 0, and G the corresponding Green function for Ω. We assume our Green functions are so normalized that their Laplacians are equal to δ on a neighborhood of 0 (thus, each Green function is subharmonic and negative on its domain of definition).

Let y^i denote a point on $\Gamma \cap \partial B_i$, $i = 1, 2$. Then $G(x) - G_1(x)$ is harmonic in B_1 and non-positive on its boundary (and not identically zero), so it is negative in B_1 and its maximum in $\overline{B_1}$ is attained at y^1. Hence, denoting by $\partial/\partial N$ the partial derivative in the direction of the outward normal vector, $(\partial/\partial N)(G - G_1)$ is non-negative at y^1 (observe that B_1 and Γ have the same normal at y^1). Indeed, by the Hopf maximum principle this normal derivative is *strictly* positive, so

$$\frac{\partial G}{\partial N} - \frac{\partial G_1}{\partial N} > 0 \qquad \text{at } y^1. \tag{3.26}$$

Now, the harmonic measure on Γ is $(\partial G/\partial N)\,dS$, and similarly for B_1 so (3.26) can be written as (in view of our assumption)

$$F(r_1) - b_n^{-1} r_1^{1-n} > 0 \tag{3.27}$$

where b_n is a positive constant depending only on n (more exactly, $b_n r_1^{n-1}$ is the "area" of ∂B_1).

In like manner, $G_2 - G$ is negative on Ω and, on $\overline{\Omega}$ attains its maximum value zero at y^2, so $(\partial G_2/\partial N) - (\partial G/\partial N)$ is positive at y^2, and we have

$$b_n^{-1} r_2^{1-n} - F(r_2) > 0. \tag{3.28}$$

But (3.27) and (3.28) imply $r_1^{n-1}F(r_1) > r_2^{n-1}F(r_2)$ which contradicts the hypothesis that $t^{n-1}F(t)$ is non-decreasing, and establishes the theorem. □

Remarks.

(i) As we have already noted, the C^1 smoothness hypothesis is crucial: many of the above equivalences fail for domains in $\mathbf{R}^n$, $n \geq 3$ that are not regular for Dirichlet's problem, e.g. "Lebesgue's spine" where the boundary is C^∞ except at one point. In those domains there are nontrivial measures supported on the boundary and annihilating all continuous harmonic functions, cf. [L].

(ii) The analog of Conjecture 3.16, even in its strong form, when integration is over the domain Ω with respect to volume measure rather than over its boundary, is well known to be true, and even for arbitrary radially symmetric weights, not merely $|x|^\alpha$: the only domains satisfying the corresponding quadrature formulae are balls centered at the origin, see [ASZ], [Ku].

(iii) There is still much to be said concerning functional equations characterizing the Poisson kernel of a ball. The content of Poisson's formula for a ball Ω in $\mathbf{R}^n$ with boundary Γ is: for every $y \in \Omega$, the harmonic measure on Γ with respect to y has a density with respect to surface measure dS that is, at x, inversely proportional to the $|x-y|^{-n}$. In other words, the Poisson kernel is given by

$$P(y;x^0) = \frac{|x^0 - y|^{-n}}{\int_\Gamma |x-y|^{-n}\,dS(x)}$$

for $x^0 \in \Gamma$, $y \in \Omega$. And, in view of our previous analysis, Reichel's theorem implies that in a domain Ω with sufficiently smooth boundary Γ, the Poisson kernel being of this form *even for one fixed point y of Ω* implies Ω is a ball. In this regard, the following theorem due to one of the present authors (HSS, unpublished) may be of interest (it neither implies, nor is implied by, Reichel's theorem):

Theorem. *Let Ω be a smoothly bounded, simply connected domain in $\mathbf{R}^n$, and suppose that, for every $x^0 \in \Omega$ the function*

$$h(y) := \frac{|x^0 - y|^{-n}}{\int_\Gamma |x-y|^{-n}\, dS(x)}, \qquad y \in \Omega$$

is harmonic on Ω. Then, Ω is a ball.

We conclude this section by settling Conjecture 3.16 (for all real α) when $n = 2$. It is more convenient for us, however, to work with harmonic measures, i.e. to formulate our result in the setting of Theorem 3.25.

Theorem 3.29. *Let Ω be a Jordan domain in $\mathbf{R}^2 \cong \mathbf{C}$ containing 0, and with rectifiable boundary Γ satisfying the Smirnov condition. Suppose the harmonic measure of Ω with respect to 0 equals $c|z|^\alpha ds$ for $z \in \Gamma$, where ds denotes arc length measure on Γ, $\alpha \in \mathbf{R}$ and c is a positive constant. Then*

(i) *For $\alpha = -2$, the solutions are precisely all balls Ω containing 0 (with arbitrary centers!).*
(ii) *For $\alpha = -3, -4, -5, \ldots$ there are solutions Ω which are not balls (these will be given explicitly).*
(iii) *For all other values of α, the only solutions are balls centered at 0.*

Proof. Let $z = \varphi(w)$ denote the Riemann map of $\{|w| < 1\}$ on Ω such that $\varphi(0) = 0$ and $\varphi'(0) > 0$. Then, writing $w = \rho e^{it}$, we have $ds(z) = |\varphi'(e^{it})|\, dt$ for $z = \varphi(e^{it}) \in \Gamma$, and the harmonic measure μ on Γ with respect to 0 satisfies $d\mu(z) = (1/2\pi)\, dt$, so our hypothesis is equivalent to

$$(1/2\pi)\, dt = c|z|^\alpha\, ds = c\big|\varphi\big(e^{it}\big)\big|^\alpha \big|\varphi'\big(e^{it}\big)\big|\, dt,$$

i.e.

$$\big|\varphi\big(e^{it}\big)\big|^\alpha \big|\varphi'\big(e^{it}\big)\big| = C, \tag{3.30}$$

a constant, a.e. for $0 \le t < 2\pi$. Now, $\varphi(w)/w$ is holomorphic in $\{|w| < 1\}$ and non-vanishing, so there is a holomorphic branch of $(\varphi(w)/w)^\alpha$ (choose e.g. that branch which is positive for $w = 0$), and (3.30) implies:

(3.31) $\big(\varphi(w)/w\big)^\alpha \varphi'(w)$ *is holomorphic in the unit disk, and has constant modulus on the unit circle.*

Now, since $\big(\varphi(w)/w\big)^\alpha$ is bounded away from 0 and ∞ near $\{|w| = 1\}$ it is an outer function, as is φ' by virtue of the hypothesis that Ω is Smirnov. Since an outer function in the disk with constant modulus on the boundary must be constant, we conclude

$$\big(\varphi(w)/w\big)^\alpha \varphi'(w) = A \tag{3.32}$$

where A is a constant (in fact, $A = \varphi'(0)^{\alpha+1}$). Thus,

$$\varphi(w)^{\alpha}\,\varphi'(w) - Aw^{\alpha} = 0. \tag{3.33}$$

We now consider cases. Suppose first $\alpha = -1$. Then, (3.33) reads

$$\frac{\varphi'(w)}{\varphi(w)} = \frac{A}{w},$$

integration of which yields $\varphi(w) = Bw^{A}$ for some constant B. The only choice of A for which φ is analytic and univalent in the unit disk is $A = 1$, and we conclude in this case that Ω is a ball centered at 0 (which also follows from Theorem 3.25 subject to the stronger regularity assumptions there, but for the unity of the present proof we have deduced it from (3.20)).

If $\alpha \neq -1$, integration of (3.33) yields

$$\varphi(w)^{\alpha+1} = B + Aw^{\alpha+1} \qquad (B = \text{ constant}). \tag{3.34}$$

In the crucial case $\alpha = -2$, this reads

$$\varphi(w) = w/(Bw + A),$$

which is a Möbius function and Ω is a ball containing 0.

To study the remaining cases, write (3.34) as

$$\left(\frac{\varphi(w)}{w}\right)^{\alpha+1} = Bw^{-\alpha-1} + A. \tag{3.35}$$

This implies $w^{-\alpha-1}$ is holomorphic on a neighborhood of 0 if $B \neq 0$, so unless $-\alpha - 1$ is a non-negative integer, B must vanish. Therefore, unless α takes one of the values $\{-1, -2, -3, -4, \ldots\}$ Ω certainly is a disk centered at 0. It remains to look at the cases where α takes the value $-k-1$ where $k \geq 2$ is an integer. Then (3.35) reads

$$\left(\frac{w}{\varphi(w)}\right)^{k} = Bw^{k} + A \qquad (k \in \{2, 3, 4, \ldots\}). \tag{3.36}$$

All these cases lead to nontrivial solutions, in which Ω is not a disk. For, if we choose $|B| < A$ the right side of (3.36) has a holomorphic k^{th} root in the unit disk and (3.36) can be written,

$$\varphi(w) = \frac{w}{(A + Bw^{k})^{1/k}}. \tag{3.37}$$

For each k, taking say $A = 1$, it is clear that for all choices of B with $|B|$ sufficiently small, φ given by (3.37) is a small perturbation of the identity map, and hence univalent on the closed disk. In these cases Ω is a solution of the original problem that is a Jordan domain with analytic boundary, and not a disk. This concludes the proof. □

Remark. Concerning assertions (i) and (iii) of the theorem, we do not know whether the assumption that Γ satisfies the Smirnov condition is necessary, except for the case $\alpha = 0$, when it is indeed necessary.

4. Equilibrium charge distribution, and the "Infinitesimal Form" of Newton's and Dive's theorems

Let Ω be an (open) ellipsoid in $\mathbf{R}^n$, centered at 0, and $t\Omega$ (for $t > 0$) a homothetic image of it. According to a classical theorem of Newton, a uniform mass distribution on the "shell" $t\Omega \setminus \overline{\Omega}$, where $t > 1$, exerts no gravitational force at points of the "cavity" Ω. (References will be given below; we begin by recapitulating some well-known relations.)

Equivalently, the Newtonian potentials of uniform mass distributions with equal densities on two homothetic ellipsoids differ by a constant inside the smaller one. One can put this in another, equivalent way: Let u be the Newtonian potential of a uniform mass distribution on Ω, and u_t the corresponding potential for $t\Omega$. Thus (assuming that $n \geq 3$) u is uniquely characterized by

$$\Delta u = -\chi_\Omega, \qquad u(\infty) = 0 \tag{4.1}$$

and u_t by

$$\Delta u_t = -\chi_{t\Omega}, \qquad u_t(\infty) = 0 \tag{4.2}$$

where χ_E denotes the characteristic function of a set E. Note that $\chi_{t\Omega}(x) = \chi_\Omega(t^{-1}x)$. Of course, (4.1) and (4.2) are to hold in the distributional sense. Hence, for $\varphi \in C_0^\infty(\mathbf{R}^n)$, from (4.2):

$$\int u_t \Delta\varphi \, dx = -\int \chi_{t\Omega}\varphi \, dx = -\int \chi_\Omega(t^{-1}x)\varphi(x) \, dx = -\int \chi_\Omega(y)\varphi(ty)t^n \, dy$$

and from (4.1) the last integral equals

$$t^n \int u(y)\Delta[\varphi(ty)] \, dy = t^{n+2} \int u(y)(\Delta\varphi)(ty) \, dy = t^2 \int u(t^{-1}x)(\Delta\varphi)(x) \, dx.$$

Hence $\int [u_t(x) - t^2 u(t^{-1}x)]\Delta\varphi \, dx = 0$ for all $\varphi \in C_0^\infty \ (\mathbf{R}^n)$, so the bracketed expression is harmonic on $\mathbf{R}^n$ and, vanishing at ∞, is identically 0. Thus

$$u_t(x) = t^2 u(t^{-1}x). \tag{4.3}$$

On a neighborhood of $x = 0$, u is real-analytic with a Taylor expansion $u = \sum_{m=0}^\infty u_m$, u_m being a homogeneous polynomial of degree m, so *in case u and u_t differ only by a constant $C(t)$ near* 0, we get from (4.3)

$$\sum_0^\infty u_m(x) + C(t) = u(x) + C(t) = u_t(x) = \sum_0^\infty t^2 u_m(t^{-1}x) = \sum_0^\infty t^{2-m} u_m(x). \tag{4.4}$$

This holds identically for x near 0, and t near 1. Setting $x = 0$ gives (note that $u_0 = u(0)$): $u(0) + C(t) = t^2 u(0)$, so $C(t) = u(0)(t^2 - 1)$. Substituting this in (4.4) and simplifying, we get

$$\sum_{m=1}^{\infty} u_m(x) = \sum_{m=1}^{\infty} t^{2-m} u_m(x).$$

Since this holds identically in t, we have $u_m = 0$ for all $m \neq 2$, and consequently $u = u_0 + u_2$, i.e.

$$u(x) = A + Q(x)$$

where A is a constant, and Q a homogeneous quadratic polynomial. This holds on a neighborhood of 0, and so by virtue of real-analyticity of u in Ω, throughout Ω. Thus, Newton's theorem on ellipsoidal shells implies that the potential of a uniform mass distribution on an ellipsoid Ω is a quadratic polynomial in Ω. More generally, as remarked by Ferrers over 100 years ago, the potential of a mass distribution over an ellipsoid whose density is given by a polynomial of degree m (in the Cartesian coordinates of the point), is, at points of the ellipsoid, a polynomial of degree $m + 2$; cf. [Kh3].

Much later, P. Dive [D] proved a converse of Newton's theorem: if Ω contains 0, and is strongly star-shaped with respect to 0, and for all $t > 1$ and sufficiently close to 1, the uniform mass distribution on the shell $t\Omega \setminus \overline{\Omega}$ produces a constant potential in Ω, then Ω must be an ellipsoid. As we just observed, the hypothesis of Dive's theorem implies that the potential of a uniform mass distribution on Ω is a quadratic polynomial on Ω, and it is easy to show that the converse is true; so Dive's theorem can be formulated without reference to homothety: *A body such that the Newtonian potential of a uniform mass distribution on it equals a quadratic polynomial on the body, is an ellipsoid.* This is, in fact, the form in which Dive reformulated, and proved his theorem (in $\mathbf{R}^3$; however, there are nontrivial complications in extending the proof to $\mathbf{R}^n$). There is, however, another way to reformulate the theorems of Newton and Dive, which may be called their *infinitesimal analogues.*

Namely, for $t > 1$, consider the probability measure μ_t, defined as Lebesgue measure on $t\Omega \setminus \overline{\Omega}$ renormalized to have total mass 1, where Ω is any smoothly bounded domain strictly star shaped with respect to 0. It is easy to see that as $t \searrow 1$, μ_t tends in the weak* topology of bounded measures on $\mathbf{R}^n$, to a limit measure μ supported on $\Gamma = \partial\Omega$. Let us compute this limit.

Let φ be any function in $C^1(\mathbf{R}^n)$ with compact support. Then

$$\int_{t\Omega \setminus \Omega} \varphi(x)\, dx = \int \varphi(x) \chi_\Omega(t^{-1}x)\, dx - \int \varphi(x) \chi_\Omega(x)\, dx$$

where integrations are over $\mathbf{R}^n$ where not otherwise indicated. The first integral on the right equals $\int \varphi(ty)\chi_\Omega(y)t^n\,dy$, so

$$\int_{t\Omega\setminus\Omega} \varphi(x)\,dx = \int \big[t^n\varphi(tx) - \varphi(x)\big]\chi_\Omega(x)\,dx. \tag{4.5}$$

Putting $\varphi = 1$ in (4.5) gives

$$|t\Omega\setminus\Omega| = (t^n - 1)|\Omega| \tag{4.6}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbf{R}^n$. Hence

$$|t\Omega\setminus\Omega|^{-1}\int_{t\Omega\setminus\Omega} \varphi(x)\,dx = \big((t^n-1)|\Omega|\big)^{-1}\int \big[t^n\varphi(tx) - \varphi(x)\big]\chi_\Omega(x)\,dx. \tag{4.7}$$

Now, the right-hand term in (4.7) equals

$$\begin{aligned}
&\big((t^n-1)|\Omega|\big)^{-1}\int \big\{\big[t^n\varphi(tx) - t^n\varphi(x)\big] + [t^n\varphi(x) - \varphi(x)]\big\}\chi_\Omega(x)\,dx\\
&\quad = |\Omega|^{-1}\left[t^n\Big(\frac{t-1}{t^n-1}\Big)\int \frac{\big(\varphi(tx)-\varphi(x)\big)}{t-1}\,\chi_\Omega(x)\,dx + \int \varphi(x)\chi_\Omega(x)dx\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{t\searrow 1}|t\Omega\setminus\Omega|^{-1}\int_{t\Omega\setminus\Omega}\varphi(x)\,dx\\
&\qquad = |\Omega|^{-1}\left(\frac{1}{n}\int\Big[\sum_{j=1}^n x_j\varphi_j(x)\Big]\chi_\Omega(x)\,dx + \int \varphi(x)\chi_\Omega(x)\,dx\right)
\end{aligned}$$

where φ_j denotes $\partial\varphi/\partial x_j$. In terms of the limit measure μ defined earlier, we thus have

$$\int \varphi\,d\mu = |\Omega|^{-1}\int_\Omega \Big[\varphi(x) + \frac{1}{n}\sum_{j=1}^n x_j\varphi_j(x)\Big]\,dx. \tag{4.8}$$

Now, $\partial_j(x_j\varphi) = x_j\varphi_j + \varphi$ so

$$n\varphi + \sum_{j=1}^n x_j\varphi_j = \sum_{j=1}^n \partial_j(x_j\varphi) = \operatorname{div}(\varphi x).$$

Hence, from (4.8)

$$n|\Omega|\int\varphi\,d\mu = \int_\Omega \operatorname{div}(\varphi x)\,dx = \int_\Gamma \langle x, N\rangle\varphi(x)\,dS(x) \tag{4.9}$$

by Gauss' theorem, where $\langle x, N\rangle$ denotes the component of the vector joining 0 to x, in the direction of the outward normal vector to Γ at x or, in other words, *the distance from* 0 *to the tangent plane to* Γ *at* x. From (4.9) we conclude: μ is the measure

$$(n|\Omega|)^{-1}\langle x, N\rangle\, dS.$$

From this, we see that Newton's theorem *implies* that μ is the equilibrium measure on Γ, and so: *the equilibrium measure on an ellipsoid centered at* 0 *is a constant multiple of* $\langle x, N\rangle\, dS$. (This is also proved, for $n = 3$, in [Ke, Chapter VII].) Also, Dive's theorem (at least for domains with C^1 boundaries) would be implied by the following.

Proposition 4.10. *If the equilibrium measure on* Γ *is a constant multiple of the measure* $\langle x, N\rangle\, dS$, *then* Ω *is an ellipsoid centered at* 0.

This we have been able to prove directly (with strong regularity assumptions) in $\mathbf{R}^2$, and so obtain a new proof of Dive's theorem for this case. It would be of considerable interest to prove Proposition 4.10 independently (i.e. without using Dive's theorem) also in n dimensions, $n \geq 3$ but this we have been unable to do.

On the other hand, it is of interest that (modulo certain technical points we shall gloss over concerning the regularity assumptions) *Proposition* 4.10 *is deducible from Dive's theorem.* Here is a sketch of the proof. We suppose, then, that Ω contains 0 and is strictly star-shaped with respect to 0, and on its boundary Γ, the measure $\langle x, N\rangle\, dS$ is a constant multiple of the equilibrium measure. We shall deduce from this that for the *homeoid figure* $G_t := \Omega \setminus t\overline{\Omega}$, where $0 < t < 1$, the potential of a uniform mass distribution on G_t is constant in $t\Omega$. (Hence, by Dive's theorem, Ω is an ellipsoid.)

Let k_n denote the Newtonian kernel. We have to show that

$$\int_{G_t} \big[k_n(x-y) - k_n(x)\big]\, dx = 0 \tag{4.11}$$

for $y \in t\Omega$. Now, (4.11) can be evaluated by using a kind of "polar coordinates" (also called the co-area formula; see e.g. [EG]), which for the integral over G_t of a function f gives the formula

$$\int_{G_t} f\, dx = \int_t^1 \left(\int_{s\Gamma} f\ \langle x, N\rangle\ dS(x)\right) \frac{ds}{s}.$$

Since, at every point of the ray sx, where x is a fixed point of Γ, the normal vector to $s\Gamma$ has the same direction it is easy to see that the result of the first integration, over $s\Gamma$, with f being the integrand in (4.11) is always zero if this is so (for all $y \in \Omega$) when $s = 1$. But

$$\int_\Gamma \big[k_n(x-y) - k_n(x)\big]\langle x, N\rangle\, dS(x) = 0, \quad \text{when } y \in \Omega$$

because of the assumed equilibrium property of the measure $\langle x, N\rangle\, dS(x)$. Hence (4.11) holds. $\square$

We conclude this section by proving directly the two-dimensional version of 4.10:

Theorem 4.12. *Let Ω be a plane domain containing 0, and bounded by an analytic Jordan curve Γ. If the equilibrium density (with respect to arc length ds) on Γ at each point z equals the inner product of the radius vector from 0 to z and a unit normal to Γ at z, then Γ is an ellipse centered at 0.*

Remark. The referee pointed out that this result was also proved, using a different method, in [HP].

Proof of Theorem 4.12. Denote by f the equilibrium density for Γ, so that f is a non-negative smooth function on Γ such that

$$\int_\Gamma \log|z-\zeta| f(z)\, ds_z = C = \text{ constant}, \qquad \zeta \in \Omega.$$

Thus

$$\operatorname{Re} \int_\Gamma \log(z-\zeta) f(z)\, ds_z = C$$

for $\zeta = \xi + i\eta$ in Ω. Applying $\partial/\partial\xi$ to this gives

$$\operatorname{Re} \int_\Gamma \frac{f(z)}{z-\zeta}\, ds_z = 0, \qquad \zeta \in \Omega.$$

The integral defines a holomorphic function of ζ in Ω which, having vanishing real part, is constant, so

$$\int_\Gamma \frac{f(z)}{z-\zeta}\, ds_z = \int_\Gamma \frac{f(z)}{z}\, ds_z$$

and hence

$$\int_\Gamma \frac{f(z)}{z(z-\zeta)}\, ds_z = 0, \qquad \zeta \in \Omega. \tag{4.13}$$

Now, the inner product of the vector $z \in \Gamma$ and the unit outer normal $N(z) = -iT(z)$ at z, is

$$\operatorname{Re}\bigl(-iT(z)\bar{z}\bigr) = \tfrac{1}{2}\bigl(-iT(z)\bar{z} + i\,\overline{T(z)}\,z\bigr)$$

which equals $f(z)$, by assumption. Substituting this for $f(z)$ in (4.13) and simplifying, using the relation $dz = T(z)\, ds_z$, and introducing the Schwarz function S of Γ (which satisfies $S(z) = \bar{z}$, $S'(z) = \overline{T(z)}^2$ for $z \in \Gamma$; see [Sh2]) gives

$$\int_\Gamma \left[\frac{S(z)}{z} - S'(z)\right] \frac{dz}{z-\zeta} = 0, \qquad \zeta \in \Omega.$$

As is well known (see [M]) this implies there is a holomorphic function on Ω_e which vanishes at ∞ and equals $(S(z)/z) - S'(z)$ on Γ. Thus $(d/dz)\big(S(z)/z\big)$ extends holomorphically from Γ to Ω_e and has a zero at ∞ of order at least 2.

It follows easily that S extends to Ω_e so as to be regular at ∞, or have a simple pole there. As is well known (see [Sh3] for a simple proof) Γ must be a circle centered at 0 in the former case, and an ellipse centered at 0 in the latter case. This proves Theorem 4.12. ◻

5. A general regularity problem and some final remarks

Looking over the proofs of Proposition 2.10 or Theorem 2.14, one can easily extract the following local statements.

Proposition 5.1. (i) *Let Γ be a rectifiable curve and Ω a Jordan domain that has Γ as part of its boundary. Let $T(z)$ be the unit tangent vector on $\partial\Omega$. Suppose there exists a bounded analytic function h in Ω, such that*

$$h(z) = \overline{T(z)} \qquad \text{a.e. on } \Gamma. \tag{5.2}$$

Then, Γ must be an analytic curve.

(ii) *Let Γ be a C^1-curve with Ω, h, T as in* (i). *If*

$$h(z) = T(z) \qquad \text{a.e. on } \Gamma,$$

Γ must be an analytic curve.

Remark. Note the difference between the hypotheses in (i) and (ii), which is unavoidable in view of Example 2.12.

Proof of Proposition 5.1. (i) Repeating with obvious modifications the calculations done in (2.5)–(2.8) we obtain from (5.2) that on an arc $\gamma \subset \mathbf{T}$, $\gamma := \varphi^{-1}(\Gamma)$, $\varphi\colon \mathbf{D} \to \Omega$, we have

$$z^2(h\circ\varphi)^2\varphi' = -\overline{\varphi'} = -(\varphi')^*\left(\frac{1}{z}\right) \qquad \text{a.e. on } \gamma \tag{5.3}$$

where g^* denotes the analytic function obtained from g by conjugating its Taylor coefficients. The left-hand side in (5.3) is an $H^1(\mathbf{D})$ function, while an easy calculation shows that the right-hand side belongs to $H^1(\mathbf{D}_e)$ $(\mathbf{D}_e = \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}})$. It is well known (and essentially follows from the Morera theorem) that both sides are analytically continuable with one another, hence φ' is analytically continuable across γ and so is φ. Thus, Γ is an analytic arc.

(ii) The difference between (i) and (ii) is that for the latter the calculation (2.16) implies

$$-\frac{z^2\varphi'}{(h\circ\varphi)^2} = \overline{\varphi'} \qquad \text{a.e. on } \gamma,$$

where we keep the same notation as in (i). For the conclusion to follow we must have $(h \circ \varphi)$ bounded away from 0 near γ. This does hold under our hypothesis since representing $(h \circ \varphi)$ by the Poisson integral in $\mathbf{D}$ we conclude that $(h \circ \varphi)$ is continuous near γ. Hence, since $|h \circ \varphi| = 1$ on γ, we arrive at the desired conclusion. □

In view of the discussion in the beginning of Section 2, the above proposition suggests several questions that one may raise in this context.

Questions 5.4. Is Theorem 2.3 true for all rectifiable curves? Can we replace the class N^+ in Theorem 2.14 by a larger class? This in turn leads to the following question. Does there exist a function f say in the Nevanlinna class, but not N^+ in a smoothly bounded domain Ω such that on the boundary f coincides almost everywhere with a continuous unimodular function $T(z)$? Perhaps surprisingly, the answer is yes. Indeed, even in the unit disk there are functions of bounded characteristic, not in $N^+(\mathbf{D})$, whose boundary values on the unit circle coincide almost everywhere with a continuous (even with an infinitely differentiable) unimodular function. Let us sketch the construction for the continuous example, the more refined one is done is a similar way, but slightly more technical. Let

$$\varphi(z) = \exp\big[(z+1)/(z-1)\big], \qquad z \in \mathbf{D}.$$

Then $h(z) := (z-1)/\varphi(z)$ is holomorphic and in $N(\mathbf{D}) \setminus N^+(\mathbf{D})$. Its values on the unit circle $\mathbf{T}$ are continuous. Choose $A > 0$ so large that $h + A$ is different from zero on $\mathbf{T}$. Let g be an outer function with $|g(z)| = |h(z) + A|$ for $z \in \mathbf{T}$. Then $(h + A)/g$ has continuous unimodular boundary values on $\mathbf{T}$, yet it is not in $N^+(\mathbf{D})$ since that would imply its boundedness in $\mathbf{D}$, whereas it is violently unbounded, growing exponentially in $1/(1-|z|)$ as $z \to 1$ along the radius vector.

We would also like to direct the reader to the book [Fr], where related regularity problems are discussed.

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References

[AB] Aftalion, A. and J. Busca: Radial symmetry of overdetermined boundary value problems in exterior domains. - Arch. Rational Mech. Anal. 143, 1998, 195–196.

[ASZ] Aharonov, D., M.M. Schiffer and L. Zalcman: Potato kugel. - Israel J. Math. 40, 1981, 331–339.

[CH] Courant, R., and D. Hilbert: Methods of Mathematical Physics, Vol. 2. - Wiley, 1989.

[DL] Dautray, R., and J.-L. Lions: Analysis and Numerical Methods for Science and Technology, Vol. 1. - Springer-Verlag, 1990.

[Du] Duren, P.: Theory of H^p-spaces. - Academic Press, 1970.

[EG] Evans, L.C., and R.F. Gariepy: Measure Theory and Fine Properties of Functions. - Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.

[Fo] Folland, G.B.: Introduction to Partial Differential Equations. - Math. Notes, Princeton University Press, Princeton, NJ, 1976.

[Fr] Friedman, A.: Variational Principles and Free-Boundary Problems. - Pure Appl. Math., A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, NY, 1982.

[GS] Garofalo, N., and E. Sartori: Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates. - Preprint.

[GK] Gohberg, I.C., and M.G. Krein: Introduction to the Theory of Linear Nonselfadjoint Operators. - Amer. Math. Soc., 1969 (Translations of Math. Monographs, Vol. 18).

[Go] Goluzin, G.M.: Functions of a Complex Variable. - Amer. Math. Soc., Providence, RI, 1969 (translated from Russian).

[HP] Henrot, A., and G.A. Philippin: Some overdetermined boundary value problems with elliptical free boundaries. - SIAM J. Math. Anal. 29, 1998, 309–320.

[H] Hörmander, L.: The Analysis of Linear Partial Differential Operators; Vols. I, II. - Springer-Verlag, 1983.

[Je] Jeans, J.: The Mathematical Theory of Electricity and Magnetism. - 5th ed. reprinted, Cambridge Univ. Press, 1966.

[JS] Jones, P., and S. Smirnov: On Smirnov domains. - Ann. Acad. Sci. Fenn. Math. 24, 1999, 105–108.

[Ke] Kellogg, O.: Foundations of Potential Theory. - Ungar, New York, 1970,

[Kh1] Khavinson, D.: Factorization theorems for certain classes of analytic functions in multiply connected domains. - Pacific. J. Math. 108, 1983, 295–318.

[Kh2] Khavinson, D.: Remarks concerning boundary properties of Ep-classes. - Indiana Univ. Math. J. 31, 1982, 779–787.

[Kh3] Khavinson, D.: Holomorphic Partial Differential Equations and Classical Potential Theory. - Lecture Notes, Universidad de la Laguna Press, La Laguna, Tenerife, 1996.

[Ku] Kuran, Ü.: On the mean-value property of harmonic functions. - Bull. London Math. Soc. 4, 1972, 311–312.

[L] Landkof, N.S.: Foundations of Modern Potential Theory. - Springer-Verlag, 1972.

[LV] Lewis, J.L., and A. Vogel: On pseudospheres. - Rev. Mat. Iberoamericana 7, 1991, 25–54.

[Mar] Martensen, E.: Eine Integralgleichung für die log. Gleichgewichtsverteilung und die Krümmung der Randkurve eines ebenes Gebiets. - Z. Angew. Math. Mech. 72, 1992, 596–599.

[Max] Maxwell, I.C.: A Treatise on Electricity and Magnetism, Vol. 1, third edition reprinted. - Dover, 1954.

[MR] Mendez, O., and W. Reichel: Electrostatic characterization of spheres. - Forum Math. 12, 2000, 223–245.

[M] Mushkhelishvili, N.I.: Singular Integral Equations, 3rd ed. - Nauka, Moscow (Russian).

[Pr] Privalov, I.I.: Boundary Properties of Analytic Functions. - Moscow, 1950 (Russian; German translation: Deutscher Verlag der Wiss., Berlin, 1956).

[R] Reichel, W.: Radial symmetry for elliptic boundary value problems on exterior domains. - Arch. Rational Mech. Anal. 137, 1997, 381–394.

[S] Serrin, J.: A symmetry problem in potential theory. - Arch. Rational Mech. Anal. 43, 1971, 304–318.

[Shah] Shahgholian, H.: A characterization of the sphere in terms of single-layer potentials. - Proc. Amer. Math. Soc. 115, 1992, 1167–1168.

[Sh1] Shapiro, H.S.: Remarks concerning domains of Smirnov type. - Michigan Math. J. 13, 1966, 341–348.

[Sh2] Shapiro, H.S.: The Schwarz Function and its Generalizations to Higher Dimensions. - Wiley, 1992.

[Sh3] Shapiro, H.S.: Domains allowing exact quadrature identities for harmonic functions—an approach based on p.d.e. - In: Anniversary Volume on Approximation Theory and Functional Analysis, edited by P.L. Butzer et al., Birkhäuser, 1984, pp. 335–354.

[V] Vainikko, G.: Multidimensional Weakly Singular Integral Equations. - Lecture Notes in Math. 1549, Springer-Verlag.

[W] Wermer, J.: Potential Theory. - Lecture Notes in Math. 408, Springer-Verlag, 1974.

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