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ON A PROBLEM IN COMPLEX OSCILLATION THEORY OF PERIODIC SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS AND SOME RELATED PERTURBATION RESULTS

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Abstract. It was conjectured in a previous work [7] that every non-trivial solution of

$$y'' + Ay = 0$$

has infinite exponent of convergence of zeros, where $A(z) = B(e^z)$, $B(\zeta) = \sum_{j=1}^{p} b_{-j}\zeta^{-j} + g(\zeta)$, p is an odd positive integer and $g(\zeta)$ is an entire function of order not equal to a positive integer. We give an affirmative answer to this conjecture and obtain generalizations of some previous results. In addition, perturbation results for periodic equations are found. Some new properties of periodic equations have been found in order to solve the above problems.

1. Introduction

We use standard notation from Nevanlinna theory in this paper (see [8], [9] and [10]). In addition, we use the notation $\sigma(f)$ and $\lambda(f)$ respectively to denote the *order of growth* and *exponent of convergence* of the zeros of a meromorphic function f.

The following result was proved in [7].

Theorem A. *Let* $B(\zeta) = g(1/\zeta) + \sum_{j=1}^{p} b_j\zeta^j$, *where* $g(\zeta)$ *is a transcendental entire function with* $\sigma(g) < 1$, p *is an odd positive integer and* $b_p \neq 0$. *Let* $A(z) = B(e^z)$. *Then any non-trivial solution* f *of*

$$f'' + A(z)f = 0 \tag{1.1}$$

must have $\lambda(f) = +\infty$. *In fact, the stronger conclusion*

$$\log^+ N(r, f) \neq o(r) \tag{1.2}$$

holds.

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We remark that Theorem A remains valid when we define

$$B(\zeta) = g(\zeta) + \sum_{j=1}^{p} b_{-j}\zeta^{-j}, \qquad b_{-p} \neq 0.$$

An example is given in [7] showing that Theorem A does not hold when $\sigma(g)$ is any positive integer. It was conjectured in [7] that the conclusion of Theorem A remains valid when the order $\sigma(g) > 1$ but is not a positive integer. We now give an affirmative answer to this problem in this paper. In fact, we show that Theorem A remains valid when $\sigma(g)$ is not a positive integer or infinity. We also give examples showing that Theorem A is no longer valid when $\sigma(g)$ is infinity. Thus our results are the best possible. The second author acknowledges useful discussions with Jim Langley while he was visiting the Mathematics Department of St Andrews University in 1990.

The proof of the above conjecture is based on a generalization of the following result.

Theorem B ([7, Lemma 5] or [8, Lemma 3.6]). *Let* $A(z) = B(e^z)$ *where* $B(\zeta)$ *is transcendental and analytic in* $0 < |\zeta| < +\infty$. *Suppose* $A(z)$ *satisfies*

$$\overline{\lim_{r\to+\infty}} \frac{\log T(r,A)}{r} < 1. \tag{1.3}$$

Then for any non-trivial solution $f(z)$ *of* (1.1) *with*

$$\log^+ N(r, 1/f) = o(r), \tag{1.4}$$

$f(z)$ *and* $f(z+2\pi i)$ *must be linearly dependent.*

The generalization of Theorem B we obtain is when (1.1) has the coefficient of the form $A(z) = B(e^z)$ where

$$B(\zeta) = g_1(1/\zeta) + g_2(\zeta), \tag{1.5}$$

and $g_1(\zeta)$ and $g_2(\zeta)$ are entire functions. Thus the equation (1.1) we consider here is related to the well-known equation of G.H. Hill in mathematical physics concerning lunar theory (see [1, Chapter VII], [11] and [12]). We also prove some related perturbation results for equation (1.1) in the second half of this paper.

The main tools that we use in this paper are Nevanlinna theory in $\mathbf{C}_0$ (see [3, p. 4], [6, pp. 97–107] and [10, pp. 101–102]) where $\mathbf{C}_0 = \mathbf{C}\backslash\{z : |z| \le R_0\}$, Valiron representation for functions analytic in $\mathbf{C}_0$ [14, p. 15] and the fact that if f_1 and f_2 are two non-trivial, linearly independent solutions of (1.1), then the product $E(z) = f_1(z)f_2(z)$ satisfies the differential equation

$$4A(z) = \left(\frac{E'}{E}(z)\right)^2 - 2\frac{E''}{E}(z) - \frac{c^2}{E(z)^2} \tag{1.6}$$

where $c \neq 0$ is the Wronskian of f_1 and f_2 [1, p. 354] (see also [6, p. 81] or [8, p. 5]), and

$$T(r,E) = N(r,1/E) + \tfrac{1}{2}T(r,A) + O\big(\log rT(r,E)\big) \qquad \text{n.e.} \tag{1.7}$$

In general, we use "n.e." to denote that an asymptotic relation holds except possibly outside a set of finite linear measure (i.e., finite length). Our argument actually depends on an analogue of (1.7) on the Valiron representation of periodic functions. We also make use of the following properties of periodic entire coefficient in (1.1). Let $A(z) = B(e^z)$ where $A(z)$ and $B(\zeta)$ are defined in Theorem B. Then $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$, where both g_1 and g_2 are entire functions in $\mathbf{C}$. Using this decomposition and coupled with the above tools, we show that the oscillation properties of solutions of (1.1) depend largely on the properties of g_1 or g_2. The proof of the conjecture is based on the above tools as well as on new concepts and methods which we shall develop in Sections 2 and 3. This also allows us to obtain new perturbation results for (1.1).

This paper is organized as follows. Some new concepts and preparatory results will be introduced in Sections 2 and 3. The main results are stated in Section 4, and their proofs are given in Section 5. In Section 6, we discuss perturbation type results for periodic equations of the form (1.1). We note that the notations c, c_1, $c_2, \ldots,$ are used to denote constants that may have different values at different occurrences in this paper.

2. Preliminaries for the proof of the main results

Let $A(z)$ be an entire function. We define

$$\sigma_e(A) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,A)}{r} \tag{2.1}$$

to be the *e-type order of $A(z)$*. We also define

$$\overline{\lim_{r\to+\infty}} \frac{\log^+ N(r,1/A)}{r}, \tag{2.2}$$

denoted by $\lambda_e(A)$, to be the *e-type exponent of convergence of the zeros of $A(z)$*. We shall have occasions to consider the zeros of $A(z)$ in the right-half plane only.

In that case, we define the upper limit in (2.2) by $\lambda_{eR}(A)$ when we only count the zeros of $A(z)$ in the right-half plane. Similarly, we define $\lambda_{eL}(A)$ to be the upper limit in (2.2) when we only count the zeros of $A(z)$ in the left-half plane. We shall derive some new relations for (2.1) and (2.2) when $A(z)$ is in the form $B(e^z)$ for some function $B(\zeta)$ below.

Let $B(\zeta)$ be analytic in $0<|\zeta|<+\infty$. Hence we have a representation $B(\zeta)=g_1(1/\zeta)+g_2(\zeta)$, where both $g_1(\zeta)$ and $g_2(\zeta)$ are entire functions. Let $A(z)=B(e^z)=A_1(z)+A_2(z)$, where $A_1(z)=g_1(e^{-z})$ and $A_2(z)=g_2(e^z)$. Observe that the transformation $\zeta=e^z$ is a one-one correspondence between the sets $\{z:-\log\varrho\le\operatorname{Re}z\le\log\varrho,\ -\pi<\operatorname{Im}z\le\pi\}$ and $\{\zeta:\varrho^{-1}\le|\zeta|\le\varrho\}$. By the periodicity of e^z, we have

$$\begin{aligned}\max_{\varrho^{-1}\le|\zeta|\le\varrho}|B(\zeta)| &= \max_{\substack{-\log\varrho\le\operatorname{Re}z\le\log\varrho\\ -\pi<\operatorname{Im}z\le\pi}}|A(z)|\le\max_{|z|\le\log\varrho+\pi}|A(z)|\\ &\le \max_{\substack{-(\log\varrho+\pi)\le\operatorname{Re}z\le\log\varrho+\pi\\ -\pi<\operatorname{Im}z\le\pi}}|A(z)| = \max_{(e^\pi\varrho)^{-1}\le|\zeta|\le e^\pi\varrho}|B(\zeta)|.\end{aligned}$$

We deduce that

$$\sigma_e(A)=\varlimsup_{\varrho\to+\infty}\frac{\log\log\max\limits_{\varrho^{-1}\le|\zeta|\le\varrho}|B(\zeta)|}{\log\varrho}. \tag{2.3}$$

From

$$\max_{\varrho^{-1}\le|\zeta|\le\varrho}B(\zeta)=\max\Big\{\max_{|\zeta|=\varrho^{-1}}|B(\zeta)|,\max_{|\zeta|=\varrho}|B(\zeta)|\Big\},$$

and the fact that

$$\begin{aligned}\varlimsup_{\varrho\to+\infty}\frac{\log\log\max\limits_{|\zeta|=\varrho^{-1}}|B(\zeta)|}{\log\varrho}&=\sigma(g_1),\\ \varlimsup_{\varrho\to+\infty}\frac{\log\log\max\limits_{|\zeta|=\varrho}|B(\zeta)|}{\log\varrho}&=\sigma(g_2),\end{aligned} \tag{2.4}$$

we deduce that

$$\varlimsup_{\varrho\to+\infty}\frac{\log\log\max\limits_{\varrho^{-1}\le|\zeta|\le\varrho}|B(\zeta)|}{\log\varrho}=\max\{\sigma(g_1),\sigma(g_2)\}.$$

This together with (2.3) yields

$$\sigma_e(A)=\max\{\sigma(g_1),\sigma(g_2)\}. \tag{2.5}$$

In particular,

$$\sigma_e(A_1) = \sigma(g_1), \qquad \sigma_e(A_2) = \sigma(g_2). \tag{2.6}$$

Let us now turn to the discussion of zeros. Let $n(D, 1/F)$ be the number of zeros of $F(z)$ in the set D. Then we deduce

$$\begin{aligned} n\big(\varrho^{-1} \le |\zeta| \le \varrho, 1/B(\zeta)\big) &= n\left(\left\{\begin{matrix} -\log\varrho \le \operatorname{Re} z \le \log\varrho \\ -\pi < \operatorname{Im} z \le \pi \end{matrix}\right\}, 1/A(z)\right) \\ &\le n\big(|z| \le \log\varrho + \pi, 1/A(z)\big) \\ &\le 2\left(\frac{(\log\varrho+\pi)-\pi}{2\pi}+1\right) \\ &\quad \times n\left(\left\{\begin{matrix} -(\log\varrho+\pi) \le \operatorname{Re} z \le \log\varrho+\pi \\ -\pi < \operatorname{Im} z \le \pi \end{matrix}\right\}, 1/A(z)\right) \\ &= \left(\frac{\log\varrho}{\pi}+2\right) n\Big((e^{\pi}\varrho)^{-1} \le |\zeta| \le e^{\pi}\varrho, 1/B(\zeta)\Big). \end{aligned} \tag{2.7}$$

Thus

$$\lambda_e(A) = \varlimsup_{\varrho\to+\infty} \frac{\log n\big(\varrho^{-1} \le |\zeta| \le \varrho, 1/B(\zeta)\big)}{\log\varrho}. \tag{2.8}$$

From

$$n\big(\varrho^{-1} \le |\zeta| \le \varrho, 1/B(\zeta)\big) = n\big(1 < |\zeta| \le \varrho, 1/B(\zeta)\big) + n\big(\varrho^{-1} \le |\zeta| \le 1, 1/B(\zeta)\big),$$

we deduce

$$\begin{aligned} &\varlimsup_{\varrho\to+\infty} \frac{\log n\big(\varrho^{-1} \le |\zeta| \le \varrho, 1/B(\zeta)\big)}{\log\varrho} \\ &= \max\left\{\varlimsup_{\varrho\to+\infty} \frac{\log n\big(1 < |\zeta| \le \varrho, 1/B(\zeta)\big)}{\log\varrho}, \varlimsup_{\varrho\to+\infty} \frac{\log n\big(\varrho^{-1} \le |\zeta| \le 1, 1/B(\zeta)\big)}{\log\varrho}\right\}. \end{aligned}$$

As in (2.8), we have

$$\begin{aligned} \lambda_{eR}(A) &= \varlimsup_{\varrho\to+\infty} \frac{\log n\big(1 < |\zeta| \le \varrho, 1/B(\zeta)\big)}{\log\varrho}, \\ \lambda_{eL}(A) &= \varlimsup_{\varrho\to+\infty} \frac{\log n\big(\varrho^{-1} \le |\zeta| \le 1, 1/B(\zeta)\big)}{\log\varrho}. \end{aligned} \tag{2.9}$$

Thus

$$\lambda_e(A) = \max\{\lambda_{eR}(A),\ \lambda_{eL}(A)\}. \tag{2.10}$$

If f is analytic in $\mathbf{C}_0$, then [14, p. 15] implies that

$$f(z) = z^n \Theta(z) u(z), \tag{2.11}$$

where n is an integer, $\Theta(z)$ is analytic and non-vanishing on $\mathbf{C}_0 \cup \{\infty\}$, and $u(z)$ is an entire function with

$$u(z) = \pi(z) e^{h(z)}. \tag{2.12}$$

The function $\pi(z)$ is a Weierstrass product formed from the zeros of f in $\mathbf{C}_0$, and $h(z)$ is an entire function. We remark that the assumption that f has an essential singularity at infinity made in [14, p. 15] in order for f to have (2.11) is redundant (see [5, Section 2]).

Letting $R_0 = 1$, we may regard $B(\zeta)$ to be analytic in $\mathbf{C}^*$, where $\mathbf{C}^* := \mathbf{C} \backslash \{z : |z| \le 1\}$. By (2.11) we have a similar representation

$$B(\zeta) = \zeta^n R(\zeta) b(\zeta), \tag{2.13}$$

where n is an integer, $R(\zeta)$ is analytic and non-vanishing on $\mathbf{C}^* \cup \{\infty\}$, and $b(\zeta)$ is an entire function. From (2.13) we easily deduce

$$\varlimsup_{\varrho \to +\infty} \frac{\log\log \max\limits_{|\zeta|=\varrho} |B(\zeta)|}{\log \varrho} = \varlimsup_{\varrho \to +\infty} \frac{\log\log M\big(\varrho, b(\zeta)\big)}{\log \varrho} = \sigma\big(b(\zeta)\big). \tag{2.14}$$

From (2.4) we obtain

$$\sigma(g_2) = \sigma(b). \tag{2.15}$$

Let us now consider $B(\zeta)$ as an analytic function in $0 < |\zeta| < 1$. If $t = 1/\zeta$, then the function $B^*(t) = B(1/t)$ is analytic in $\mathbf{C}^*$. Thus we have a similar Valiron representation in $\mathbf{C}^* \cup \{\infty\}$ as (2.13) with $b(\zeta)$ replaced by another entire function, denoted by $b^*(t)$. We deduce from (2.4) again that

$$\sigma(g_1) = \sigma(b^*). \tag{2.16}$$

We further deduce, from (2.5), (2.15) and (2.16), that

$$\sigma_e(A) = \max\big\{\sigma\big(b(\zeta), \sigma\big(b^*(t)\big)\big\}. \tag{2.17}$$

We also deduce, from (2.6), (2.15) and (2.16), that

$$\sigma_e(A_1) = \sigma\big(b^*(t)\big), \qquad \sigma_e(A_2) = \sigma\big(b(\zeta)\big). \tag{2.18}$$

Since the zeros of $B(\zeta)$ and $b(\zeta)$ coincide in $1 < |\zeta| < +\infty$, we deduce from (2.9) that

$$\lambda_{eR}(A) = \lambda\big(b(\zeta)\big). \tag{2.19}$$

Similarly

$$\lambda_{eL}(A) = \lambda\big(b^*(t)\big). \tag{2.20}$$

It follows from (2.10) that

$$\lambda_e(A) = \max\big\{\lambda\big(b(\zeta)\big), \lambda\big(b^*(t)\big)\big\}. \tag{2.21}$$

3. Nevanlinna characteristic functions in $|z| > R_0$

Suppose $w(z)$ is meromorphic in $\mathbf{C}_0 := \{z : R_0 < |z| < +\infty\}$. By a similar argument as in Valiron [14, p. 15], $w(z)$ has a representation

$$w(z) = z^n \Theta(z) f(z), \tag{3.1}$$

where n is an integer, $\Theta(z)$ is analytic and non-vanishing on $\mathbf{C}_0 \cup \{\infty\}$, f is a meromorphic function in $\mathbf{C}$. In fact we may write

$$f(z) = \frac{u(z)}{v(z)} e^{g(z)}, \tag{3.2}$$

where $u(z)$ and $v(z)$ are Weierstrass products formed respectively from the zeros and poles of w in $\mathbf{C}_0$, and $g(z)$ is an entire function. Thus we can apply the Nevanlinna theory to (3.1) in the region $\mathbf{C}_0$ (see [6, pp. 97–107]). Let $T(r,w)$ denote the usual Nevanlinna characteristic function in $\mathbf{C}$ and $T_1(r,w)$ denote the Nevanlinna characteristic function (see [3, p. 4]) for $w(z)$ in $\mathbf{C_0}$, which is defined by

$$T_1(r,w) = m_1(r,w) + N_1(r,w),$$

where

$$m_1(r,w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\varphi})| \, d\varphi, \tag{3.3}$$

and $N_1(r,w)$ is the counting function for the poles of w in $\mathbf{C}_0$.

We deduce, from (3.1), that

$$m_1(r,w) = m(r,f) + O(\log r), \tag{3.4}$$

and

$$N_1(r,w) = N(r,f). \tag{3.5}$$

Thus

$$T_1(r,w) = T(r,f) + O(\log r). \tag{3.6}$$

But $T(r,f) = T(r,1/f) + O(1)$, so

$$\begin{aligned} T_1(r,1/w) &= T(r,1/f) + O(\log r) \\ &= T(r,f) + O(\log r) \\ &= T_1(r,w) + O(\log r). \end{aligned} \tag{3.7}$$

That is,

$$T_1(r,1/w) = T_1(r,w) + O(\log r). \tag{3.8}$$

Note that (3.8) is similar to a special case of the first fundamental theorem of the usual Nevanlinna characteristic function.

As in [3, p. 4], we define the order of w in $\mathbf{C}_0$ by

$$\sigma_1(w) = \overline{\lim_{r\to+\infty}} \frac{\log T_1(r,w)}{\log r}. \tag{3.9}$$

4. Main results

Theorem 1. *Let $A(z) = B(e^z)$, where $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$, g_1 and g_2 are entire functions with g_2 transcendental and $\sigma(g_2)$ not equal to a positive integer or infinity, and g_1 arbitrary.*

(i) *Suppose $\sigma(g_2) > 1$.* (a) *If f is a non-trivial solution of* (1.1) *with $\lambda_e(f) < \sigma(g_2)$, then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.* (b) *If f_1 and f_2 are any two linearly independent solutions of* (1.1), *then $\lambda_e(f_1f_2) \geq \sigma(g_2)$.*

(ii) *Suppose $\sigma(g_2) < 1$.* (a) *If f is a non-trivial solution of* (1.1) *with $\lambda_e(f) < 1$ then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.* (b) *If f_1 and f_2 of* (1.1) *are any two linearly independent solutions, then $\lambda_e(f_1f_2) \geq 1$.*

We remark that the conclusion of Theorem 1 remains valid if we assume $\sigma(g_1)$ is not equal to an integer or infinity, and g_2 arbitrary and still assume $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$. In the case when g_1 is transcendental with its order not equal to an integer or infinity and g_2 is arbitrary, we need only consider $B^*(\eta) = B(1/\eta) = g_1(\eta) + g_2(1/\eta)$ in $0 < |\eta| < +\infty$, $\eta = 1/\zeta$.

From Part (ii)(a) of Theorem 1, and (2.5), we immediately generalize Theorem B on the growth restriction on $A(z)$, and on $\lambda_e(f)$. Part (i) and (ii)(b) of Theorem 1 are new results.

We easily deduce the following result from Theorem 1.

Corollary 1. *Under the assumption of* $A(z) = B(e^z)$ *in Theorem* 1, *any two linearly independent solutions* f_1 *and* f_2 *of* (1.1) *must have* $\lambda_e(f_1 f_2) \geq 1$, *and hence* $\lambda(f_1 f_2) = +\infty$.

Theorem 1 also leads to an affirmative answer to the conjecture in [7] mentioned in Section 1.

Theorem 2. *Let* $g(\zeta)$ *be a transcendental entire function and its order be not a positive integer or infinity. Let* $A(z) = B(e^z)$, *where* $B(\zeta) = g(1/\zeta) + \sum_{j=1}^{p} b_j \zeta^j$ *and* p *is an odd positive integer, then* $\lambda(f) = +\infty$ *for each non-trivial solution* f *to* (1.1). *In fact, the stronger conclusion* (1.2) *holds.*

We remark that the above conclusion remains valid if

$$B(\zeta) = g(\zeta) + \sum_{j=1}^{p} b_{-j}\zeta^{-j}.$$

The following examples show that the above results are the best possible.

Example 1. Let $E(z) = e^{pz/2}\exp\left(-\frac{1}{2}e^{mz}\right)$, where m is a positive integer and p is an odd positive integer. We further let

$$f_j(z) = E(z)^{1/2}\exp\left(\int_0^z \frac{(-1)^j}{E(t)}\,dt\right), \tag{4.1}$$

$j = 1, 2$. Then f_1 and f_2 are two linearly independent zero-free entire solutions of (1.1). In fact, the Wronskian of f_1 and f_2 is $W(f_1, f_2) = 2$. The coefficient

$A(z)$ can be calculated from (1.5):

$$\begin{aligned}
-4A(z) &= \frac{2^2}{E^2} - \left(\frac{E'}{E}\right)^2 + 2\frac{E''}{E} = \frac{4}{E^2} + 2\left(\frac{E'}{E}\right)' + \left(\frac{E'}{E}\right)^2 \\
&= \frac{4}{e^{pz}}\exp(e^{mz}) + \frac{p^2}{4} - \left(m^2 + \frac{pm}{2}\right)e^{mz} + \frac{m^2}{4}e^{2mz} \\
&= \frac{4}{\zeta^p}e^{\zeta^m} + \frac{p^2}{4} - \left(m^2 + \frac{pm}{2}\right)\zeta^m + \frac{m^2}{4}\zeta^{2m} \\
&= \sum_{j=1}^{p} b_{-j}\zeta^{-j} + g(\zeta) = -4B(\zeta),
\end{aligned}$$

where $\zeta = e^z$ and b_{-j} are constants, $b_{-p} \neq 0$, and $B(\zeta)$ has a pole of odd order at $\zeta = 0$. Moreover, $\sigma(g) = m$.

This example shows that Theorem 2 is the best possible in the sense that the assumption that $\sigma(g)$ is not a positive integer cannot be dropped. The case when $\sigma(g) = +\infty$ will be discussed in the next example. Since the solutions f_1 and f_2 are zero-free, the example also shows that both (i)(b) and (ii)(b) of Theorem 1 under the assumption that $\sigma(g)$ is not a positive integer are the best possible. The same situation applies to Corollary 1. Moreover, since $f_1(z+2\pi i) = cf_2(z)$, the example shows that (i)(a) and (ii)(a) of Theorem 1 are the best possible.

Example 2. Let $E(z) = e^{pz/2}\exp\left(-\frac{1}{2}e^{e^z}\right)$, where p is an odd positive integer. If f_1 and f_2 are given by (4.1), then $W(f_1, f_2) = 2$. Thus, they are linearly independent, zero-free solutions of (1.1) with $A(z)$ given by

$$\begin{aligned}
-4A(z) &= \frac{4}{e^{pz}}\exp(e^{e^z}) + \frac{p^2}{4} - \left(1 + \frac{p}{2}\right)e^{e^z}e^z - e^{e^z}e^{2z} + \frac{1}{4}e^{2e^z}e^{2z} \\
&= \frac{4}{\zeta^p}\exp(e^\zeta) + \frac{p^2}{4} - \left(1 + \frac{p}{2}\right)e^\zeta\zeta - e^\zeta\zeta^2 + \frac{1}{4}e^{2\zeta}\zeta^2 \\
&= \sum_{j=1}^{p} b_{-j}\zeta^{-j} + g(\zeta) = -4B(\zeta),
\end{aligned}$$

where $\zeta = e^z$, b_{-j} are again constants, $b_{-p} \neq 0$, and $B(\zeta)$ has a pole of odd order at $\zeta = 0$. Moreover, $\sigma(g) = +\infty$.

This example shows that Theorem 2, (i)(a), (ii)(a), (i)(b), (ii)(b) of Theorem 1 and Corollary 1 are the best possible, in the sense that none of the above holds if the order of $g(\zeta)$ is replaced by infinity.

5. Proofs of Theorems 1 and 2

Proof of Theorem 1. (i)(a) Let us assume that $f(z)$ and $f(z+2\pi i)$ are linearly independent. Since $\lambda_e(f) < \sigma(g_2) < +\infty$, [5, Theorem 1] implies that $f(z)$ and $f(z+4\pi i)$ must be linearly dependent. Let $E(z) = f(z)f(z+2\pi i)$, then $E(z+2\pi i) = c_1 E(z)$ for some non-zero constant c_1. Clearly E'/E and E''/E are both periodic functions with period $2\pi i$, while $A(z)$ is periodic by definition. Hence (1.6) shows that $E(z)^2$ is also periodic with period $2\pi i$. Thus we can find an analytic function $\Phi(\zeta)$ in $0 < |\zeta| < +\infty$, so that $E(z)^2 = \Phi(e^z)$. Substituting this expression into (1.6) yields

$$-4B(\zeta) = \frac{c^2}{\Phi} + \zeta\frac{\Phi'}{\Phi} - \frac{3}{4}\zeta^2\left(\frac{\Phi'}{\Phi}\right)^2 + \zeta^2\frac{\Phi''}{\Phi}. \tag{5.1}$$

Since both $B(\zeta)$ and $\Phi(\zeta)$ are analytic in $\mathbf{C}^* := \{z : 1 < |\zeta| < +\infty\}$, the Valiron theory gives their representations as

$$B(\zeta) = \zeta^n R(\zeta)\, b(\zeta), \qquad \Phi(\zeta) = \zeta^{n_1} K_1(\zeta)\, \phi(\zeta), \tag{5.2}$$

where n, n_1 are some integers, $R(\zeta)$ and $K_1(\zeta)$ are functions that are analytic and non-vanishing on $\mathbf{C}^* \cup \{\infty\}$ and $b(\zeta)$ and $\phi(\zeta)$ are entire functions. We deduce from (5.1) that

$$m_1(\varrho, 1/\Phi) = m_1(\varrho, B) + S_1(\varrho, \Phi), \tag{5.3}$$

where

$$S_1(\varrho, \Phi) = 3m_1\left(\varrho, \frac{\Phi'}{\Phi}\right) + m_1\left(\varrho, \frac{\Phi''}{\Phi}\right) + O(\log \varrho).$$

It is not difficult to show that

$$S_1(\varrho, \Phi) = S(\varrho, \phi) = o\big(T(\varrho, \phi)\big) \qquad \text{n.e.} \tag{5.4}$$

Thus, (5.3) implies

$$T_1(\varrho, 1/\Phi) = N_1(\varrho, 1/\Phi) + T_1(\varrho, B) + S_1(\varrho, \Phi). \tag{5.5}$$

Applying (3.6), (3.7), (5.4) to (5.5) and using the fact that $N_1(\varrho, 1/\Phi) = N(\varrho, 1/\phi)$, we deduce

$$T(\varrho, \phi) = N(\varrho, 1/\phi) + T(\varrho, b) + S(\varrho, \phi). \tag{5.6}$$

Notice that (5.6) satisfied by ϕ is an analogous formula to (1.7) satisfied by E. It is easy to see that $\lambda_e(f) = \lambda_e(E) = \lambda_e(E^2)$. Since $\lambda_e(f) < \sigma(g_2)$, so $\lambda_{eR}(E^2) \le \lambda_e(E^2) < \sigma(g_2)$. As in (2.19), $\lambda(\phi) = \lambda_{eR}(E^2)$. But $\sigma(g_2) = \sigma\big(b(\zeta)\big)$ by (2.15). Hence $\lambda(\phi) < \sigma(b)$. It follows from (5.6) and Fact (A) of [2, Section 2] (see also [8, Lemma 1.5] or [10, Lemma 1.1.1]) that $\sigma(\phi) = \sigma(b)$. Thus $\lambda(\phi) < \sigma(\phi)$. So $\sigma(\phi)$ is either a positive integer or infinity. So is $\sigma(b)$. But $\sigma(g_2) = \sigma(b(\zeta))$ which is a contradiction. Hence $f(z)$ and $f(z+2\pi i)$ must be linearly dependent.

(i)(b) Let us assume that $\lambda_e(f_1f_2) < \sigma(g_2)$. Hence we have $\lambda_e(f_1) < \sigma(g_2)$ and $\lambda_e(f_2) < \sigma(g_2)$. From (i)(a), $f_j(z)$ and $f_j(z+2\pi i)$ are linearly dependent, for $j = 1, 2$. Let $E(z) = f_1(z)f_2(z)$. Then, as in (i)(a), $E(z+2\pi i) = c_2E(z)$ for some non-zero constant c_2, and $E(z)^2$ is a periodic function with period $2\pi i$. Following the argument in (i)(a) yields $\sigma(g_2)$ to be a positive integer or infinity, which is a contradiction. Hence $\lambda_e(f_1f_2) \geq \sigma(g_2)$.

(ii)(a) Let us assume that $\sigma(g_2) < 1$ and that f is a non-trivial solution of (1.1) with $\lambda_e(f) < 1$. Suppose, as in the proof of (i)(a), that $f(z)$ and $f(z+2\pi i)$ are linearly independent, and we obtain (5.6) by following the same argument there.

As in (i)(a), we deduce from $\lambda_e(f) = \lambda_e(E) = \lambda_e(E^2)$ and (2.19) that $\lambda(\phi) = \lambda_{eR}(E^2) \leq \lambda_e(E^2) = \lambda_e(f) < 1$. We further deduce from the hypothesis on g_2 and (2.15) that $\sigma(b) = \sigma(g_2) < 1$. Using this result, (5.6) and Fact (A) in [2, Section 2] (see also [8, Lemma 1.5] or [10, Lemma 1.1.1]), we deduce $\sigma(\phi) < 1$.

The remaining proof now closely parallels certain parts of [7, Lemma 5]. We offer the full details here for the sake of completeness.

We turn to the representation (5.2). Since $K_1(\zeta)$ is analytic at ∞, we deduce $K_1^{(k)}(\zeta)/K_1(\zeta) = o(1)$, as $|\zeta| \to +\infty$. It follows from (5.2) and a standard estimate on the logarithmic derivative (see [8, Section 3.6] or [10, Proposition 5.12]) that there exists a positive constant M such that

$$\left|\zeta\frac{\Phi'}{\Phi} - \frac{3}{4}\zeta^2\left(\frac{\Phi'}{\Phi}\right)^2 + \zeta^2\frac{\Phi''}{\Phi}\right| \leq |\zeta|^M,$$

for $\zeta \notin V$, where V is an R-set (see [8, Section 3.6] or [10, p. 84]).

In addition to $\zeta \notin V$, let us further assume that $|\phi(\zeta)| > 1$. Then we easily see from (5.1) that there exists a positive integer N so that

$$|B(\zeta)| \leq |\zeta|^N. \tag{5.7}$$

On the other hand, we have the expansion

$$B(\zeta) = \sum_{k=-\infty}^{+\infty} b_k\zeta^k, \qquad 0 < |\zeta| < +\infty,$$

where $g_2(\zeta) = \sum_{k=0}^{+\infty} b_k\zeta^k$ is a transcendental entire function with $\sigma(g_2) < 1$. We now rewrite $B(\zeta) = h_1(\zeta) + h_2(\zeta)$, where

$$h_1(\zeta) = \sum_{k\leq N} b_k\zeta^k, \qquad h_2(\zeta) = \sum_{k>N} b_k\zeta^k.$$

Clearly $|h_1(\zeta)| = O(|\zeta|^N)$. Let $u_1(\zeta) = \phi(\zeta)$, and by the earlier choice that $\zeta \notin V$ and $|u_1(\zeta)| = |\phi(\zeta)| > 1$ we deduce from (5.7) that $|h_2(\zeta)| = O(|\zeta|^N)$ holds. We can therefore find a positive constant K such that $|\zeta^{-N} h_2(\zeta)/K| \le 1$, where $\zeta \notin V$ and $|u_1(\zeta)| > 1$. Let $u_2(\zeta) = \zeta^{-N} h_2(\zeta)/K$, then it is easy to see that $u_2(\zeta)$ is an entire transcendental function with $\sigma(u_2) < 1$. It follows from above that if $\zeta \notin V, |u_1(\zeta)| > 1$, then $|u_2(\zeta)| \le 1$.

In addition, we note that since $g_2(\zeta)$ is transcendental, $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$ and hence $b(\zeta)$ is also transcendental by (5.2). Therefore formula (5.6) implies that $u_1(\zeta) = \phi(\zeta)$ must also be transcendental.

We define $D_j^* = \{\zeta : |u_j(\zeta)| > 1\}$, $j = 1, 2$. Clearly both D_1^* and D_2^* are open sets. We denote the boundary of D_j^* by ∂D_j^*, $j = 1, 2$, and so we have $|u_j(\zeta)| = 1$ for ζ belongs to ∂D_j^*, $j = 1, 2$. But both $u_1(\zeta)$ and $u_2(\zeta)$ are transcendental, so each D_j^* must contain an unbounded component D_j for $j = 1, 2$. Denote the boundary of D_j by ∂D_j, $j = 1, 2$. Let $E_j(\varrho) = \{\theta : \varrho e^{i\theta} \in D_j\}$, $j = 1, 2$, and $E(\varrho) = \{\theta : \varrho e^{i\theta} \in V\}$. Clearly $E_1(\varrho) \cap E_2(\varrho) \subset E(\varrho)$. We also let $\theta_j(\varrho)$, $j = 1, 2$, and $\theta(\varrho)$ respectively, to be the angular measures of $E_j(\varrho)$, $j = 1, 2$, and $E(\varrho)$.

We note that since V is an R-set, so given $\varepsilon > 0$, there exists $\varrho_0 > 0$ such that $\theta(\varrho) < \varepsilon$ for $\varrho > \varrho_0$. We also note that we can choose $\varrho > \varrho_0$ so that the circle $|\zeta| = \varrho$ intersects D_j, $j = 1, 2$.

We now define

$$\theta_j^*(\varrho) = \begin{cases} \theta_j(\varrho), & \text{if } E_j(\varrho) \ne [0, 2\pi], \\ +\infty, & \text{if } E_j(\varrho) = [0, 2\pi]. \end{cases} \tag{5.8}$$

Then the Beurling–Tsuji inequality [13, Theorem III 68, p. 117] gives

$$\pi \int_{\varrho_0}^{\varrho/2} \frac{dt}{t\theta_j^*(t)} < \log\log M(\varrho, u_j) + O(1), \qquad j = 1, 2, \tag{5.9}$$

where $M(\varrho, u_j)$, $j = 1, 2$, denotes the usual maximum modulus of u_j on $|\zeta| = \varrho$. It was shown in the remark in [7, pp. 153–154] (see also [8, pp. 96–97]) that even when $E_j(\varrho) = [0, 2\pi]$, $j = 1, 2$, the inequality (5.9) is replaced by

$$\pi \int_{\varrho_0}^{\varrho/2} \frac{dt}{t\theta_j(t)} < \log\log M(\varrho, u_j) + \frac{\varepsilon}{2\pi} K_j \log \varrho + O(1), \qquad j = 1, 2, \tag{5.10}$$

if $K_i > \sigma(u_j)$, $i \ne j$ and ϱ is sufficiently large. Since $\sigma(u_j) < 1$, $j = 1, 2$, we deduce from (5.10) that there exists a constant $0 < \beta < 1$ so that

$$\pi \int_{\varrho_0}^{\varrho/2} \frac{dt}{t\theta_j(t)} < (1 - \beta) \log \varrho,$$

for $\varrho > \varrho_1 > \varrho_0$, $j = 1, 2$. Summing the above inequalities for $j = 1, 2$ yields

$$\pi \int_{\varrho_0}^{\varrho/2} \frac{\theta_1(t) + \theta_2(t)}{\theta_1(t)\theta_2(t)} \frac{dt}{t} < 2(1-\beta)\log\varrho.$$

But $2\sqrt{ab} \le a + b$, $(a, b \ge 0)$. Hence

$$4\pi \int_{\varrho_0}^{\varrho/2} \frac{dt}{t\big(\theta_1(t) + \theta_2(t)\big)} < 2(1-\beta)\log\varrho.$$

Notice that $\theta_1(\varrho) + \theta_2(\varrho) \le 2\pi + \varepsilon$ for $\varrho > \varrho_1$. This gives

$$\frac{4\pi}{2\pi + \varepsilon} \log\frac{\varrho}{2\varrho_0} < 2(1-\beta)\log\varrho. \tag{5.11}$$

Since $\varepsilon > 0$ is arbitrary, we obtain a contradiction after dividing both sides by $\log\varrho$. Thus we conclude that $f(z)$ and $f(z+2\pi i)$ must be linearly dependent.

(ii)(b) Suppose f_1 and f_2 are linearly independent and $\lambda_e(f_1 f_2) < 1$. Then $\lambda_e(f_1) < 1$ and $\lambda_e(f_2) < 1$. We deduce from the conclusion of (ii)(a) that $f_j(z)$ and $f_j(z+2\pi i)$ are linearly dependent, $j = 1, 2$. Let $E(z) = f_1(z)f_2(z)$, then we can find a non-zero constant c_3 such that $E(z+2\pi i) = c_3 E(z)$. Repeating the same argument in (ii)(a) by using the fact that $E(z)^2$ is also periodic, we obtain a contradiction as (5.11). Hence $\lambda_e(f_1 f_2) \ge 1$.

This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose there exists a non-trivial solution f that satisfies (1.4). We deduce $\lambda_e(f) = 0$. Theorem 1(i)(a) and (ii)(a) imply that $f(z)$ and $f(z+2\pi i)$ are linearly dependent. However, [7, Lemma 6] or [8, Theorem 3.7] implies that $f(z)$ and $f(z+2\pi i)$ are linearly independent. This is impossible. Hence (1.2) holds for each non-trivial solution f. This completes the proof of Theorem 2.

6. Perturbation results

Suppose (1.1) admits a non-trivial solution that has a finite e-type exponent of convergence of zeros. If $\prod(z)$ is a periodic entire function with period $2\pi i$ and $\sigma_e(\Pi) < \sigma_e(A)$, what can we say about the e-type exponent of convergence of zeros of any two linearly independent solutions of the equation

$$f'' + \big(A(z) + \textstyle\prod(z)\big)f = 0? \tag{6.1}$$

A perturbation result for (6.1) where the coefficients are not necessarily periodic is given in [4, Theorem 3.1]. We answer the above perturbation problem based on the method used in [4] coupled with the special properties of periodic coefficients established in Section 2.

Theorem 3. *Let* $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$, $C(\zeta) = g_3(\zeta)$, *where* g_1, g_2 *and* g_3 *are entire functions of finite order such that* $\sigma(g_2)$ *is a positive integer, and* $\sigma(g_3) < \sigma(g_2)$. *Suppose* $A(z) = B(e^z)$, $\prod(z) = C(e^z)$ *and furthermore that* (1.1) *admits a non-trivial solution* f *with* $\lambda_e(f) < \sigma(g_2)$ *and that* $f(z)$ *and* $f(z+2\pi i)$ *are linearly independent. Then*

(i) *for any non-trivial solution* h *of equation* (6.1) *with* $\lambda_e(h) < \sigma(g_2)$, $h(z)$ *and* $h(z+2\pi i)$ *are linearly dependent,*

and

(ii) *for any two linearly independent solutions* h_1 *and* h_2 *of* (6.1), *we must have* $\lambda_e(h_1 h_2) \geq \sigma(g_2)$.

Let us consider Example 1 in Section 4 where $f_1(z+2\pi i) = cf_2(z)$ for some non-zero constant c. This implies that $f_1(z)$, $f_1(z+2\pi i)$ are linearly independent solutions of (1.1) with the $A(z)$ given in Example 1. Clearly $\lambda_e(f) = 0 < \sigma(g)$. Suppose h is a non-trivial solution of (6.1) with $\lambda_e(h) < \sigma(g) = m$, then Theorem 3(i) shows that $h(z)$ and $h(z+2\pi i)$ are linearly dependent. Part (ii) implies $\lambda_e(h_1 h_2) \geq \sigma(g) = m$ for any two non-trivial linearly independent solutions h_1 and h_2.

Theorem 2 investigates the oscillation properties of any non-trivial solution to (1.1) with $B(\zeta) = \sum_{j=1}^{p} b_{-j}\zeta^{-j} + g(\zeta)$, where p is an odd positive integer and $\sigma(g)$ is not a positive integer or infinity. We now investigate perturbation problem for (6.1) precisely when $\sigma(g)$ *is* a positive integer.

Theorem 4. *Let* $g(\zeta)$ *be a transcendental entire function of an integer order* $\sigma(g)$ *and* $C(\zeta) \not\equiv 0$ *be an entire function with* $\sigma(C) < \sigma(g)$. *Let* $A(z) = B(e^z)$, *where* $B(\zeta) = \sum_{j=1}^{p} b_{-j}\zeta^{-j} + g(\zeta)$, p *is an odd positive integer and* $\prod(z) = C(e^z)$. *Suppose* (1.1) *admits a non-trivial solution* f *with* $\lambda(f) < +\infty$. *Then any non-trivial solution* h *of* (6.1) *must have* $\lambda(h) = +\infty$. *In fact, the stronger conclusion* (1.2) *holds.*

Example 1 shows that $A(z)$ satifies the hypothesis of Theorem 4, and that (1.1) possesses two zero-free solutions. Thus any non-trivial solution h to (6.1) must have $\lambda(h) = +\infty$.

Proof of Theorem 3. (i) Let f be a non-trivial solution of (1.1) with $\lambda_e(f) < \sigma(g_2)$, and $f(z)$ and $f(z+2\pi i)$ be linearly independent. Let $E(z) = f(z)f(z+2\pi i)$, then (1.5) implies that $E(z)^2$ is a periodic function with period $2\pi i$ since both E'/E and E''/E are periodic functions with period $2\pi i$. Thus, we can find an analytic function $\Phi(\zeta)$ in $0 < |\zeta| < +\infty$ so that $E(z)^2 = \Phi(e^z)$. Substituting this representation into (1.5) with $\zeta = e^z$ yields (5.1). We recall that $B(\zeta)$ and $\Phi(\zeta)$ have the Valiron representations in $\mathbf{C}^*$ given in (5.2).

Although the proof now closely parallels that of [4, Theorem 3.1], the modifications needed to apply the argument there are sufficiently intricate that it warrants the inclusion of the details here. Let us now suppose (6.1) possesses a non-trivial solution $h(z)$ such that $\sigma_e(h) < \sigma(g_2)$ but $h(z)$ and $h(z+2\pi i)$ are linearly independent. Let $F(z) = h(z)h(z+2\pi i)$. By a similar argument that we have applied to $E(z)$ above, we conclude that there exists an analytic function $\Psi(\zeta)$ in $0 < |\zeta| < \infty$ such that $F(z)^2 = \Psi(e^z)$. Similary, Ψ has a Valiron representation

$$\Psi(\zeta) = \zeta^{n_2} K_2(\zeta)\psi(\zeta) \tag{6.2}$$

in $\mathbf{C}^*$, where n_2 is an integer, $K_2(\zeta)$ is analytic and non-vanishing on $\mathbf{C}^* \cup \{\infty\}$, and $\psi(\zeta)$ is an entire function in $\mathbf{C}$.

We now substitute $F(z)^2 = \Psi(e^z)$, with $\zeta = e^z$, into (1.5) with $A(z)$ replaced by $A(z) + \prod(z)$. This yields

$$-4\big(B(\zeta) + C(\zeta)\big) = \frac{c_1^2}{\Psi} + \zeta\frac{\Psi'}{\Psi} - \frac{3}{4}\zeta^2\left(\frac{\Psi'}{\Psi}\right)^2 + \zeta^2\frac{\Psi''}{\Psi}, \tag{6.3}$$

where $c_1 \neq 0$ is the Wronskian of $h(z)$ and $h(z+2\pi i)$.

Since, as in (2.19), we have $\lambda(\phi) = \lambda_{eR}(E^2) = \lambda_{eR}(f) < \sigma(g_2) = \sigma\big(b(\zeta)\big)$, so (5.6) implies that $\sigma(\phi) = \sigma\big(b(\zeta)\big) = \sigma(g_2)$, and hence $\sigma(\phi)$ is an integer with $\lambda(\phi) < \sigma(\phi)$. Thus we may write $\phi(\zeta) = \pi_1 e^P$, where $P(\zeta) = \alpha z^\sigma$, α is a constant, $\sigma = \sigma(g_2)$ and $\sigma(\pi_1) < \sigma$.

In a similar fashion, we have $\lambda(\psi) = \lambda_{eR}(F^2) = \lambda_{eR}(h) < \sigma(g_2) = \sigma\big(b(\zeta)\big)$. We then apply a similar argument to (6.3) to obtain

$$T(\varrho, \psi) = N(\varrho, 1/\psi) + T(\varrho, d) + S(\varrho, \psi), \tag{6.4}$$

as to (5.1) for (5.6), where $d(\zeta)$ is an entire function appearing in the Valiron representation of

$$B(\zeta) + C(\zeta) = \zeta^{n_3} R_d(\zeta) d(\zeta),$$

where the functions $R_d(\zeta)$ and $d(\zeta)$ play the same roles of the corresponding functions in (5.2), and it is easy to check that $\sigma(d) = \sigma(b)$. But then $\sigma(\psi) = \sigma\big(d(\zeta)\big) = \sigma(g_2) > \lambda(\psi)$. Thus we may write $\psi(\zeta) = \pi_2 e^Q$, where $Q = \beta z^\sigma$, β is a constant, $\sigma = \sigma(g_2)$, and $\sigma(\pi_2) < \sigma$.

Let $\Phi(\zeta) = R_1 e^P$ and $\Psi(\zeta) = R_2 e^Q$, where $R_1(\zeta) = \zeta^{n_1} K_1(\zeta)\pi_1(\zeta)$, $R_2(\zeta) = \zeta^{n_2} K_2(\zeta)\pi_2(\zeta)$, $\sigma_1(R_1), \sigma_1(R_2) < \sigma = \sigma(g_2)$, and substitute them into (5.1) and (6.3) respectively yielding

$$\begin{aligned} -4B(\zeta) = \frac{c^2}{R_1 e^P} + \zeta\left(\frac{R_1'}{R_1} + P'\right) - \frac{3}{4}\left(\left(\zeta\frac{R_1'}{R_1}\right)^2 + 2\zeta\frac{R_1'}{R_1}P' + \zeta^2 P'^2\right) \\ + \zeta^2\left(\frac{R''}{R_1} + 2\frac{R_1'}{R_1}P' + P'^2 + P''\right) \end{aligned} \tag{6.5}$$

and

$$
\begin{aligned}
-4\big(B(\zeta)+C(\zeta)\big) = {} & \frac{c_1^2}{R_2 e^Q} + \zeta\left(\frac{R_2'}{R_2} + Q'\right) \\
& - \frac{3}{4}\left(\left(\zeta\frac{R_2'}{R_2}\right)^2 + 2\zeta\frac{R_2'}{R_2}Q' + \zeta^2 {Q'}^2\right) \\
& + \zeta^2\left(\frac{R_2''}{R_2} + 2\frac{R_2'}{R_2}Q' + {Q'}^2 + Q''\right).
\end{aligned} \tag{6.6}
$$

Substracting (6.6) from (6.5) yields

$$
4C(\zeta) = \frac{c^2}{R_1 e^P} - \frac{c_1^2}{R_2 e^Q} + H(\zeta), \tag{6.7}
$$

where $H(\zeta)$ is meromorphic in $\mathbf{C}^*$. In fact $H(\zeta)$ is a differential polynomial in R_1'/R_1, R_2'/R_2, P', Q' and their derivatives. We can deduce from the definitions of R_1, R_2, P, Q that $\sigma_1(H) < \sigma(g_2)$. Rewriting (6.7) as

$$
e^{-P} + H_1 e^{-Q} = H_2, \tag{6.8}
$$

where H_1 and H_2 are meromorphic functions in $\mathbf{C}^*$ with $\max\{\sigma_1(H_1), \sigma_1(H_2)\} < \sigma(g_2)$ and $H_1 = -c_1^2/c^2 R_1/R_2$. Differentiating (6.8) and using the resulting equation to eliminate e^{-P} from (6.8) yields

$$
H_3 e^{-Q} = H_4, \tag{6.9}
$$

where $H_3 = H_1' + (P' - Q')H_1$ and H_4 is a meromorphic function in $\mathbf{C}^*$ with $\sigma_1(H_3), \sigma_1(H_4) < \sigma(g_2) = \sigma$. Thus $H_3 \equiv 0$, i.e., $H_1 = c_2 e^{Q-P}$, where c_2 is a non-zero constant, from a simple order consideration in (6.9). This is a contradiction to $\sigma_1(H_1) < \sigma(g_2)$ unless $P \equiv Q = \alpha z^\sigma$. Thus $R_1 = c_3 R_2$, where c_3 is a constant, and $R_1'/R_1 = R_2'/R_2$. Substituting $R_1'/R_1 = R_2'/R_2$, $P = Q$ into (6.5) and (6.6) and subtracting the resulting equations yields (6.7) with $H(\zeta) \equiv 0$. Substituting $R_1 = c_3 R_2$ into this new equation and considering its order yields immediately $C(\zeta) \equiv 0$ in $\mathbf{C}^*$, and hence in $\mathbf{C}$. This is a contradiction. Hence $h(z)$ and $h(z+2\pi i)$ must be linearly dependent.

(ii) Suppose (6.1) possesses two non-trivial solutions h_1 and h_2 that are linearly independent and $\lambda_e(h_1 h_2) < \sigma(g_2)$. Part (i) implies that $h_j(z)$, and $h_j(z+2\pi i)$ are linearly dependent, for $j = 1, 2$. Let $E(z) = f(z)f(z+2\pi i)$ and $F(z) = h_1(z)h_2(z)$. Then $F(z+2\pi i) = c_4 F(z)$, for some non-zero constant c_4. Applying a similar argument to $E(z)$ and $F(z)$ as in (i) yields $C(\zeta) \equiv 0$. This is a contradiction. Hence $\lambda_e(h_1 h_2) \geq \sigma(g_2)$. This completes the proof of Theorem 3.

Proof of Theorem 4. Let g, C and A be defined in Theorem 4. Suppose (1.1) possesses a non-trivial solution f with $\lambda(f) < +\infty$. Hence $\lambda_e(f) = 0 < \sigma(g)$. Thus [7, Lemma 6] implies that $f(z)$ and $f(z + 2\pi i)$ are linearly independent solutions of (1.1). Thus $f(z)$ and $f(z + 2\pi i)$ satisfy the hypotheses of Theorem 3. Suppose that (6.1) admits a non-trivial solution h with $\lambda_e(h) = 0 < \sigma(g)$. Then [7, Lemma 6] again implies that $h(z)$ and $h(z + 2\pi i)$ are linearly independent. Theorem 3(ii) implies that $\lambda_e\big(h(z)h(z + 2\pi i)\big) \geq \sigma(g) > 0$. But $\lambda_e(h) = \lambda_e\big(h(z)h(z + 2\pi i)\big) > 0$. Hence (1.2) holds. This completes the proof.

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