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ON THE COMPLEXIFICATION OF THE WEIERSTRASS NON-DIFFERENTIABLE FUNCTION

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Abstract. It is shown that for the Weierstrass nowhere differentiable functions $X_{a,b}(t) = \sum_{n=0}^{\infty} a^n \cos(b^n t)$ and $Y_{a,b}(t) = \sum_{n=0}^{\infty} a^n \sin(b^n t)$ the set $(X_{a,b}, Y_{a,b})([0, 2\pi])$ has a non-empty interior in $\mathbf{R}^2$, provided $b \in \mathbf{N}$, $b \geq 2$ and $a < 1$ is sufficiently close to 1. It follows that the box dimension of $\operatorname{graph}(X_{a,b}, Y_{a,b})$ is equal to $3 - 2\alpha$ where $\alpha = -\log a / \log b$ and its Hausdorff dimension is at least 2. Moreover, the level sets $L(s)$ for $X_{a,b}$ and $Y_{a,b}$ have Hausdorff dimension at least α for open sets of $s \in \mathbf{R}$, so the Hausdorff dimension of $\operatorname{graph} X_{a,b}$ and $\operatorname{graph} Y_{a,b}$ is at least $1 + \alpha$.

1. Introduction

This paper concerns the famous functions

$$X_{a,b}(t) = \sum_{n=0}^{\infty} a^n \cos(b^n t), \qquad Y_{a,b}(t) = \sum_{n=0}^{\infty} a^n \sin(b^n t)$$

for $t \in [0, 2\pi]$ and $0 < a < 1$, $b > 1$, $ab \geq 1$. The first one was introduced by Weierstrass in 1872 as an example of a continuous, nowhere differentiable function. In fact, the non-differentiability for all given above parameters a, b was proved by Hardy in [Ha]. Later, the graphs of these and related functions were studied as fractal curves. A basic question which arises in this context is computing the Hausdorff dimension (HD) of these curves. However, this problem is still unsolved for the classical functions $X_{a,b}$ and $Y_{a,b}$.

For $ab = 1$, the graphs of $X_{a,b}$ and $Y_{a,b}$ have Hausdorff dimension 1 and σ-finite 1-dimensional Hausdorff measure, as was proved by Mauldin and Williams in [MW]. For $ab > 1$, it is easy to check that the functions $X_{a,b}$ and $Y_{a,b}$ are Hölder continuous with exponent α for

$$\alpha = -\frac{\log a}{\log b}$$

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(i.e. $a = b^{-\alpha}$). Consequently, the box dimension (BD) of their graphs is at most $2-\alpha$ (see Lemma 2.2). In fact, it is equal to $2-\alpha$, as was proved in [KMY]. Hence, $\mathrm{HD}(\operatorname{graph} X_{a,b})$, $\mathrm{HD}(\operatorname{graph} Y_{a,b})$ cannot exceed $2-\alpha$. Moreover, the packing dimension of these graphs is $2-\alpha$ (see [R]).

It is believed that the Hausdorff dimension of these graphs should also be equal to $2-\alpha$. Note that

$$X_{a,b}(t) = aX_{a,b}(bt) + \cos t, \qquad Y_{a,b}(t) = aY_{a,b}(bt) + \sin t,$$

which means that the graphs are roughly self-similar for the scaling with horizontal factor b and vertical factor a.

The difficulties lie in the lower estimates of the Hausdorff dimension. There are not too many results in this direction. Mauldin and Williams in [MW] gave the lower bound of the form $2-\alpha-C/\log b$ for a constant $C>0$ independent of b, which approaches the upper bound as $b\to\infty$. Przytycki and Urbański proved in [PU] that the Hausdorff dimension of $\operatorname{graph} X_{a,b}$, $\operatorname{graph} Y_{a,b}$ is greater than 1 for $b\in\mathbf{N}$, $b\ge 2$. In [Hu], Hunt showed that the Hausdorff dimension of the graph of the function

$$\sum_{n=0}^{\infty} a^n \cos(b^n t + \theta_n)$$

with θ_n chosen independently with respect to the uniform probability measure on $[0,2\pi]$, is almost surely equal to $2-\alpha$.

It turns out that it is easier to consider the problem for the Weierstrass function with cosine replaced by some other continuous periodic function $g\colon \mathbf{R}\to\mathbf{R}$. For instance, take $g(t) = \operatorname{dist}(t,\mathbf{Z})$ (the sawtooth function), which was studied by Besicovitch and Ursell in [BU]. For $a=b$, we obtain the van der Waerden–Tagaki function, which has Hausdorff dimension 1 and σ-finite 1-dimensional Hausdorff measure. This was proved by Anderson and Pitt in [AP]. Moreover, by the work of Ledrappier [L], the case $b=2$, $a>\frac{1}{2}$ can be brought to the case of the Bernoulli convolutions $\sum \pm a^n$, where the signs are chosen independently with probability $\frac{1}{2}$ on $[0,1]$. Then the work of Solomyak [S] implies that the Hausdorff dimension of the graph is $2-\alpha$ for almost all $a\in\left(\frac{1}{2},1\right)$.

Another interesting problem is studying various measures related to these graphs. For a function $f\colon [t_0,t_1]\to\mathbf{R}^m$ denote by μ_f the image under f of the uniform probability measure on $[t_0,t_1]$. Little is known about the measures $\mu_{X_{a,b}}$ and $\mu_{Y_{a,b}}$. It is conjectured that the Hausdorff dimension of $\operatorname{graph} X_{a,b}$ (or $\operatorname{graph} Y_{a,b}$) is $2-\alpha$ if and only if the measure $\mu_{X_{a,b}}$ (or $\mu_{Y_{a,b}}$) is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$. This holds for the Bernoulli convolutions, as was proved in [PU]. Kôno showed in [K] that if $b\in\mathbf{N}$ and ab is sufficiently large and the suitable measure $\mu_{X_{a,b}}$ or $\mu_{Y_{a,b}}$ has a bounded density function with respect to the Lebesgue measure, then the Hausdorff dimension of the graph is $2-\alpha$.

In this paper we consider the complexification of the functions $X_{a,b}$ and $Y_{a,b}$, i.e.

$$F_{a,b}(z) = \sum_{n=0}^{\infty} a^n z^{b^n}, \qquad z \in \mathbf{C},\ |z| \le 1,$$

for $a \in (0,1)$ and $b \in \mathbf{N}$, $b \ge 2$. Then $F_{a,b}$ is holomorphic in the open unit disc, continuous in the closed unit disc and

$$\operatorname{Re}\bigl(F_{a,b}(e^{it})\bigr) = X_{a,b}(t), \qquad \operatorname{Im}\bigl(F_{a,b}(e^{it})\bigr) = Y_{a,b}(t).$$

We prove that if a is sufficiently close to 1, then the image of the unit circle $\mathbf{S}^1$ under $F_{a,b}$ (i.e. the image of the segment $[0,2\pi]$ under the map $(X_{a,b}, Y_{a,b})$) is a curve which has non-empty interior in the topology of the plane. More precisely, we show

Theorem 1.1. *There exist $a_0 < 1$ and $c > 0$, such that for every $a \in [a_0, 1)$ and every $b \in \mathbf{N}$, $b \ge 2$, the set $F_{a,b}(\mathbf{S}^1)$ contains a disc of radius $c/(1-a)$. Moreover, $F_{a,b}(\mathbf{S}^1)$ is the closure of its interior in the topology of the plane.*

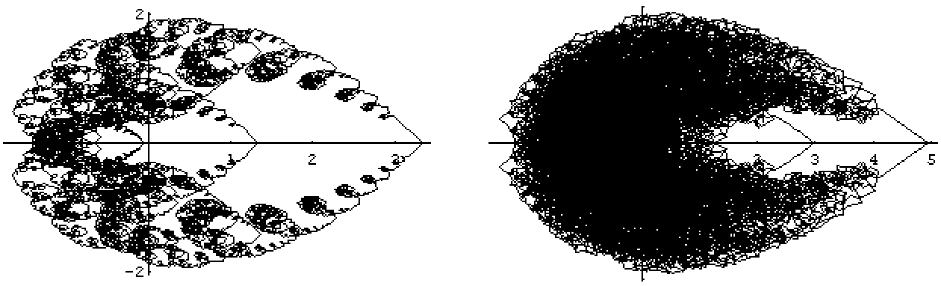

Figure 1. The curve $F_{a,2}(\mathbf{S}^1)$ for $a = 0.7$ (left) and $a = 0.8$ (right).

The idea of complexifying the Weierstrass function is not new. In [Ha] Hardy used its harmonic extension to prove non-differentiability. Our approach, however, is not analytical but relies on some elementary geometric facts (Lemma 3.3).

Apart from presenting an interesting example of a "plane-filling" curve, Theorem 1.1 has some consequences concerning the graphs of the functions $X_{a,b}$, $Y_{a,b}$. First, we can compute the exact value of the box dimension of $\operatorname{graph}(X_{a,b}, Y_{a,b})$ (as a subset of $\mathbf{R}^3$), which is equal to $3 - 2\alpha$. The Hausdorff dimension of this graph is at least 2. These results are shown in Corollary 4.1.

For $s \in \mathbf{R}$ define the level sets of $X_{a,b}$, $Y_{a,b}$ as

$$L_{X_{a,b}}(s) = \{t \in \mathbf{R} : X_{a,b}(t) = s\}, \qquad L_{Y_{a,b}}(s) = \{t \in \mathbf{R} : Y_{a,b}(t) = s\}.$$

In Corollary 4.3 we show that the Hausdorff dimension of $L_{X_{a,b}}(s)$ and $L_{Y_{a,b}}(s)$ is at least α for some open sets of $s \in \mathbf{R}$. This implies (Corollary 4.4) that

$$\operatorname{HD}(\operatorname{graph} X_{a,b}), \operatorname{HD}(\operatorname{graph} Y_{a,b}) \ge 1 + \alpha.$$

Theorem 1.1 and the corollaries are true for a close to 1, i.e. for α close to 0. The functions $X_{a,b}$, $Y_{a,b}$ are Hölder continuous with exponent α, so the map $(X_{a,b}, Y_{a,b})$ is also Hölder continuous with the same exponent. This implies (see Lemma 2.2) that

$$\mathrm{HD}\big(F_{a,b}(\mathbf{S}^1)\big) = \mathrm{HD}\big((X_{a,b}, Y_{a,b})([0, 2\pi])\big) \le \underline{\mathrm{BD}}\big((X_{a,b}, Y_{a,b})([0, 2\pi])\big) \le 3 - 2\alpha,$$

so for $\alpha > \frac{1}{2}$ we have $\mathrm{HD}(F_{a,b}(\mathbf{S}^1)) < 2$. In particular, $F_{a,b}(\mathbf{S}^1)$ has 2-dimensional Lebesgue measure 0, $\mu_{(X_{a,b},Y_{a,b})}$ is singular with respect to this measure and Theorem 1.1 cannot be true. It would be of interest to check whether Theorem 1.1 holds for every $\alpha \le \frac{1}{2}$. (See Figure 1, where the left picture shows the curve for $\alpha = 0.5145\ldots$ and the right one for $\alpha = 0.3219\ldots$.) The most interesting case is $\alpha = \frac{1}{2}$. Indeed, we have the following:

Fact. *Suppose $\alpha = \frac{1}{2}$ and $F_{a,b}(\mathbf{S}^1)$ has positive 2-dimensional Lebesgue measure. Then $\mu_{(X_{a,b},Y_{a,b})}$ is not singular with respect to this measure and the measures $\mu_{X_{a,b}}$, $\mu_{Y_{a,b}}$ are not singular with respect to 1-dimensional Lebesgue measure. Moreover,* $\mathrm{HD}(\operatorname{graph} X_{a,b}) = \mathrm{HD}(\operatorname{graph} Y_{a,b}) = 2 - \alpha = \frac{3}{2}$.

Proof. Let Leb_m be the m-dimensional Lebesgue measure. By Lemma 2.2, we have $\mathrm{Leb}_2|_{F_{a,b}(\mathbf{S}^1)} \le C\mu_{(X_{a,b},Y_{a,b})}$ for a constant C. Suppose $\mu_{(X_{a,b},Y_{a,b})}$ is singular with respect to Leb_2 and take a set $A \subset F_{a,b}(\mathbf{S}^1)$ such that $\mu_{(X_{a,b},Y_{a,b})}(A) = 1$ and $\mathrm{Leb}_2(A) = 0$. Then

$$\mathrm{Leb}_2\big(F_{a,b}(\mathbf{S}^1)\big) = \mathrm{Leb}_2\big(F_{a,b}(\mathbf{S}^1) \setminus A\big) \le C\mu_{(X_{a,b},Y_{a,b})}\big(F_{a,b}(\mathbf{S}^1) \setminus A\big) = 0,$$

which contradicts the assumption. Hence, $\mu_{(X_{a,b},Y_{a,b})}$ is not singular with respect to Leb_2. Since $\mu_{X_{a,b}}$, $\mu_{Y_{a,b}}$ are orthogonal projections of $\mu_{(X_{a,b},Y_{a,b})}$ on the coordinate axes, they are not singular with respect to Leb_1. The last part follows from Corollary 4.4, because $1 + \alpha = \frac{3}{2} = 2 - \alpha$.

2. Preliminaries

We recall some basic definitions and facts concerning the Hausdorff and box dimension.

Definition 2.1. For $A \subset \mathbf{R}^n$ and $\delta > 0$ the (outer) δ-Hausdorff measure of A is defined as

$$\mathscr{H}^\delta(A) = \lim_{\varepsilon \to 0} \inf \sum_{U \in \mathscr{U}} (\operatorname{diam} U)^\delta,$$

where infimum is taken over all countable coverings $\mathscr{U}$ of A by open sets of diameters less than ε.

The Hausdorff dimension of A is defined as

$$\mathrm{HD}(A) = \sup\{\delta > 0 : \mathscr{H}^\delta(A) = +\infty\} = \inf\{\delta > 0 : \mathscr{H}^\delta(A) = 0\}.$$

Let $N_\varepsilon(A)$ be the minimal number of balls of diameter ε needed to cover A. Define the lower and upper box dimension as

$$\underline{\mathrm{BD}}(A) = \liminf_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log\varepsilon}, \qquad \overline{\mathrm{BD}}(A) = \limsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log\varepsilon}.$$

The box dimension is also called the box-counting or Minkowski dimension. It is easy to check that

$$\mathrm{HD}(A) \le \underline{\mathrm{BD}}(A).$$

The definitions of the Hausdorff and box dimension easily imply

Lemma 2.2. *Let $A \subset \mathbf{R}^n$ and let $f\colon A \to \mathbf{R}^m$ be a map such that*

$$\|f(x) - f(y)\| \le c\|x-y\|^\beta$$

for every $x, y \in A$ and constants $c > 0$, $0 < \beta \le 1$. Then for every $\delta > 0$,

$$\mathscr{H}^{\delta/\beta}\big(f(A)\big) \le c^{\delta/\beta}\mathscr{H}^\delta(A), \qquad \text{so} \qquad \mathrm{HD}\big(f(A)\big) \le \mathrm{HD}(A)/\beta.$$

Moreover,

$$\underline{\mathrm{BD}}(\operatorname{graph} f) \le \underline{\mathrm{BD}}(A) + m(1-\beta),$$
$$\overline{\mathrm{BD}}(\operatorname{graph} f) \le \overline{\mathrm{BD}}(A) + m(1-\beta).$$

We shall use the following theorem estimating the Hausdorff dimension of a planar set by the dimensions of its level sets (for the proof see e.g. [F]).

Theorem 2.3. *Let $E \subset \mathbf{R}^2$ and $A \subset \mathbf{R}$. Suppose that there exists $\beta > 0$, such that if $x \in A$, then $\mathscr{H}^\beta\big(\{y \in \mathbf{R} : (x,y) \in E\}\big) > c$, for some constant c. Then for every $\delta > 0$,*

$$\mathscr{H}^{\delta+\beta}(E) \ge bc\mathscr{H}^\delta(A),$$

where b depends only on β and δ. In particular,

$$\mathrm{HD}(E) \ge \mathrm{HD}(A) + \inf_{x\in A} \mathrm{HD}\big(\{y \in \mathbf{R} : (x,y) \in E\}\big).$$

Notation. The symbols cl, int and ∂ denote respectively the closure, interior and boundary in the topology of the plane. The euclidean distance is denoted by dist. $\overline{AB}$ is the segment with endpoints A, B and $|\overline{AB}|$ is its length. We write $\mathbf{D}_r(x)$ for the open disc centred at $x \in \mathbf{C}$ of radius r. For $t \in \mathbf{R}$ we denote by $[t]$ the integer part of t, i.e. the largest integer not greater than t.

3. Proof of Theorem 1.1

Let $0<a<1$, $b\in\mathbf{N}$, $b\geq 2$. By the definition of $F_{a,b}$, we have

$$F_{a,b}(e^{2\pi i t_1})-F_{a,b}(e^{2\pi i t_2})=2i\sum_{n=0}^{\infty}a^n\sin\bigl(\pi b^n(t_1-t_2)\bigr)e^{\pi i b^n(t_1+t_2)}$$

for every $t_1,t_2\in[0,1]$. Let

$$z_{n,k}=e^{2\pi i k/b^n}\qquad\text{for } n\geq 0,\ k=1,\ldots,b^n$$

and fix $j\in\mathbf{Z}$. Then

$$\begin{aligned}F_{a,b}(z_{n,k+j})-F_{a,b}(z_{n,k})&=2i\sum_{l=0}^{n-1}a^l\sin(\pi j/b^{n-l})e^{\pi i(2k+j)/b^{n-l}}\\&=2ia^n\sum_{m=1}^{n}a^{-m}\sin(\pi j/b^m)e^{\pi i(2k+j)/b^m}\\&=2ia^n\sum_{m=1}^{n}u_m^{(j)}(a,b)\zeta_{m,k}^{(j)}(b),\end{aligned}$$

where

$$u_m^{(j)}(a,b)=a^{-m}\sin(\pi j/b^m),\qquad \zeta_{m,k}^{(j)}(b)=e^{\pi i(2k+j)/b^m}.$$

Note that $u_m^{(j)}(a,b)\in\mathbf{R}$, $\zeta_{m,k}^{(j)}(b)\in\mathbf{S}^1$. Moreover,

$$z_{n,k}=z_{n+n_0,b^{n_0}k}$$

and

$$\zeta_{m,b^{n_0}k}^{(j)}(b)=e^{\pi i(2b^{n_0}k+j)/b^m}=e^{\pi i j/b^m}e^{2\pi i k b^{n_0-m}}$$

for every $n_0\geq 0$. Thus,

$$\zeta_{m,b^{n_0}k}^{(j)}(b)=e^{\pi i j/b^m}\qquad\text{for } m\leq n_0$$

and

$$\begin{aligned}F_{a,b}(z_{n+n_0,b^{n_0}k+j})-F_{a,b}(z_{n,k})&=2ia^{n+n_0}\sum_{m=1}^{n+n_0}u_m^{(j)}(a,b)\zeta_{m,b^{n_0}k}^{(j)}(b)\\&=2ia^{n+n_0}\sum_{m=1}^{n_0}u_m^{(j)}(a,b)e^{\pi i j/b^m}+2ia^{n+n_0}\sum_{m=n_0+1}^{n+n_0}u_m^{(j)}(a,b)\zeta_{m,b^{n_0}k}^{(j)}(b)\\&=2ia^{n+n_0}\Biggl(\sum_{m=1}^{\infty}u_m^{(j)}(a,b)e^{\pi i j/b^m}-\sum_{m=n_0+1}^{\infty}u_m^{(j)}(a,b)e^{\pi i j/b^m}\\&\quad+\sum_{m=n_0+1}^{n+n_0}u_m^{(j)}(a,b)\zeta_{m,b^{n_0}k}^{(j)}\Biggr)\\&=2ia^{n+n_0}\bigl(U^{(j)}(a,b)+\Delta_{n,k,n_0}^{(j)}(a,b)\bigr),\end{aligned}\tag{1}$$

where

$$U^{(j)}(a,b)=\sum_{m=1}^{\infty}u_m^{(j)}(a,b)e^{\pi ij/b^m}.$$

Note that

$$(2)\quad |\Delta_{n,k,n_0}^{(j)}(a,b)|\le 2\sum_{m=n_0+1}^{\infty}|u_m^{(j)}(a,b)|\le 2\pi j\sum_{m=n_0+1}^{\infty}(ab)^{-m}=\frac{2\pi j}{ab-1}(ab)^{-n_0}.$$

Let

$$U^{(j)}(b)=\sum_{m=1}^{\infty}\sin(\pi j/b^m)e^{\pi ij/b^m}.$$

Lemma 3.1. *For every $b\in\mathbf{N}$, $b\ge 2$, there exists an integer j_0 such that*

(1) $U^{(j_0)}(b)$, $U^{(-j_0)}(b)\ne 0$ *and* $\operatorname{Arg}\big(U^{(j_0)}(b)\big)\ne\operatorname{Arg}\big(U^{(-j_0)}(b)\big)$,

(2) *if b tends to ∞, then $U^{(\pm j_0)}(b)$ tend respectively to $U^{(\pm j_0)}\ne 0$ such that* $\operatorname{Arg}\big(U^{(j_0)}\big)\ne\operatorname{Arg}\big(U^{(-j_0)}\big)$.

Proof. By definition,

$$\operatorname{Re}\big(U^{(j)}(b)\big)=\frac{1}{2}\sum_{m=1}^{\infty}\sin(2\pi j/b^m),\qquad \operatorname{Im}\big(U^{(j)}(b)\big)=\sum_{m=1}^{\infty}\sin^2(\pi j/b^m).$$

Note that for every $j\ne 0$ we have $\operatorname{Im}\big(U^{(j)}(b)\big)>0$, so $U^{(j)}(b)\ne 0$ and $\operatorname{Arg}\big(U^{(j)}(b)\big)\in(0,\pi)$. Moreover,

$$\operatorname{Re}\big(U^{(-j)}(b)\big)=-\operatorname{Re}\big(U^{(j)}(b)\big),\qquad \operatorname{Im}\big(U^{(-j)}(b)\big)=\operatorname{Im}\big(U^{(j)}(b)\big),$$

so $\operatorname{Arg}\big(U^{(j)}(b)\big)\ne\operatorname{Arg}\big(U^{(-j)}(b)\big)$ if and only if $\operatorname{Re}\big(U^{(j)}(b)\big)\ne 0$.

Let

$$j_0=\begin{cases}1 & \text{for } b<4,\\ \left[\frac{1}{4}b\right] & \text{for } b\ge 4.\end{cases}$$

Then $0<2\pi j_0/b^m\le\pi$ for all $b\ge 2$, $m\ge 1$ and the equality holds only if $b=2$, $m=1$. This implies $\operatorname{Re}\big(U^{(j_0)}(b)\big)>0$, so $\operatorname{Arg}\big(U^{(j_0)}(b)\big)\ne\operatorname{Arg}\big(U^{(-j_0)}(b)\big)$.

Note that

$$(3)\qquad 0<j_0\le\tfrac{1}{2}b.$$

Using this, we obtain

$$|U^{(\pm j_0)}(b)-\sin(\pm\pi j_0/b)e^{\pm\pi ij_0/b}|\le\sum_{m=2}^{\infty}\frac{\pi j_0}{b^m}=\frac{\pi j_0}{b(b-1)}\le\frac{\pi}{2(b-1)},$$

which tends to 0 as b tends to ∞, so $U^{(\pm j_0)}(b)$ tends to

$$U^{(\pm j_0)}=\lim_{b\to\infty}\sin\big(\pm\pi\left[\tfrac{1}{4}b\right]/b\big)e^{\pm\pi i[b/4]/b}=\tfrac{1}{2}(\pm 1+i).$$

Lemma 3.2. *Let $j_0=j_0(b)$ be the number defined in the proof of Lemma 3.1. If a tends to 1, then $U^{(\pm j_0)}(a,b)$ tend respectively to $U^{(\pm j_0)}(b)$ uniformly with respect to $b\ge 2$.*

Proof. Recall that

$$u_m^{(\pm j_0)}(a,b) = a^{-m}\sin(\pm\pi j_0/b^m) \underset{a\to 1}{\longrightarrow} \pm\sin(\pi j_0/b^m),$$

so it is sufficient to show that the series

$$\sum_{m=1}^{\infty} u_m^{(\pm j_0)}(a,b)e^{\pm\pi i j_0/b^m}$$

are convergent uniformly with respect to $b \ge 2$ and $a \in [a_1, 1)$ for some $a_1 < 1$. To check this, it is enough to notice that by (3), we have

$$|u_m^{(\pm j_0)}(a,b)e^{\pm\pi i j_0/b^m}| = a^{-m}\sin\left(\frac{\pi j_0}{b^m}\right) \le \frac{\pi}{2a}(ab)^{1-m} \le \pi\left(\frac{2}{3}\right)^m$$

for every $a \in \left[\frac{3}{4}, 1\right)$.

The proof of Theorem 1.1 is based on the following elementary planar geometric property.

Lemma 3.3. *Let A, B, C be three non-collinear points in the plane. Then there exist a point P in the interior of the triangle ABC and constants $\varepsilon, c > 0$, such that for every $q < 1$ sufficiently close to 1 there exists $r > c/(1-q)$ such that*

$$\mathbf{D}_r(P) \subset \mathbf{D}_{qr}(\tilde{A}) \cup \mathbf{D}_{qr}(\widetilde{B}) \cup \mathbf{D}_{qr}(\widetilde{C})$$

for every $\tilde{A} \in \mathbf{D}_\varepsilon(A)$, $\widetilde{B} \in \mathbf{D}_\varepsilon(B)$, $\widetilde{C} \in \mathbf{D}_\varepsilon(C)$.

Proof. Let P be the unique point in the interior of the triangle ABC, such that $\measuredangle APB = \measuredangle BPC = \measuredangle CPA = \frac{2}{3}\pi$. For $Z = A, B, C$ denote by S_Z the closed angle of measure $\frac{2}{3}\pi$ and vertex P, symmetric with respect to the line PZ and containing Z. Then

$$\mathbf{D}_r(P) = \big(\mathbf{D}_r(P) \cap S_A\big) \cup \big(\mathbf{D}_r(P) \cap S_B\big) \cup \big(\mathbf{D}_r(P) \cap S_C\big). \tag{4}$$

Take $r > |\overline{AP}|$. Let Q, Q' be the two points in $\partial\mathbf{D}_r(P)$, such that $\measuredangle APQ = \measuredangle APQ' = \frac{1}{3}\pi$ and let R be the point of intersection of the line AP with $\partial\mathbf{D}_r(P) \cap S_A$ (see Figure 2).

Then

$$\max\big\{\mathrm{dist}(A,Z) : Z \in \partial\big(\mathbf{D}_r(P) \cap S_A\big)\big\} = |\overline{AQ}|. \tag{5}$$

To see this, observe that $\max\big\{\mathrm{dist}(A,Z) : Z \in \overline{PQ}\big\}$ is achieved for $Z \in \{P, Q\}$. Moreover, it is easy to check that $\mathrm{dist}(A,Z)$ decreases as Z goes along

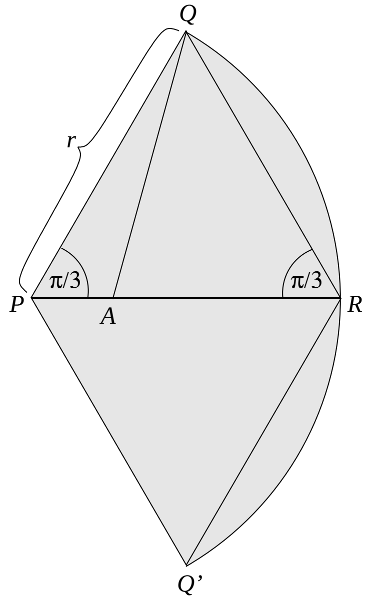

Figure 2. The set $\mathbf{D}_r(P) \cap S_A$.

$\partial \mathbf{D}_r(P) \cap S_A$ from Q to R. Since $\measuredangle PQA < \frac{1}{3}\pi = \measuredangle APQ$, we have $|\overline{AQ}| > |\overline{AP}|$. This shows (5). By (5) and the triangle inequality, if

$$qr > |\overline{AQ}| + \varepsilon, \tag{6}$$

then

$$\mathbf{D}_r(P) \cap S_A \subset \mathbf{D}_{qr}(\tilde{A})$$

for every $\tilde{A} \in \mathbf{D}_\varepsilon(A)$. Since

$$|\overline{AQ}| = \sqrt{r^2 - |\overline{AP}|r + |\overline{AP}|^2},$$

the condition (6) is equivalent to

$$(1 - q^2)r^2 - (|\overline{AP}| - 2\varepsilon q)r + |\overline{AP}|^2 - \varepsilon^2 < 0. \tag{7}$$

Solving the quadratic inequality, it is easy to check that if $\varepsilon > 0$ is sufficiently small and q is sufficiently close to 1, then (7) holds for $r \in \left[c'_A/(1-q), c_A/(1-q)\right]$, where $c_A > 0$ depends only on $|\overline{AP}|$ and $c'_A > 0$ is arbitrarily small if ε and $1-q$ are sufficiently small. Replacing A by B and C and repeating the above arguments, we obtain by (4)

$$\mathbf{D}_r(P) \subset \mathbf{D}_{qr}(\tilde{A}) \cup \mathbf{D}_{qr}(\widetilde{B}) \cup \mathbf{D}_{qr}(\widetilde{C})$$

for every $\tilde{A} \in \mathbf{D}_\varepsilon(A)$, $\widetilde{B} \in \mathbf{D}_\varepsilon(B)$, $\widetilde{C} \in \mathbf{D}_\varepsilon(C)$ and

$$r \in \left[\max(c'_A, c'_B, c'_C)/(1-q), \min(c_A, c_B, c_C)/(1-q)\right]$$

(if ε is sufficiently small and q is sufficiently close to 1). Hence, the lemma holds for $c = \min(c_A, c_B, c_C)/2$ and $r = 2c/(1-q)$. $\square$

Now we can prove the main lemma which is used in the proof of Theorem 1.1.

Lemma 3.4. *There exist $a_0 < 1$, $n_0 > 0$ and $c > 0$, such that for every $a \in [a_0, 1)$ and every $b \in \mathbf{N}$, $b \geq 2$ there exist $z_0 \in \mathbf{C}$ and $\varrho > c/(1-a)$, such that for every $n \geq 0$ and $k \in \{1, \ldots, b^n\}$,*

$$\mathbf{D}_{\varrho a^n}\big(F_{a,b}(z_{n,k}) + z_0 a^n\big) \subset \bigcup_{l=b^{n_0}(k-1)}^{b^{n_0}(k+1)} \mathbf{D}_{\varrho a^{n+n_0}}\big(F_{a,b}(z_{n+n_0,l})\big).$$

Proof. Let j_0 be the number defined in the proof of Lemma 3.1. Take $b \geq 2$ and define $A, B, C \in \mathbf{C}$ setting

$$A = 0, \qquad B = U^{(j_0)}(b), \qquad C = U^{(-j_0)}(b).$$

By Lemma 3.1, the points A, B, C are not collinear. For $a < 1$, $n \geq 0$, $k \in \{1, \ldots, b^n\}$ and $n_0 > 0$ let

$$\begin{aligned}
\tilde{A} &= 0,\\
\widetilde{B} &= a^{n_0}\big(U^{(j_0)}(a,b) + \Delta^{(j_0)}_{n,k,n_0}(a,b)\big),\\
\widetilde{C} &= a^{n_0}\big(U^{(-j_0)}(a,b) + \Delta^{(-j_0)}_{n,k,n_0}(a,b)\big).
\end{aligned}$$

Take a small $\varepsilon > 0$. By (2) and (3) we obtain

$$a^{n_0}|\Delta^{(\pm j_0)}_{n,k,n_0}(a,b)| < \pi\frac{1}{a - 1/b}(ab)^{-n_0} \leq 4\pi\big(\tfrac{2}{3}\big)^{n_0}$$

for every $a \in \big[\frac{3}{4}, 1\big)$, every $b \geq 2$, every n_0 and every k. Fix n_0 such that

$$4\pi\big(\tfrac{2}{3}\big)^{n_0} < \tfrac{1}{3}\varepsilon.$$

By Lemma 3.2, there exists $\frac{3}{4} < \tilde{a}_0 < 1$ such that for every $a \in [\tilde{a}_0, 1)$ and every $b \geq 2$,

$$|a^{n_0}U^{(\pm j_0)}(a,b) - U^{(\pm j_0)}(b)| < \tfrac{1}{3}\varepsilon. \tag{8}$$

This implies $|\widetilde{B} - B|, |\widetilde{C} - C| < \varepsilon$. Apply Lemma 3.3 for the points A, B, C, $\tilde{A}$, $\widetilde{B}$, $\widetilde{C}$. By this lemma, there exist a point $P \in \mathbf{C}$ and constants $c_b > 0$, $q_0 < 1$ (depending only on b), such that for every $q \in [q_0, 1)$ there exists $r > c_b/(1-q)$ such that

$$\mathbf{D}_r(P) \subset \mathbf{D}_{qr}(\tilde{A}) \cup \mathbf{D}_{qr}(\widetilde{B}) \cup \mathbf{D}_{qr}(\widetilde{C}) \tag{9}$$

Take $a_0(b) < 1$ such that $a_0(b) > \tilde{a}_0$ and $\big(a_0(b)\big)^{n_0} > q_0$. Let

$$z_0 = 2iP.$$

Take $a \in \big[a_0(b), 1\big)$ and $q = a^{n_0}$. Then $q \geq \big(a_0(b)\big)^{n_0} > q_0$, so by (9) and (1), we have

$$\begin{aligned}\mathbf{D}_r(z_0/(2i)) \subset \mathbf{D}_{ra^{n_0}}(0) & \\ \cup \mathbf{D}_{ra^{n_0}} & \left(\frac{F_{a,b}(z_{n+n_0, b^{n_0}k+j_0}) - F_{a,b}(z_{n,k})}{2ia^n}\right) \\ \cup \mathbf{D}_{ra^{n_0}} & \left(\frac{F_{a,b}(z_{n+n_0, b^{n_0}k-j_0}) - F_{a,b}(z_{n,k})}{2ia^n}\right),\end{aligned}$$

where

$$r > \frac{c_b}{1-a^{n_0}} > \frac{c_b}{n_0(1-a)}. \tag{10}$$

Multiplying by $2ia^n$ and adding $F_{a,b}(z_{n,k})$ we obtain

$$\begin{aligned}\mathbf{D}_{2ra^n}(F_{a,b}(z_{n,k}) + z_0 a^n) \subset & \ \mathbf{D}_{2ra^{n+n_0}}\big(F_{a,b}(z_{n,k})\big) \\ & \cup \mathbf{D}_{2ra^{n+n_0}}\big(F_{a,b}(z_{n+n_0, b^{n_0}k+j_0})\big) \\ & \cup \mathbf{D}_{2ra^{n+n_0}}\big(F_{a,b}(z_{n+n_0, b^{n_0}k-j_0})\big).\end{aligned} \tag{11}$$

In this way we have proved the lemma with a_0, c and r depending on b. To get the independence of b, let

$$A = 0, \qquad B = U^{(j_0)}, \qquad C = U^{(-j_0)}$$

and define $\tilde{A}$, $\widetilde{B}$, $\widetilde{C}$, n_0 and $\tilde{a}_0$ as previously. Note that by Lemma 3.1, for given $\varepsilon > 0$ there exists b_0 such that for $b > b_0$ we have

$$|U^{(\pm j_0)}(b) - U^{(\pm j_0)}| < \tfrac{1}{3}\varepsilon.$$

Using this together with (8), we have

$$|a^{n_0} U^{(\pm j_0)}(a,b) - U^{(\pm j_0)}| < \tfrac{2}{3}\varepsilon$$

for $a \in [\tilde{a}_0, 1)$ and $b > b_0$. Repeating the previous arguments, we show that there exist $a_0(\infty) < 1$, $c_\infty > 0$, $z_0 \in \mathbf{C}$ and $r > 0$ such that

$$r > \frac{c_\infty}{n_0(1-a)} \tag{12}$$

and (11) holds for every $a \in [a_0(\infty), 1)$ and every $b > b_0$.

Define

$$a_0 = \max\big(a_0(2), \ldots, a_0(b_0), a_0(\infty)\big), \qquad c = \frac{\min(c_2, \ldots, c_{b_0}, c_\infty)}{2n_0}.$$

Take $a \in [a_0, 1)$, $b \geq 2$ and let $\varrho = 2r$ for r from (11). Then by (10) and (12) we have $\varrho > c/(1-a)$ and

$$\begin{aligned}\mathbf{D}_{\varrho a^n}(F_{a,b}(z_{n,k}) + z_0 a^n) \subset \mathbf{D}_{\varrho a^{n+n_0}}&\big(F_{a,b}(z_{n,k})\big)\\ \cup\, \mathbf{D}_{\varrho a^{n+n_0}}&\big(F_{a,b}(z_{n+n_0, b^{n_0}k+j_0})\big)\\ \cup\, \mathbf{D}_{\varrho a^{n+n_0}}&\big(F_{a,b}(z_{n+n_0, b^{n_0}k-j_0})\big).\end{aligned}$$

By (3), this implies

$$\mathbf{D}_{\varrho a^n}\big(F_{a,b}(z_{n,k}) + z_0 a^n\big) \subset \bigcup_{l=b^{n_0}(k-1)}^{b^{n_0}(k+1)} \mathbf{D}_{\varrho a^{n+n_0}}\big(F_{a,b}(z_{n+n_0,l})\big). \ \square$$

Remark. In fact, the symmetry between B and C gives $z_0 \in \mathbf{R}$, $z_0 < 0$.

Proof of Theorem 1.1. Let

$$A_n(\varrho, p, q) = \bigcup_{l=p}^{q} \mathbf{D}_{\varrho a^n}(F_{a,b}(z_{n,l})).$$

Take $n \geq 0$, $k \in \{1, \ldots, b^n\}$ and $m > 0$. Applying Lemma 3.4 a number of times we obtain

$$\begin{aligned}&\mathbf{D}_{\varrho a^n}\big(F_{a,b}(z_{n,k}) + z_0 a^n\big) \subset A_{n+n_0}(\varrho, b^{n_0}k - b^{n_0}, b^{n_0}k + b^{n_0}),\\ &\quad \mathbf{D}_{\varrho a^n}\big(F_{a,b}(z_{n,k}) + z_0 a^n + z_0 a^{n+n_0}\big)\\ &\quad \subset A_{n+2n_0}(\varrho, b^{2n_0}k - b^{2n_0} - b^{n_0}, b^{2n_0}k + b^{2n_0} + b^{n_0}),\\ &\quad \ldots\\ &\mathbf{D}_{\varrho a^n}\left(F_{a,b}(z_{n,k}) + z_0 a^n \frac{1-a^{mn_0}}{1-a^{n_0}}\right)\\ &\quad \subset A_{n+mn_0}\left(\varrho, b^{mn_0}k - b^{n_0}\frac{b^{mn_0}-1}{b^{n_0}-1}, b^{mn_0}k + b^{n_0}\frac{b^{mn_0}-1}{b^{n_0}-1}\right)\\ &\quad \subset A_{n+mn_0}\big(\varrho, b^{mn_0}(k-2), b^{mn_0}(k+2)\big)\end{aligned}$$

for every $a \in [a_0, 1)$, $b \geq 2$ and suitable $z_0 \in \mathbf{C}$, $\varrho > c/(1-a)$. Hence,

$$\mathbf{D}_{\varrho a^n/2}\left(F_{a,b}(z_{n,k}) + \frac{z_0 a^n}{1-a^{n_0}}\right) \subset \bigcap_{m=m_0}^{\infty} A_{n+mn_0}\big(\varrho, b^{mn_0}(k-2), b^{mn_0}(k+2)\big)$$

for sufficiently large m_0. This means

$$\mathbf{D}_{\varrho a^n/2}\left(F_{a,b}(z_{n,k})+\frac{z_0a^n}{1-a^{n_0}}\right)\subset\bigcap_{m=m_0}^{\infty}\ \bigcup_{t\in[(k-2)/b^n,(k+2)/b^n]}\mathbf{D}_{\varrho a^{mn_0}}\big(F_{a,b}(e^{2\pi it})\big).$$

By the compactness of $F_{a,b}(\mathbf{S}^1)$, we have

$$\text{(13)}\quad \mathbf{D}_{\varrho a^n/2}\left(F_{a,b}(z_{n,k})+\frac{z_0a^n}{1-a^{n_0}}\right)\subset F_{a,b}\big(\{e^{2\pi it}:t\in\big[(k-2)/b^n,(k+2)/b^n\big]\}\big),$$

which easily implies both parts of the theorem. □

4. Corollaries

Corollary 4.1. *For every $a\in[a_0,1)$ and every $b\in\mathbf{N}$, $b\geq 2$,*

$$\underline{\mathrm{BD}}\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)=\overline{\mathrm{BD}}\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)=3-2\alpha.$$

Moreover, $\mathscr{H}^2\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)>0$, so $\mathrm{HD}\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)\geq 2$.

Proof. Consider the first part of the corollary. Since the map $(X_{a,b},Y_{a,b})$ is Hölder continuous with exponent α, we have by Lemma 2.2,

$$\overline{\mathrm{BD}}\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)\leq 3-2\alpha,$$

so it is sufficient to show the opposite inequality. Take $\varepsilon>0$. Let n be the maximal number for which $2\pi/b^n>\varepsilon$ and let

$$I_k=\left[\frac{2\pi 2k}{b^n},\frac{2\pi(2k+1)}{b^n}\right]\qquad\text{for } k=0,\dots,\left[\frac{b^n}{2}-1\right].$$

Then $|I_k|>\varepsilon$ and $\mathrm{dist}(I_{k_1},I_{k_2})>\varepsilon$ for $k_1\neq k_2$. By (13), the set $(X_{a,b},Y_{a,b})(I_k)$ contains a disc of diameter ca^n for a constant $c>0$ independent of k, n. Hence, to cover $\mathrm{graph}(X_{a,b},Y_{a,b})|_{I_k}$ we need at least $c^2a^{2n}\varepsilon^{-2}$ balls of diameter ε with non-empty intersections with $\mathrm{graph}(X_{a,b},Y_{a,b})|_{I_k}$. Since for $k_1\neq k_2$ we have $\mathrm{dist}(I_{k_1},I_{k_2})>\varepsilon$, such balls for k_1 and k_2 are disjoint. Hence,

$$N_\varepsilon\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)\geq c_1a^{2n}\varepsilon^{-3}=c_1b^{-2\alpha n}\varepsilon^{-3}\geq c_2\varepsilon^{2\alpha-3}$$

for some constants $c_1,c_2>0$. This implies $\underline{\mathrm{BD}}\big(\mathrm{graph}(X_{a,b},Y_{a,b})\big)\geq 3-2\alpha$.

To prove the second part, note that $F_{a,b}(\mathbf{S}^1)$ is the orthogonal projection of $\mathrm{graph}(X_{a,b},Y_{a,b})$. Since the projection is a Lipschitz map, it is enough to use Theorem 1.1 and Lemma 2.2 for $\beta=1$.

The next corollary shows that for the functions $X_{a,b}$, $Y_{a,b}$ we can improve the general estimates from Lemma 2.2.

Corollary 4.2. *For every $a \in [a_0, 1)$ and every $b \in \mathbf{N}$, $b \geq 2$ there exist $U_{X_{a,b}}, U_{Y_{a,b}} \subset \mathbf{R}$, such that $U_{X_{a,b}}$ (or $U_{Y_{a,b}}$) is open and dense in $X_{a,b}([0,2\pi])$ (or $Y_{a,b}([0,2\pi])$) and for every $\delta > 0$,*

$$\text{if } \mathscr{H}^{\delta}(A) > 0, \quad \text{then } \mathscr{H}^{\alpha(\delta+1)}\big(X_{a,b}^{-1}(A)\big) > 0 \text{ for every set } A \subset U_{X_{a,b}},$$
$$\text{if } \mathscr{H}^{\delta}(A) > 0, \quad \text{then } \mathscr{H}^{\alpha(\delta+1)}\big(Y_{a,b}^{-1}(A)\big) > 0 \text{ for every set } A \subset U_{Y_{a,b}}.$$

In particular,

$$\mathrm{HD}\big(X_{a,b}^{-1}(A)\big) \geq \alpha\big(\mathrm{HD}(A) + 1\big) \quad \text{for every set } A \subset U_{X_{a,b}},$$
$$\mathrm{HD}\big(Y_{a,b}^{-1}(A)\big) \geq \alpha(\mathrm{HD}(A) + 1) \quad \text{for every set } A \subset U_{Y_{a,b}}.$$

Proof. Let $U_{X_{a,b}}$ (or $U_{Y_{a,b}}$) be the orthogonal projection of $\operatorname{int} F_{a,b}(\mathbf{S}^1)$ on the real (or imaginary) axis. By Theorem 1.1, $U_{X_{a,b}}$ (or $U_{Y_{a,b}}$) is open and dense in $X_{a,b}([0,2\pi])$ (or $Y_{a,b}([0,2\pi])$). Take $A \subset U_{X_{a,b}}$ such that $\mathscr{H}^{\delta}(A) > 0$. By definition, for every $s \in A$ the set

$$\big(\{s\} \times \mathbf{R}\big) \cap (X_{a,b}, Y_{a,b})\big([0,2\pi]\big)$$

contains a non-trivial interval. Hence, by Theorem 2.3, we have

$$\mathscr{H}^{\delta+1}\big((A \times \mathbf{R}\big) \cap (X_{a,b}, Y_{a,b})\big([0,2\pi]\big)\big) > 0.$$

Since the map $(X_{a,b}, Y_{a,b})$ is Hölder with exponent α, we have by Lemma 2.2,

$$\mathscr{H}^{\alpha(\delta+1)}\big((X_{a,b})^{-1}(A)\big) = \mathscr{H}^{\alpha(\delta+1)}\big((X_{a,b}, Y_{a,b})^{-1}(A \times \mathbf{R})\big) > 0.$$

The case $A \subset U_{Y_{a,b}}$ is symmetric.

Taking $A = \{s\}$ in Corollary 4.2, we obtain the following result on level sets $L_{X_{a,b}}(s)$, $L_{Y_{a,b}}(s)$.

Corollary 4.3. *For every $a \in [a_0, 1)$ and every $b \in \mathbf{N}$, $b \geq 2$,*

$$\mathscr{H}^{\alpha}\big(L_{X_{a,b}}(s)\big) > 0 \quad \text{for every } s \in U_{X_{a,b}},$$
$$\mathscr{H}^{\alpha}\big(L_{Y_{a,b}}(s)\big) > 0 \quad \text{for every } s \in U_{Y_{a,b}}.$$

In particular,

$$\mathrm{HD}\big(L_{X_{a,b}}(s)\big) \geq \alpha \quad \text{for every } s \in U_{X_{a,b}},$$
$$\mathrm{HD}\big(L_{Y_{a,b}}(s)\big) \geq \alpha \quad \text{for every } s \in U_{Y_{a,b}}.$$

Moreover,

$$\operatorname{int}_{\mathbf{R}}\big(Y_{a,b}\big(L_{X_{a,b}}(s)\big)\big) \neq \emptyset \quad \text{for every } s \in U_{X_{a,b}},$$
$$\operatorname{int}_{\mathbf{R}}\big(X_{a,b}\big(L_{Y_{a,b}}(s)\big)\big) \neq \emptyset \quad \text{for every } s \in U_{Y_{a,b}}.$$

By Corollary 4.3 and Theorem 2.3, we get

Corollary 4.4. *For every* $a \in [a_0, 1)$ *and every* $b \in \mathbf{N}$, $b \geq 2$,

$$\mathscr{H}^{1+\alpha}(\operatorname{graph} X_{a,b}), \mathscr{H}^{1+\alpha}(\operatorname{graph} Y_{a,b}) > 0.$$

In particular, $\mathrm{HD}(\operatorname{graph} X_{a,b}), \mathrm{HD}(\operatorname{graph} Y_{a,b}) \geq 1 + \alpha$.

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