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ON A PAPER OF CARLESON

Melkana A. Brakalova and James A. Jenkins

The Hotchkiss School, Department of Mathematics
Lakeville, CT 06039, U.S.A.; mbrakalo@hotchkiss.org
Current address: Institute for Mathematical Sciences SUNY
Stony Brook, NY 11794-3660, U.S.A.; melkana@math.sunysb.edu

Washington University, Department of Mathematics
Campus Box 1146, St. Louis, MO 63130-4899, U.S.A.

Abstract. We study the boundary correspondence under sense-preserving homeomorphic self-mapping of the upper half-plane. Under assumptions that involve integrals of the complex dilatation we prove that the natural extension of the mapping is conformal at every point on the boundary. If the integrals involving the complex dilatation are locally uniformly convergent we prove that the derivative of the boundary extension is continuous. The results obtained are closely related to well-known results in a paper by L. Carleson.

1. Introduction

It is well known that a quasiconformal self-mapping of the open upper half-plane admits an extension as a homeomorphism to the closed half-plane. If the points at infinity correspond, this gives a continuous, monotone increasing real-valued function. Beurling and Ahlfors [1] have given a necessary and sufficient condition for such a function to be the extension of a quasiconformal mapping. In particular they deduced that such a function need not be Lebesgue measurable. On the other hand, Carleson [3] showed that if the complex dilatation of a quasiconformal self-mapping of the upper-half plane satisfies in two instances subsidiary conditions in a strip neighborhood of the real axis the extension will be measurable. In one of the cases the extension is proved to have a continuous derivative.

There have been various alternative proofs and extensions of these results. In particular Gutlyanskii and Vuorinen [4] have shown, in the context of quasiconformal self-mappings of the unit disk, that the uniform convergence of certain integrals associated with the points of the circumference implies the existence of a continuously differentiable extension.

In this paper we consider a sense-preserving homeomorphic self-mapping of the upper half-plane (without the assumption of quasiconformality). We assume that this mapping possesses, in a neighborhood of the real axis, a complex dilatation that satisfies pointwise convergence of certain integrals associated with

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points on the real axis. Under such assumptions the homeomorphism admits a differentiable homeomorphic extension mapping the real axis onto itself. The natural extension of the mapping is conformal at every point on the boundary. If the convergence of the above-mentioned integrals is locally uniform the derivative is continuous.

2. Main results

Let $f(z)$ be a sense preserving homeomorphism of the open upper half-plane H onto itself. Let $g(w)$ be the inverse of $f(z)$ in the open upper half-plane H. Suppose that as $w \to \infty$, $w \in H$, then $g(w) \to \infty$ and as $z \to \infty$, $z \in H$, then $f(z) \to \infty$. We assume that $f(z)$ has complex dilatation $k(z) = f_{\bar{z}}/f_z$ a.e. in a neighborhood $N \in H$ of the real axis R. Denote by x a real number. For a fixed x and $0 < r_1 < r_2$, we define the semiannulus $Q(x, r_1, r_2) = \{\operatorname{Im} z > 0\} \cap \{r_1 < |z - x| < r_2\}$ and assume r_2 sufficiently small so that $Q(x, r_1, r_2) \subset N$. For every $x \in R$

$$h_1^*(\phi, x) = \int_{r_1}^{r_2} \frac{|1 + e^{-2i\phi} k(re^{i\phi})|^2}{1 - |k(re^{i\phi})|^2} \frac{dr}{r}, \quad h_2^*(r, x) = \int_0^{\pi} \frac{|1 - e^{-2i\phi} k(re^{i\phi})|^2}{1 - |k(re^{i\phi})|^2} \, d\phi,$$

are to exist for almost all $r \in [r_1, r_2]$ and $\phi \in [0, \pi]$.

Denote

$$J(x, r_1, r_2) = \iint\limits_{Q(x, r_1, r_2)} \frac{|k(z)|^2 + |\operatorname{Re} e^{-2i\phi} k(z)|}{1 - |k(z)|^2} \frac{dA}{|z - x|^2},$$

where $\phi = \arg(z - x)$.

Theorem 1. *If $f(z)$ satisfies the assumptions outlined above and*

(2.1) $\quad J(x, r_1, r_2)$ *converges for every real x as $r_1 \to 0$,*

then $f(z)$ admits a differentiable increasing homeomorphic extension $f(x)$ from the real axis to itself. In addition for $z \in N$ and $x \in R$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} > 0. \tag{2.2}$$

Theorem 2. *If*

(2.3) $\quad J(x, r_1, r_2)$ *converges locally uniformly for every real x as $r_1 \to 0$,*

then $f'(x)$ is continuous.

3. Auxiliary results

Here we restate results from [2] essential to our proof.

Let $w = w(z)$ be a homeomorphism of the unit disc U_0 onto itself, $w(0) = 0$, with a complex dilatation $k(z) = w_{\bar z}(z)/w_z(z)$ defined a.e. in U_0. Assume that

$$h_1^*(\phi) = \int_{r_1}^{r_2} \frac{|1 + e^{-2i\phi}k(x + re^{i\phi})|^2}{1 - |k(x + re^{i\phi})|^2} \frac{dr}{r},$$

$$h_2^*(r) = \int_0^{2\pi} \frac{|1 - e^{-2i\phi}k(x + re^{i\phi})|^2}{1 - |k(x + re^{i\phi})|^2} \, d\phi,$$

exist a.e. for $r \in [r_1, r_2]$, $0 < r_1 < r_2 < 1$ and $\phi \in [0, 2\pi]$. In addition we assume that $h_1^*(\phi) \in L^1[0, 2\pi]$, $h_2^*(r) \in L^1[r_1, r_2]$. We denote by $M^*(r_1, r_2)$ the module of the image under $w(z)$ of the annulus $r_1 < |z| < r_2$. Under the above assumptions we have shown the following results.

Lemma 1 ([2, Lemma 3.1]). *We have*

$$\left| M^*(r_1, r_2) - \frac{1}{2\pi} \log \frac{r_2}{r_1} \right| < \frac{1}{2\pi^2} \iint\limits_{r_1 < |z| < r_2} \frac{|k(z)|^2 + |\operatorname{Re} e^{-2i\phi}k(z)|}{1 - |k(z)|^2} \frac{dA}{|z|^2}, \tag{3.1}$$

where $\phi = \arg z$.

Lemma 2 ([2, Corollary 1.1]). *If*

$$\iint\limits_{U_0} \frac{|k(z)|^2 + |\operatorname{Re} e^{-2i\phi}k(z)|}{1 - |k(z)|^2} \frac{dA}{|z|^2} < \infty \tag{3.2}$$

where $\phi = \arg z$, *and if*

$$\lim_{r \to 0} \arg w(re^{i\theta_0}) = a \tag{3.3}$$

for a particular value θ_0, *then* $w(z)$ *is conformal at* $z = 0$, *i.e.*

$$\lim_{z \to 0} \frac{w(z)}{z} = C, \quad C \neq 0. \tag{3.4}$$

A well-known result in the theory of quasiconformal mappings is the Teichmüller–Belinskii theorem according to which (3.4) holds under the restriction that $w(z)$ is K-quasiconformal and:

$$\iint\limits_{U_0} \frac{|k(z)|}{1 - |k(z)|} \frac{dA}{|z|^2} < \infty. \tag{3.5}$$

Since (3.5) is a stronger condition than (3.2) the use of Lemma 2 allows the proof of the differentiability of the boundary mapping under a weaker condition than a condition similar to (3.5), as used by Carleson, without the assumption of K-quasiconformality. We, naturally, take advantage of the fact that when mapping the upper half-plane onto itself (3.3) holds with $\theta_0 = 0$ and $a = 0$.

4. Proofs of the main results

Proof of Theorem 1. The semiannulus $Q(x,r_1,r_2)$ becomes a quadrangle after assigning as a pair of opposite sides the boundary segments on the real axis. Denote by $M^*(x,r_1,r_2)$ the module of the image $Q^*(x,r_1,r_2)$ of the quadrangle $Q(x,r_1,r_2)$ under $f(z)$. In a method similar to the proof of Lemma 1 (or Lemma 3.1 from [2]) one can show that:

$$\left|M^*(x,r_1,r_2) - \frac{1}{\pi}\log\frac{r_2}{r_1}\right| < \frac{1}{\pi^2}\iint\limits_{Q(x,r_1,r_2)} \frac{|k(z)|^2 + |\operatorname{Re} e^{-2i\phi}k(z)|}{1-|k(z)|^2}\,\frac{dA}{|z-x|^2},$$

where where $\phi = \arg(z-x)$. Since (2.1) holds we see that

$$M^*(x,r_1,r_2) \to \infty, \text{ as } r_1 \to 0.$$

This means that the image of the inner semicircle with center x and radius r_1 tends to a point. We will define this point as $f(x)$. Thus it is obvious that $\lim_{z\to x} f(z) = f(x)$. Denote by $S_{x,d}$ a semicircle in the upper half plane with center x and diameter d. Let $C_{x,d} = \left[x-\frac{1}{2}d, x+\frac{1}{2}d\right] \cup S_{x,d}$. Consider a sufficiently small interval I of length d centered at $f(x_0)$ and take a semicircle $S_{x_0,2r_0}$ such that its image lies in the interior $\operatorname{int} C_{f(x_0),d}$. Take any point x in $[x_0-r_0, x_0+r_0]$. The construction of $f(x)$ as outlined above assures that $f(x)$ falls in the interval I. Therefore the image of the interval $[x_0-r_0, x_0+r_0]$ falls inside I and $f(x)$ is continuous. Now take two different points x_1 and x_2. We can choose semicircles S_{x_1,d_1} and S_{x_2,d_2} so that $\operatorname{int} C_{x_1,d_1} \cap \operatorname{int} C_{x_2,d_2} = \emptyset$. Since $f(z)$ is one to one the images of $\operatorname{int} C_{x_i,d_i}$, $i = 1,2$ are disjoint. Therefore $f(x_1) \neq f(x_2)$, thus $f(x)$ is one to one.

Consider the images under f of $C_{x,d}$ and a closed Jordan curve $\gamma \in \operatorname{int} C_{x,d}$. Since $f(z)$ is sense preserving, the orientation of $f(\gamma)$ is the same as γ. By the argument principle the orientation of the image of $f(C_{x,d})$ is the same as the orientation of $C_{x,d}$. That shows that f is increasing.

Now we are going to show that f is differentiable. We extend f to a function in a neighborhood N_0 of R, $N_0 = N \cup \overline{N} \cup R$ by reflection. Here $\overline{N} = \{z : \bar{z} \in N\}$. Then for $z \in \overline{N}$ the extension $f(z)$ is defined as $f(z) = \bar{f}(\bar{z})$. The so obtained new mapping will be called f. The complex dilatation $k(z)$ of the extended mapping is defined a.e. in N_0. Because of (2.1) for all x

$$\iint\limits_{|z-x|<r_2} \frac{|k(z)|^2 + |\operatorname{Re} e^{-2i\phi}k(z)|}{1-|k(z)|^2}\,\frac{dA}{|z-x|^2} < \infty,$$

where $\phi = \arg(z-x)$. Since $\lim\limits_{r\to 0} \arg f(x+r) = 0$, by Lemma 2 (or by Corollary 1.1 from [2])

$$\lim_{z\to x} \frac{f(z)-f(x)}{z-x} = C, \qquad C \neq 0.$$

This implies that $f(x)$ has a nonzero derivative that must be positive since f is increasing and (2.2) follows.

Proof of Theorem 2. Let $f(z)$ and $k(z)$ be the extended map and its complex dilatation N_0, as constructed in Theorem 1. Because of (2.3) the integral

$$\iint\limits_{r_1<|z-x|<r_2} \frac{|k(z)|^2+|\operatorname{Re} e^{-2i\phi}k(z)|}{1-|k(z)|^2}\,\frac{dA}{|z-x|^2},$$

where $\phi=\arg(z-x)$, is convergent locally uniformly with respect to x as $r_1\to 0$. Now we can consider the module $M(x,r_1,r_2)$ of the image of the annulus $\{z : r_1<|z-x|<r_2\}$ under $f(z)$. Denote by $M(x,r_2)$ the reduced module of the image of the circle $|z-x|<r_2$ under $f(z)$. For fixed x_0 and $\varepsilon>0$ by Lemma 1, (2.3) and the continuity of the reduced module $M(x,r_2)$ in x there exists $\delta>0$ such that for $|x-x_0|<\delta$ and $r_1<r_2$, r_2 sufficiently small

$$|M(x,r_1,r_2)-M(x_0,r_1,r_2)|<\varepsilon$$

and

$$|M(x,r_2)-M(x_0,r_2)|<\varepsilon.$$

On the other hand by (2.2) and the definition of reduced module (see [5])

$$M(x,r_2)=\lim_{r_1\to 0}\Big(M(x,r_1,r_2)+\frac{1}{2\pi}\log\big(r_1 f'(x)\big)\Big).$$

For r_1 sufficiently small

$$\Big|M(x,r_2)-\Big(M(x,r_1,r_2)+\frac{1}{2\pi}\log\big(r_1 f'(x)\big)\Big)\Big|<\varepsilon$$

and

$$\Big|M(x_0,r_2)-\Big(M(x_0,r_1,r_2)+\frac{1}{2\pi}\log\big(r_1 f'(x_0)\big)\Big)\Big|<\varepsilon.$$

Therefore

$$\begin{aligned}
&\Big|\frac{1}{2\pi}\log\big(r_1 f'(x)\big)-\frac{1}{2\pi}\log\big(r_1 f'(x_0)\big)\Big|\\
&\qquad\le\Big|M(x,r_2)-\Big(M(x,r_1,r_2)+\frac{1}{2\pi}\log\big(r_1 f'(x)\big)\Big)\\
&\qquad\quad-\Big(M(x_0,r_2)-\Big(M(x_0,r_1,r_2)+\frac{1}{2\pi}\log\big(r_1 f'(x_0)\big)\Big)\Big)\\
&\qquad\quad+\big(M(x_0,r_2)-M(x,r_2)\big)+\big(M(x,r_1,r_2)-M(x_0,r_1,r_2)\big)\Big|\\
&\qquad<4\varepsilon
\end{aligned}$$

Thus $\log(f'(x))$ is continuous and so is $f'(x)$.

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