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BOUNDARY BEHAVIOUR OF POSITIVE HARMONIC FUNCTIONS ON LIPSCHITZ DOMAINS

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Abstract. Christopher Bishop (1991) proved an extension to higher dimensions of a result of Bishop, Carleson, Garnett and Jones (1989) concerning the harmonic measure of the intersection of a disk of radius r and a Jordan curve Γ both with respect to the interior domain for Γ and with respect to the exterior domain. Their result provides the upper bound Cr^2 for the product of these harmonic measures. We apply the extended result to study the boundary behaviour of positive harmonic functions on Lipschitz domains in $\mathbf{R}^d$ and obtain results that, in two dimensions, are related to the angular derivative problem.

A Jordan curve Γ divides the Riemann sphere into two simply connected domains. Fixing a point in each and considering the resulting harmonic measures, Bishop, Carleson, Garnett and Jones [3] answer the question of when these two measures on Γ are mutually singular. The question was motivated by a result of Browder and Wermer who found that the class $A(\Gamma)$ of functions bounded and continuous on the sphere and analytic off Γ is a Dirichlet algebra if and only if these measures are mutually singular, and it is proved in [3] that this is the case if and only if the set of points on the curve Γ at which a tangent exists has one-dimensional Hausdorff measure 0. The main ingredients in their proof are a refinement due to Pommerenke of Makarov's result on the support of harmonic measure and a local estimate, for which they give credit to Beurling, for the product of the harmonic measures with respect to the two complementary domains to the curve.

This latter estimate was subsequently extended by Bishop [2] to higher dimensions. We state it here in a form that suits our purposes. For a regular domain D in $\mathbf{R}^d$ and $x \in D$, we will denote by $\omega_D(x)$ the harmonic measure at x and with respect to D of that part of the boundary of D lying outside the closed unit ball.

Theorem A. *There is a constant K depending only on the dimension d such that, for disjoint regular domains D_1 and D_2 in $\mathbf{R}^d$ and points x_1 in D_1 and x_2 in D_2 with $|x_1| \le r$ and $|x_2| \le r$,*

$$\omega_{D_1}(x_1)\,\omega_{D_2}(x_2) \le Kr^2.$$

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Bishop is more interested in an analogous situation where one considers the harmonic measure of a ball of variable radius r centred on a common boundary point of the disjoint domains D_1 and D_2. The upper bound on the product of the harmonic measures of this ball with respect to D_1 at a fixed point x_1 in D_1 and with respect to D_2 at a fixed point x_2 in D_2 is then Cr^{2d-2}. Bishop essentially obtains his result by first proving Theorem A using a convexity result of Huber for the Carleman mean of a subharmonic function and a lower bound due to Friedland and Hayman for the characteristic constant of a set on a sphere, and then applying the Kelvin transform.

Our goal is to demonstrate some consequences of Bishop's harmonic measure estimate for the boundary behaviour of positive harmonic functions in Lipschitz domains. By $x=(X,y)$ we denote a point in $\mathbf{R}^d$ with $X\in\mathbf{R}^{d-1}$ and $y\in\mathbf{R}$. We suppose that $h(X)$ is a Lipschitz function on $\mathbf{R}^{d-1}$ with constant c, so that for X_1 and $X_2\in\mathbf{R}^{d-1}$ we have $|h(X_1)-h(X_2)|\le c|X_1-X_2|$, and we suppose that $h(0)=0$. We write D for the component, that contains the point $(0,1)$, of the intersection of $\{(X,y): y>h(X)\}$ with the ball $B(0,2)$, this being a bounded domain lying above the graph of h. With the notation $h^+(X)=\max\{0,h(X)\}$ and $h^-(X)=-\min\{0,h(X)\}$, the local boundary behaviour near the point $(0,0)$ of positive harmonic functions in D depends, at least in part, on the convergence or divergence of the integrals

$$I^+=\int_{|T|<1}\frac{h^+(T)}{|T|^d}\,dT \quad\text{and}\quad I^-=\int_{|T|<1}\frac{h^-(T)}{|T|^d}\,dT.$$

For example, there are now at least four proofs of the following theorem.

Theorem B. *Suppose that the integral I^+ is finite and that the integral I^- is infinite. Suppose that u is a positive harmonic function on D that vanishes continuously on that part of the boundary of D formed by the graph of h. Then*

$$\frac{u(0,y)}{y}\to\infty \quad \textit{as } y\to 0^+. \tag{1}$$

Theorem B relates the normal derivative of the function u at 0 to the geometry of D near 0. In fact, the two dimensional case led to important progress on the angular derivative problem. The original probabilistic proof is in Burdzy [5] for $d=2$ and in Burdzy and Williams [6] for higher dimensions. The present author proposed a classical proof in [7]. Gardiner [10] has simplified the proof of Theorem B considerably. He uses results of Beurling and Dahlberg together with results of Naïm on minimal thinness to put together a very neat proof. A short proof of the two dimensional case is due to Sastry [15]. Her proof is based on extremal length.

The harmonic measure estimate from Theorem A together with Theorem B immediately imply a companion result to Theorem B. We write D_1 for the component, that contains the point $(0,-1)$, of the intersection of the region lying below the graph of h with the ball $B(0,2)$. Then, by Theorem A,

$$\frac{\omega_{D_1}(0,-y)}{y}\,\frac{\omega_D(0,y)}{y} \le K. \tag{2}$$

If I^+ is finite and I^- is infinite then, with u replaced by ω_D, we may deduce from (1) that $\omega_D(0,y)/y \to \infty$ and then from (2) that

$$\frac{\omega_{D_1}(0,-y)}{y} \to 0 \quad \text{as } y \to 0^+.$$

It follows from the boundary Harnack principle [16], [8] that if u is positive and harmonic in D_1 and vanishes continuously on the boundary of D_1 near 0 then $u(0,-y)/y$ also has limit 0 as $y \to 0^+$. On reflecting D_1 in the X-hyperplane, the roles of I^+ and I^- are reversed and we obtain

Theorem 1. *Suppose that the integral I^+ is infinite and that the integral I^- is finite. Suppose that u is a positive harmonic function on D that vanishes continuously on that part of the boundary of D formed by the graph of h. Then*

$$\frac{u(0,y)}{y} \to 0 \quad \textit{as } y \to 0^+.$$

The argument cannot be reversed to deduce Theorem B from Theorem 1—if $\omega_D(0,y)/y$ has limit 0 we cannot deduce from (2) that $\omega_{D_1}(0,-y)/y$ tends to infinity. In this sense, Theorem B is a more difficult result than Theorem 1.

Theorem A and arguments similar to those used by Gardiner permit us to characterize the behaviour of positive harmonic functions when both integrals are convergent. Note that the minimal fine limit in Theorem 2 is also a non tangential limit [10], [4].

Theorem 2. *Suppose that both integrals I^+ and I^- are finite. Suppose that u is a positive harmonic function on D that vanishes continuously on that part of the boundary of D formed by the graph of h. Then*

$$l = \underset{(X,y)\to(0,0)}{\text{m.f.}\lim} \frac{u(X,y)}{G_{\mathrm{H}}\big((X,y),(0,1)\big)}$$

exists, where this minimal fine limit is with respect to the upper half-space H *and where $G_{\mathrm{H}}\big((X,y),(0,1)\big)$ is the Green's function for* H *with pole at $(0,1)$ and evaluated at (X,y). Moreover, l is finite and non-zero.*

Theorems 1 and 2, together with Theorem B, settle the problem of the local growth of a positive harmonic function vanishing on the boundary of a Lipschitz domain when at least one of the integrals I^+ or I^- is finite. Little is known if both integrals are infinite. Theorem 1 in itself is not new, though the short proof made possible by the harmonic measure estimate in Theorem A is new. The two dimensional case was first proved by Rodin and Warschawski [13, Theorem 3] and the general result by Burdzy and Williams in [6] using probabilistic techniques. Theorem 2 is a significant improvement on earlier results. Burdzy [5, Theorem 7.1] is a result on the angular derivative similar to Theorems B and 2 where h arises as a local Lipschitz majorant to the boundary of a simply connected domain in the complex plane. Rodin and Warschawski [14] gave a proof of Burdzy's result that does not use probability in the case when both I^+ and I^- are finite. Theorem 2 in higher dimensions, again with approach along the normal and using probabilistic techniques, was proved by Burdzy and Williams in [6].

We now state some results that will be needed in the proof of Theorem 2. A subset E of a domain D is said to be minimally thin with respect to D at a minimal Martin boundary point ζ of D if the reduced function over E of the Martin kernel $K(\zeta,\cdot)$ with pole at ζ differs from $K(\zeta,\cdot)$ [9, 1.XII.11]. The Martin compactification of a Lipschitz domain is its closure in the one point compactification of $\mathbf{R}^d$ and the minimal Martin boundary points of D are precisely the ordinary boundary points of D together with, possibly, the point at infinity. The following theorem is then a special case of a theorem of Naïm [12, Théorème 15].

Theorem C. *Suppose that the Lipschitz domain D_1 contains the Lipschitz domain D_2 and that 0 is a common boundary point of D_1 and D_2. If $D_1 \setminus D_2$ is minimally thin at 0 with respect to D_1 then a subset A of D_2 is minimally thin at 0 with respect to D_1 if and only if it is minimally thin at 0 with respect to D_2.*

The next lemma is a consequence of results of Beurling and Dahlberg and is Lemma 1 in [10]. Lemma B is a direct consequence of [9, Theorem 1.XII.14] and [12, Théorème 11], see [6, Lemma 4.1].

Lemma A. *If $h(X)$ is a positive Lipschitz function on $\mathbf{R}^{d-1}$ then the set $\{(X,y) : 0 < y < h(X)\}$ is minimally thin at 0 with respect to the upper half-space H if and only if*

$$\int_{|T|<1} \frac{h(T)}{|T|^d}\, dT < \infty. \tag{3}$$

Lemma B. *Suppose that D_1 and D_2 are Lipschitz domains, that D_2 is contained in D_1 and that 0 is a common boundary point of D_1 and D_2. Suppose that y_0 is in D_2 and $G_1(x,y_0)$ and $G_2(x,y_0)$ are Green's functions for D_1 and D_2 respectively with poles at y_0. Then the minimal fine limit with respect to D_2 of $G_1(x,y_0)/G_2(x,y_0)$ exists as $x \to 0$ and this limit is finite if and only if $D_1 \setminus D_2$ is minimally thin at 0 with respect to D_1.*

Proof of Theorem 2. The notation $G^-(x)$ stands for the Green's function with pole at $y_0 = (0,1)$ for the domain $D^- = \{(X,y) : y > -h^-(X)\}$. We will use Theorem A to show that $E = D^- \setminus \mathrm{H} = \{(X,y) : -h^-(X) < y \le 0\}$ is minimally thin at 0 with respect to D^-. Suppose that this was not the case. Then, by Lemma B,

$$\text{m.f.}\lim_{x\to 0} \frac{G^-(x)}{G_{\mathrm{H}}(x)} = \infty$$

where this minimal fine limit is with respect to H. It follows that the ratio of Green's functions $G^-(x)/G_{\mathrm{H}}(x)$ has nontangential limit $+\infty$ as $x \to 0$, [9, Theorem 1.XII.21]. We conclude that $\omega_{D^-}(0,y)/y \to \infty$ as $y \to 0^+$. This is because $G_{\mathrm{H}}(0,y) \sim y$ and it follows from the boundary Harnack principle that $G^-(0,y) \sim \omega_{D^-}(0,y)$ as $y \to 0^+$. From Theorem A we conclude that $\omega_R(0,y)/y \to 0$ as $y \to 0^-$, where $R = \{(X,y) : y < -h^-(X)\}$. An application of Harnack's inequality improves this to $\omega_R(X,y)/y \to 0$ as $(X,y) \to (0,0)$ in the cone $\{(X,y) : y < -|X|\}$. Thus $G_{\overline{\mathrm{H}}}(x)/G_R(x)$ has limit ∞ as x tends to 0 in the cone, this being the ratio of the Green's functions for the lower half-space $\overline{H} = \{(X,y) : y < 0\}$ and the domain R, each with pole at $(0,-1)$. Now $\overline{\mathrm{H}} \setminus R$ is minimally thin at 0 with respect to $\overline{\mathrm{H}}$ because of Lemma A and the assumption that I^- is finite. It then follows from Theorem C that the cone in question is not minimally thin at 0 with respect to R because it is not minimally thin at 0 with respect to $\overline{\mathrm{H}}$. By Lemma B, $G_{\overline{\mathrm{H}}}(x)/G_R(x)$ has a minimal fine limit at 0 with respect to R and, because the cone is not minimally thin at 0 with respect to R, we must conclude that this limit is $+\infty$. But then we conclude from Lemma B that $\overline{\mathrm{H}} \setminus R$ is not minimally thin at 0 with respect to $\overline{\mathrm{H}}$. This is a contradiction and so E is minimally thin at 0 with respect to D^-.

We write $F = \{(X,y) : 0 < y \le h^+(X)\}$. Since, by Lemma A, F is minimally thin at 0 with respect to H, we conclude from Theorem C with $D_1 = D^-$ and $D_2 = \mathrm{H}$ (so that $D_1 \setminus D_2 = E$ is minimally thin at 0 with respect to D_1) that F is minimally thin at 0 with respect to D^-. This in turn allows us to apply Theorem C with $D_1 = D^-$ and $D_2 = D$ and to conclude that E is minimally thin at 0 with respect to D since it is minimally thin at 0 with respect to D^-.

We write D^+ for the domain $\{(X,y) : y > h^+(X)\}$. Since $D \setminus D^+$ also equals E and since E is minimally thin at 0 with respect to D we deduce from Lemma B that $0 < l_1 < \infty$ where

$$l_1 = \text{m.f.}\lim_{x\to 0} \frac{G_D(x)}{G^+(x)}. \tag{4}$$

Here $G_D(x)$ and $G^+(x)$ are the Green functions for D and D^+ respectively, with poles at $(0,1)$, and the minimal fine limit is with respect to D^+. Since I^+ is finite it follows from Lemma A that $\mathrm{H} \setminus D^+$ is minimally thin at 0 with respect to H and so, by Lemma B, $0 < l_2 < \infty$ where

$$l_2 = \text{m.f.}\lim_{x\to 0} \frac{G^+(x)}{G_{\mathrm{H}}(x)}. \tag{5}$$

We obtain from (4) and (5) that $0 < l < \infty$ where

$$l = \text{m.f.}\lim_{x\to 0} \frac{G_D(x)}{G_{\mathrm{H}}(x)}.$$

The minimal fine limits (4) and (5) are initially with respect to D^+. But since we have seen that minimal thinness of a set at 0 with respect to D^+ and H are equivalent, the minimal fine limit l can be taken to be with respect to H. Finally, [1, Theorem 2] (see also [11]) yields that $u(x)/G_D(x)$ has a finite, positive limit as $x \to 0$. The theorem follows.

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