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EXTENSION OF BILINEAR FORMS FROM SUBSPACES OF $\mathscr{L}_1$-SPACES

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Abstract. We study the extension of bilinear forms from a given subspace of an $\mathscr{L}_1$-space to the whole space. Precisely, an isomorphic embedding $j\colon E \to X$ is said to be (linearly) 2-exact if bilinear forms on E can be (linear and continuously) extended to X through j. We present some necesary and some sufficient conditions for an embedding $j\colon E \to X$ to be 2-exact when X is an $\mathscr{L}_1$-space.

1. Introduction and preliminaries

In this paper we shall consider the problem of the extension of bilinear forms from a subspace of an $\mathscr{L}_1$-space, continuing with the algebraic approach initiated in [9]. Other papers containing different views of the extension problem are: Carando [5]; Carando and Zalduendo [6]; Galindo, García, Maestre and Mujica [11]; Kirwan and Ryan [14] and Zalduendo [17].

Definition. Let $j\colon E \to X$ be an into isomorphism; a bilinear form b on E is said to be extendable through j if there exists a bilinear form B on X such that $B\big(j(\cdot), j(\cdot)\big) = b(\cdot,\cdot)$.

A simple observation is that a bilinear form b on E can be extended through j if and only if there exists an operator $T \in L(X, X^*)$ making commutative the diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\tau_b"] & X \arrow[d, "T"] \\ E^* & X^* \arrow[l, "j^*"] \end{tikzcd}$$

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in which $\tau_b \in L(E, E^*)$ is the operator associated to b, given by $\tau_b(x)(y) = b(x, y)$. We shall say that the embedding $j\colon E \to X$ is 2-exact if all bilinear forms on E can be extended through j.

An exact sequence of Banach spaces and operators is a diagram

$$0 \longrightarrow E \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0$$

with the property that the kernel of each arrow coincides with the image of the preceding. The open mapping theorem guarantees that E is a subspace of X through the into isomorphism j and Z is the corresponding quotient X/E through the quotient map q. From now on we shall maintain the notation j for the embedding $E \to X$ and q for the quotient map $X \to X/E$. The sequence is said to split (also called a trivial exact sequence) if j admits a linear and continuous left-inverse; i.e., if $j(E)$ is complemented in X. The sequence is said to locally split (following [13]) if the dual sequence

$$0 \longrightarrow Z^* \xrightarrow{q^*} X^* \xrightarrow{j^*} E^* \longrightarrow 0$$

splits. We suggest [7] for everything we shall use about exact sequences and functors in the category of Banach spaces; a sounder, more general, background can be found in [12]. What we need to know is that given an exact sequence $0 \to E \to X \to Z \to 0$ and an operator $T\colon M \to Z$ there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "T"] & 0.
\end{tikzcd}$$

The lower exact row is called the pull-back sequence; it splits if and only if T can be linear and continuously lifted to an operator $M \to X$. Also, given an exact sequence $0 \to E \to X \to Z \to 0$ and an operator $T\colon E \to M$ there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "T"] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & PO \arrow[r] & Z \arrow[r] & 0.
\end{tikzcd}$$

The lower exact row is called the push-out sequence; it splits if and only if T can be linear and continuously extended to an operator $X \to M$.

Given two Banach spaces A and B we use the notation $\mathrm{Ext}(B,A)=0$ with the meaning that all exact sequences $0\to A\to X\to B\to 0$ split. Further information about the pull-back and push-out constructions as well as about the functor $\mathrm{Ext}(\,\cdot\,,\cdot\,)$ can be found in [7] (in the context of Banach spaces) and in [12] (in general). Recall from [9] that an embedding $j\colon E\to X$ is said to be linearly 2-exact if it is 2-exact and there exists an operator L so that for every bilinear form b on E, $L(b)$ is an extension of b through j. The following result is in [9]:

Lemma 1.1. *An embedding $E\xrightarrow{j}X$ is linearly 2-exact if and only if the exact sequence $0\longrightarrow E\xrightarrow{j}X\xrightarrow{j}Z\longrightarrow 0$ locally splits.*

We shall adopt the slightly incorrect custom of writing $\mathscr{L}_p$ to denote an unspecified $\mathscr{L}_p$-space.

2. 2-exact embeddings

We present now our algebraic approach to the problem, and it is our hope that this will justify the special interest of the case in which X is an $\mathscr{L}_1$-space.

Lemma 2.1. *Consider an exact sequence $0\to E\xrightarrow{j}X\to X/E\to 0$. If $\mathrm{Ext}\big(E,(X/E)^*\big)=0$ and $\mathrm{Ext}(X/E,X^*)=0$ then j is 2-exact.*

Proof. The pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & (X/E)^* \arrow[r] & X^* \arrow[r] & E^* \arrow[r] & 0 \\
0 \arrow[r] & (X/E)^* \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & E \arrow[r] \arrow[u, "T"'] & 0
\end{tikzcd}$$

and the hypothesis $\mathrm{Ext}\big(E,(X/E)^*\big)=0$ yield that T can be lifted to an operator $T_1\colon E\to X^*$. Now, the push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "T_1"] & X \arrow[r] \arrow[d] & X/E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^* \arrow[r] & PO \arrow[r] & X/E \arrow[r] & 0
\end{tikzcd}$$

and the hypothesis $\mathrm{Ext}(X/E,X^*)=0$ yield that T_1 can be extended to an operator $T_2\colon X\to X^*$. Clearly $j^*T_2j=T$ and the result is proven. ▫

Reasoning first with a push-out diagram and then a pull-back diagram one gets:

Lemma 2.2. *Consider an exact sequence $0\to E\to X\to X/E\to 0$. If $\mathrm{Ext}(X/E,E^*)=0$ and $\mathrm{Ext}\big(X,(X/E)^*\big)=0$ then j is 2-exact.*

Very few instances of couples A, B of Banach spaces such that $\operatorname{Ext}(B,A)=0$ are currently known. One simple instance is when A is an injective space. Since the dual of an $\mathscr{L}_1$-space is injective, the hypothesis $\operatorname{Ext}(X/E,X^*)=0$ of Lemma 2.1 holds when X is an $\mathscr{L}_1$-space. Another instance is when A is a dual space and B is an $\mathscr{L}_1$-space, as was proved by Lindenstrauss [15]. Hence, also the hypothesis $\operatorname{Ext}(X,(X/E)^*)=0$ of Lemma 2.2 holds when X is an $\mathscr{L}_1$-space. Therefore, this approach seems to fit especially the case in which X is an $\mathscr{L}_1$-space. A special feature of this case, that the previous conditions are actually necessary, is proved now.

Theorem 2.3. *Let $j\colon E\to\mathscr{L}_1$ be an embedding. The following are equivalent.*

(1) *j is 2-exact;*
(2) $\operatorname{Ext}(\mathscr{L}_1/E,E^*)=0$;
(3) $\operatorname{Ext}\big(E,(\mathscr{L}_1/E)^*\big)=0$.

Proof. We prove the equivalence between (1) and (2) using Lemma 2.1, the proof of the equivalence between (1) and (3) being analogous to using Lemma 2.2. Let us consider an exact sequence $0\to E\xrightarrow{j}\mathscr{L}_1\to Z\to 0$. The sufficiency of the conditions is clear after Lemma 2.2; so we prove the necessity. The first terms of the exact homology sequence obtained fixing the second variable at E^* (see [12] or [3]) is:

$$0\to L(Z,E^*)\to L(\mathscr{L}_1,E^*)\to L(E,E^*)\to \operatorname{Ext}(Z,E^*)\to \operatorname{Ext}(\mathscr{L}_1,E^*)\to\cdots.$$

The hypothesis that j is 2-exact yields that $L(\mathscr{L}_1,E^*)\to L(E,E^*)$ is surjective, hence $L(E,E^*)\to\operatorname{Ext}(Z,E^*)$ is the 0-map. But since $\operatorname{Ext}(\mathscr{L}_1,E^*)=0$, it clearly follows from the exactness of the sequence that $\operatorname{Ext}(Z,E^*)=0$. □

Observe that what Theorem 2.3 seems to assert is that an embedding $j\colon E\to\mathscr{L}_1$ is 2-exact if and only if *every* embedding $E\to\mathscr{L}_1$ is 2-exact. The analogous result for embeddings into $\mathscr{L}_\infty$-spaces was proved in [9, Proposition 2.1]; the corresponding result for general embeddings does not hold.

3. Applications

We close the paper applying these results to different situations.

$\mathscr{L}_1$-subspaces of $\mathscr{L}_1$-spaces: These embeddings are always 2-exact because condition 3 of the theorem: $\operatorname{Ext}\big(E,(\mathscr{L}_1/E)^*\big)=0$ is automatically satisfied. This, however, only yields a different proof for an essentially known fact that can be derived as follows: if E and X are $\mathscr{L}_1$-spaces and E is a subspace of X then $E\otimes_\pi E$ is a subspace of $E\otimes_\pi X$ which, in turn, is a subspace of $X\otimes_\pi X$. The Hahn–Banach theorem yields now the extension of bilinear forms. Bourgain [1] constructed an uncomplemented copy of l_1 inside l_1, providing in this way a

nontrivial exact sequence $0 \to l_1 \to l_1 \to Z \to 0$ that does not locally split. Therefore, although this embedding is 2-exact it is not linearly 2-exact.

$\mathscr{L}_p$-subspaces of $\mathscr{L}_1$-spaces: Let us first consider $L_1 = L_1(0,1)$. It is well known that L_1 contains $L_p = L_p(0,1)$ for $1 < p \le 2$ and contains no complemented copies of Hilbert spaces. We will show that an embedding $j\colon L_p \to L_1$ is never 2-exact. Equivalently, we shall show that $\mathrm{Ext}(L_1/L_p, L_{p^*}) \neq 0$ for $1 < p < 2$. To do that, let H be the closure of the span of the Rademacher functions, which is isomorphic to a Hilbert space and complemented in each L_p for $p \neq 1$. We shall call $j_p\colon H \to L_p$ the embedding and $\pi_p\colon L_p \to H$ the corresponding projection. The push-out diagram:

$$\begin{tikzcd}
0 \arrow[r] & L_p \arrow[r, "j"] \arrow[d, "j_{p^*}\circ\pi_p"] & L_1 \arrow[r] \arrow[d] & L_1/L_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_{p^*} \arrow[r] & PO \arrow[r] & L_1/L_p \arrow[r] & 0
\end{tikzcd}$$

provides the nontrivial desired sequence. The operator $j_{p^*}\pi_p$ cannot be extended to an operator $T\colon L_1 \to L_{p^*}$ since then $jj_p\pi_{p^*}T : L_1 \to L_1$ would be a projection onto H:

$$jj_p\pi_{p^*}Tjj_p\pi_{p^*}T = jj_p\pi_{p^*}j_{p^*}\pi_p j_p\pi_{p^*}T = jj_p\pi_{p^*}T$$

which is impossible.

The uniform boundedness results presented in [9, Proposition 3.2] imply that bilinear forms on l_p^n cannot be extended to bigger l_1^m superspaces with uniformly bounded norms when $1 < p \le 2$. Hence, embeddings $\mathscr{L}_p \to \mathscr{L}_1$ cannot be 2-exact when $1 < p \le 2$.

We have chosen this somewhat longer way of proof to make evident that whenever E is a subspace of $\mathscr{L}_1$ that contains a complemented copy of l_2 then the embedding $E \to \mathscr{L}_1$ cannot be 2-exact. A localization of this argument gives the following.

Reflexive subspaces of $\mathscr{L}_1$-spaces: A variation of this argument works for reflexive subspaces R of $\mathscr{L}_1$. The subspace R must contain (see [16]) uniformly complemented copies of l_2^n for all n. Appealing to the uniform boundedness result [9, Proposition 3.2] and the fact that embeddings $l_2 \to \mathscr{L}_1$ are not 2-exact (by the previous paragraph) it follows that the embedding $R \to \mathscr{L}_1$ cannot be 2-exact.

Kernels of quotients: Since every Banach space B can be written as a quotient L_1/K_B, Theorem 2.3 yields the following nice “symmetry” result:

Corollary 3.1. *Let B be a Banach space. If B is written as $B = X/K_B$ for some $\mathscr{L}_1$-space X then* $\mathrm{Ext}(B, K_B^*) = 0$ *if and only if* $\mathrm{Ext}(K_B, B^*) = 0$.

As a particular case, the kernel D of a quotient map $l_1 \to L_1$ (see [15]) is an $\mathscr{L}_1$-space, and thus the embedding $D \to l_1$ is linearly 2-exact.

Problem. Does there exist a non-$\mathscr{L}_1$ subspace E of an $\mathscr{L}_1$ space such that the embedding $j\colon E \to \mathscr{L}_1$ is 2-exact? In particular, if K_0 denotes the kernel of a quotent map $l_1 \to c_0$: is the embeding $K_0 \to l_1$ 2-exact?

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