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# LIMIT FUNCTIONS IN WANDERING DOMAINS OF MEROMORPHIC FUNCTIONS

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**Abstract.** Let $f$ be a function which is meromorphic outside a sufficiently small, non-empty, totally-disconnected compact set of essential singularities, and let $U$ be a wandering component of the Fatou set of $f$. We prove that any limit function of a subsequence of iterates of $f$ in $U$ is a constant which lies in the derived set of the forward orbit of the set of singular points of the inverse $f$. This extends results of Bergweiler et al. and Zheng about transcendental entire or meromorphic functions in $\mathbf{C}$.

## 1. Introduction

Let $g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be a transcendental meromorphic function and $g^n$ the $n$-th iterate of $g$, $n = 0, 1, 2, \ldots$ (where $g^0 = \mathrm{Id}$). The Fatou set $F(g) = \{z : \{g^n\}$ is meromorphic and normal in some neighbourhood of $z\}$ and the Julia set $J(g)$ is $\widehat{\mathbf{C}} \setminus F(g)$. See [2] for the basic results about meromorphic iteration.

It is sometimes inconvenient that the iterates of a function such as $g$ are not in general meromorphic. In [1] a more general theory is developed without this disadvantage. For a set $A$, let $A^c = \widehat{\mathbf{C}} \setminus A$. We consider the class $M$ of functions $f$ for which there is a compact, totally-disconnected set $E = E(f) \subset \widehat{\mathbf{C}}$ such that $f$ is meromorphic in $E^c$, and for each $z_0 \in E$ the cluster set $C(f, E^c, z_0) = \{w : w = \lim_{n\to\infty} f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$ is $\widehat{\mathbf{C}}$. If $E = \emptyset$ we make the further assumption that $f$ is neither constant nor univalent in $\widehat{\mathbf{C}}$.

For technical reasons we introduce a subset $MA$ of $M$. We say that $f \in M$ has the $k$-island property at $z_0$ in $E(f)$ if, given any neighbourhood $U$ of $z_0$ and $k$ simply-connected domains $\Delta_i$ in $\widehat{\mathbf{C}}$ which have disjoint closures and which are bounded by sectionally analytic Jordan curves, there is a simply-connected subdomain $D$ in $U \setminus E(f)$ which maps univalently under $f$ onto one of the $\Delta_i$. $MA_k$ is the set of $f \in M$ such that $E(f) \neq \emptyset$ and for each $z_0 \in E(f)$ the function $f$ has the $k$-island property at $z_0$. $MA$ is the union of all $MA_k$, $k \in \mathbf{N}$.

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A simpler but smaller class of functions introduced by A. Bolsch [4] is $K = \{f \in M : E(f) \text{ is compact and countable}\}$. We have $K \subset MA_k \subset M$, and all three classes are closed under composition of functions. Thus $K$ includes any function of the type $f_1 \circ f_2 \circ \cdots \circ f_n$, where $f_i$ are meromorphic in $\mathbf{C}$.

It is shown in [1] that applying the definitions given above for $F(g)$, $J(g)$ to functions $f$ in $M$ yields a completely invariant $F(f)$ and a perfect set $J(f)$. Moreover $f \in M$ implies that $f^m \in M$ and $F(f^m) = F(f)$. If, in addition, $f \in MA$, then repelling cycles are dense in $J(f)$.

If $U$ is a component of $F(f)$, $f \in M$, then for each $k \in \mathbf{N}$, $f^k(U)$ is contained in some component $U_k$ of $F(f)$. If the sequence $U_k$ is eventually periodic we have only to understand the behaviour of iterates in some periodic component, indeed by replacing $f$ by an iterate we could study only invariant components. Here only a few cases arise and are reasonably well understood (see below). The other case is when all $U_k$ are different and $U$ is said to be a wandering component. In such a component any convergent sequence of iterates has a limit function which is a constant in $J(f)$ (see [1]).

The possible constant limits are connected with the singular values of the inverse $f^{-1}$. If $f \in M$, the set $S(f)$ of singular values of some branch of $f^{-1}$ consists of the critical values $f(c)$, where $f'(c) = 0$, together with the set of all asymptotic values of $f$: $w$ is an asymptotic value of $f$ if there is some $z_0 \in E(f)$ and a path $\gamma(t)$, $0 \le t < 1$, in $E(f)^c$ such that $\gamma(t) \to z_0$ and $f\big(\gamma(t)\big) \to w$ as $t \to 1$. Further, $E_j(f) = \bigcup_{k=0}^{j-1} f^{-k}\big(E(f)\big)$ is the set of essential singularities of $f^j$, and the set where for some $n \in \mathbf{N}$ some branch of $f^{-n}$ has a singularity is

$$P(f) = \bigcup_{j=0}^{\infty} f^j\big(S(f) \setminus E_j(f)\big), \qquad \text{where } E_0(f) = \emptyset.$$

Thus $P(f)$ consists of the forward orbit of $S(f)$, so far as this is defined. For a set $A$, the derived set of $A$ is denoted by $A'$.

**Theorem 1.** *If $f \in MA$ and $U$ is a wandering component of $F(f)$, then any limit function of a sequence of iterates in $U$ is a constant which lies in $P(f)'$.*

**Remarks.** $MA$ includes $K$ and in particular any (non-constant, non-univalent) function meromorphic in $\mathbf{C}$. Rational functions have no wandering components but transcendental ones may do so. The result was proved for transcendental entire functions by Bergweiler et al. [3]. Zheng [9] extended it to transcendental meromorphic functions but with an additional hypothesis. In [10] he has removed this restriction.

The theorem can be used to prove the non-existence of wandering domains given suitable information about $S(f)$ and its forward orbit.

Recall that a fixed point $a$ of $f$ satisfies $f(a) = a$. If $a$ is finite the multiplier is $\lambda(a) = f'(a)$. If $a$ is $\infty$ we define the multiplier by conjugating $f$ so that $a$

becomes finite. The fixed point is attracting if $|\lambda(a)| < 1$ and parabolic if $\lambda(a)$ is a primitive $p$-th root of unity for some $p \in \mathbf{N}$. Concerning the behaviour of iterates in a non-wandering component it is enough to know the following.

If $f \in M$ and $U$ is a component of $F(f)$ such that $f(U) \subset U$, then precisely one of the following is true; see [1].

(i) *$U$ contains an attracting fixed point $a$. For $z \in U$ we have $f^n(z) \to a$ as $n \to \infty$. $U$ is called the immediate attractive basin of $a$.*

(ii) *$U$ is a domain of attraction of a parabolic fixed point $a \in \partial U$ and for $z \in U$ we have $f^n(z) \to a$ as $n \to \infty$.*

(iii) *There is an analytic homeomorphism $\psi\colon U \to D$, where $D$ is the unit disc, such that $\psi\big(f\big(\psi^{-1}(z)\big)\big) = e^{2\pi i\alpha}z$, for some $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. In this case $U$ is called a Siegel disc.*

(iv) *There is an analytic homeomorphism $\psi\colon U \to A$ where $A$ is an annulus $A = \{z : 1 < |z| < r\}$ such that $\psi\big(f\big(\psi^{-1}(z)\big)\big) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. In this case $U$ is called a Herman ring.*

(v) *There exists $a \in \partial U$ such that $f^n(z) \to a$, for $z \in U$ as $n \to \infty$, but $f$ is not meromorphic at $a$. There exists a path $\gamma \subset U$ leading to $a$ such that $f(\gamma) \subset \gamma$ and the spherical distance between $f(z)$ and $z$ tends to zero as $z \to a$ on $\gamma$. In this case $U$ is called essentially parabolic at $a$ or alternatively a 'Baker domain'.*

We may note that if $a$ is a parabolic fixed point whose multiplier is a primitive $p$-th root of unity, then it has multiplier 1 as a fixed point of $f^p$. In fact there are then $kp$ domains $U_i$ of $F(f)$ (for some $k \in \mathbf{N}$), each invariant under $f^p$, in which $f^n \to a$. We can now state the following.

**Theorem 2.** *If $f \in MA$ and $U$ is a component of $F(f)$ such that $f^n \to a$ in $U$, where $a \in \widehat{\mathbf{C}}$, then either* (i) *$a$ is an attracting fixed point of $f$,* (ii) *$a$ is a parabolic fixed point of $f$, or* (iii) *$a \in E(f) \cap S(f)'$.*

Thus the only cases where $f$ is meromorphic at $a$ are (i) and (ii) which correspond to the first two non-wandering cases described above. In (iii) $U$ is either a wandering domain or a Baker domain. The idea of the proof goes back to Eremenko and Lyubich [5] who deal with the case where $f$ is transcendental entire and $S(f)$ is bounded ($E(f) = \{\infty\}$). Their work was taken up in the meromorphic case by Bergweiler [2], Rippon and Stallard [8] and Zheng [10], who proved Theorem 2 for functions meromorphic in $\mathbf{C}$ (by a somewhat different method). So far as Theorem 2 relates to the case of a Baker domain it is proved in [1, Theorem F].

An advantage of working in the class $MA$ or $K$ is that we can apply our results automatically to $f^p$ without special discussion of periodic cycles etc. Also we have no need to separate finite and infinite limits.

There are of course many examples in which case (iii) of Theorem 2 applies, for example entire functions with wandering components in which $f^n \to \infty$.

## 2. Proof of Theorem 1

Suppose that $f \in MA$ has a wandering component $U$ of $F(f)$ such that for some integers $n_k$ increasing to $\infty$ the sequence $f^{n_k} \to a$ in $U$, where $a$ is a constant which we may assume to be 0 (for $f$ may be conjugated by a Möbius transform). Recall that $0 \in J(f)$. We assume that there is a positive $\varepsilon$ such that the set $N = D(0,\varepsilon) \setminus \{0\}$ does not meet $P(f)$ and shall derive a contradiction.

**I.** Since periodic cycles are dense in $J(f)$ it follows that $f$ has some cycle $\{\alpha_1, \dots, \alpha_j\}$ of length $j \geq 3$ such that no $\alpha_i = 0$. The constant $\varepsilon$ will be chosen so that all $\alpha_i$ are outside $N$. For a disc $D = D(z_0, r)$ with $\overline{D} \subset U$ we have $D_k = f^{n_k}(D) \subset N$ for all sufficiently large $k$. We may replace $U$ by a suitable $f^j(U)$ and adjust the notation to get $D_k \subset N$ for all $k$ and $D \subset N$. We have $w_k = f^{n_k}(z_0) \to 0$ as $k \to \infty$.

Let $g_k$ be the branch of $f^{-n_k}$ such that $g_k(w_k) = z_0$. Fix a value of $u_k = \log(w_k)$, noting that $w_k \neq 0$ since $w_k \in F(f)$. Then $h_k(t) = g_k(\exp t)$ is analytic near $t = u_k$, $h_k(u_k) = z_0$, and $h_k$ continues throughout $H = \{t : \operatorname{Re} t < \log \varepsilon\}$ to give a single-valued analytic function in $H$. Further, $h_k$ takes none of the values $\alpha_i$ for $t \in H$. By Montel's theorem $q_k(v) = h_k\big(u_k + (\log\varepsilon - \operatorname{Re} u_k)v\big)$ is a normal family in $D(0,1)$, so by Marty's criterion the spherical derivatives $q_k^{\#}(0) = |q_k'(0)|/\big(1 + |q_k(0)|^2\big)$ are bounded by some constant $B$. Hence

$$\frac{(\log\varepsilon - \operatorname{Re} u_k)|w_k g_k'(w_k)|}{1 + |z_0|^2} \leq B,$$

so

$$|(f^{n_k})'(z_0)| \geq \frac{|f^{n_k}(z_0)|(\log\varepsilon - \operatorname{Re} u_k)}{B(1 + |z_0|^2)}, \tag{1}$$

where

$$\operatorname{Re} u_k = \log|w_k| = \log|f^{n_k}(z_0)|.$$

Suppose now that there are arbitrarily large $k$ such that every closed path in $D_k$ is null-homotopic in $N$. Then the analytic continuation of $g_k$ from $w_k$ within $D_k$ is single-valued and maps $D_k$ to $D$ (univalently). Thus $f^{n_k}$ maps $D$ to $D_k$ univalently. By Koebe's $\frac{1}{4}$-theorem $D_k$ contains a disc of centre $w_k$ and radius $\frac{1}{4}r|(f^{n_k})'(z_0)|$. Since zero is not in $D_k$ we have $\frac{1}{4}r|(f^{n_k})'(z_0)| < |w_k|$. Since $\operatorname{Re} u_k \to -\infty$ as $k \to \infty$, this conflicts with (1) for large $k$.

**II.** We conclude from the above that, for all large $k$, $D_k$ and the component $U_k$ of $F(f)$ with $U_k \supset D_k$ contain a simple closed curve $\gamma_k$ with $\gamma_k \not\sim 0$ in $N$. We may assume (by replacing $n_k$ by a subsequence) that for $l > k$ the component of $\gamma_k^c$ which contains zero also contains $U_l \supset D_l$. Fix such $k$, $l$ with $U_l$, $U_k$ in $N$. Write $\gamma = \gamma_k$, $m = n_l - n_k$ and take a point $z' \in \gamma$. Denote by $g$

the branch of $f^{-m}$ which maps $w' = f^m(z')$ to $z'$. Take a value $t'$ of $\log w'$. Then $h(t) = g(\exp t)$, where $h(t') = z'$, continues in $H$ to give a single-valued meromorphic function. As shown e.g. in [6, Chapter 11] either (a) $h$ is univalent in $H$ or (b) $h$ has period $2\pi ip$ for some minimal positive integer $p$. In case (b) $g(w^p)$ is univalent in $N' = \{w : 0 < |w| < \varepsilon^{1/p}\}$.

Now $f^m$ maps $\gamma$ to a closed path $\gamma'$ in $U_l$. In case (a) there is some simple closed path $\gamma''$ in $H$ such that $h\colon \gamma'' \to \gamma$ and we have $\gamma' = \exp\gamma''$, so $\gamma''$ is a lift of $\gamma'$. Hence $\gamma' \sim 0$ in $N$. If $\Delta$ denotes the interior of $\gamma''$, then $h(\Delta)$ is either the interior or exterior of $\gamma$, in fact the interior because values of $h(\Delta)$ belong to $g(\exp\Delta) \subset g(N)$ and so do not include the points $\alpha_i$. Thus in $\operatorname{Int}\gamma$, $h^{-1}$ is univalent with values in $\Delta$ and $f^m = \exp\circ h^{-1}$ is analytic with $\partial\big(f^m(\operatorname{Int}\gamma)\big) \subset f^m(\gamma) = \gamma'$ and $f^m(\operatorname{Int}\gamma) = \exp\Delta \subset N$. Thus $f^m$ maps $\operatorname{Int}\gamma$ into itself which implies that $\operatorname{Int}\gamma \subset F(f^m) = F(f)$. This contradicts $0 \in J(f)$.

It remains to discuss case (b). Let $\tilde\gamma$ be the simple curve in $N'$ which $G(w) = g(w^p)$ maps to $\gamma$. Since $G$ is univalent in $N'$, zero is a removable singularity. Assume that $G(0)$ has been defined so as to make $G$ analytic at zero. Then $G$ remains univalent in $N'' = N' \cup \{0\}$. Now $\tilde\Delta = \operatorname{Int}\tilde\gamma$ maps under $G$ into a subset of $g(N) \cup \{G(0)\}$ with boundary values in $\gamma$ and omits at least one of the points $\alpha_i$. Thus $G(\tilde\Delta) = \operatorname{Int}\gamma$ and $G^{-1}(\operatorname{Int}\gamma) = \tilde\Delta$. We obtain that $f^m = (G^{-1})^p$ is analytic in $\operatorname{Int}\gamma$ with values in $(\tilde\Delta)^p \subset N \cup \{0\}$ and boundary values in $(\tilde\gamma)^p = \gamma'$ in the interior of $\gamma$. Thus $f^m$ maps $\operatorname{Int}\gamma$ into itself and we obtain a contradiction as before. The proof is complete.

## 3. Proof of Theorem 2

Suppose that $f \in MA$ and $f^n \to a$ in the component $U$ of $F(f)$ as $n \to \infty$. We may suppose that $a = 0$. If $f$ is meromorphic at $0$ we have $f(0) = 0$ and $|f'(0)| \le 1$. If $f'(0) = e^{2\pi i\alpha}$, where $\alpha$ is real and irrational, then it is proved in [7] that an orbit $f^n(z) \to 0$ only if $f^n(z) = 0$, for some $n \in \mathbf{N}$. Since there must be an uncountable set of $z$ in $U$ for which $f^n(z) = 0$ with the same $n$ this implies that $f^n$ is identically zero, which is impossible. Thus we have cases (i) or (ii) of the theorem if $f$ is meromorphic at $0$, and otherwise $0 \in E(f)$.

Thus we assume that $0 \in E(f)$ and have to prove that $0 \in S(f)'$. Suppose on the contrary that there is some positive $\varepsilon$ such that $N = D(0,\varepsilon) \setminus \{0\}$ does not meet $S(f)$. As in the proof of Theorem 1 we may assume that there is a periodic cycle $\{\alpha_1, \dots, \alpha_j\}$ of length $j \ge 3$ outside $N$. By a suitable conjugation we may also assume that $\alpha_1 = \infty$.

By assumption there is some disc $D = D(z_0, r)$ such that $D$ and all $f^n(D)$ are in $N$ and $f^n \to 0$ uniformly in $D$ as $n \to \infty$. Let $z_n = f^n(z_0)$. For any branch $g$ of $f^{-1}$ analytic at some point $w'$ in $N$ and any choice $t'$ of $\log w'$, the function $h(t) = g(\exp t)$, where $h(t') = g(w')$, can be continued throughout $H = \{t : \operatorname{Re} t < \log\varepsilon\}$. Either (a) $h$ is univalent in $H$ or (b) $h$ has period $2\pi ip$

for some minimal $p \in \mathbf{N}$, and $g(w^p)$ is univalent in $\{0 < |w| < \varepsilon^{1/p}\}$ with a univalent extension $G$ to $D(0, \varepsilon^{1/p})$.

In case (a) $h(H) = W$, where $W$ is a simply-connected domain consisting of those values taken by the continuation of $g$ within $N$. Thus $W$ contains neither 0 (where $f$ has an essential singularity) nor $\infty = \alpha_1$, since $f(\alpha_1) = \alpha_2 \notin N$.

In case (b) the values of $g$ in $N$ form a subset of $W = G\bigl(\{|w| < \varepsilon^{1/p}\}\bigr)$. The set $W$ is a simply-connected domain bounded by the Jordan curve which is the image under $G$ of $\{|w| = \varepsilon^{1/p}\}$. Again $W$ contains neither 0 nor $\infty$. So in either case $h(H) \subset W$.

Take $g$ to be the branch of $f^{-1}$ such that $g(z_{n+1}) = z_n$ and define $h(t) = g(\exp t)$ as above, with $w' = z_{n+1}$ and any choice $t_{n+1}$ of $\log w'$. Then, taking a branch of log which is analytic in $W$, the function $\phi(t) = \log\bigl(h(t)\bigr) = \log\bigl(g(\exp t)\bigr)$ is analytic in $H$ and $\phi(H)$ contains no disc of radius exceeding $\pi$. Bloch's theorem states that if $\psi$ is analytic in $D(0,r)$, then $\psi\bigl(D(0,r)\bigr)$ contains a disc of radius $cr|\psi'(0)|$, where $c$ is some positive absolute constant. Thus

$$c(\log \varepsilon - \operatorname{Re} t_{n+1})|\phi'(t_{n+1})| \le \pi,$$

which yields

$$|f'(z_n)| \ge \frac{c|z_{n+1}|}{\pi |z_n|} \log\left(\frac{\varepsilon}{|z_{n+1}|}\right),$$

so

$$|(f^n)'(z_0)| \ge \left(\frac{c}{\pi}\right)^n \frac{|z_n|}{|z_0|} \prod_{j=1}^{n} \left(\log \frac{\varepsilon}{|z_j|}\right). \tag{2}$$

Note that each $D_n = f^n(D)$ lies in some domain of the type $W$ so that $\log f^n$ is analytic in $D$ and the image domain $\log f^n(D)$ contains no disc of radius greater than $\pi$. But this implies by Bloch's theorem that

$$rc\frac{|(f^n)'(z_0)|}{|f^n(z_0)|} \le \pi. \tag{3}$$

Since the product on the right in (2) tends to $\infty$ as $n \to \infty$, we have a contradiction between (2) and (3) if $n$ is sufficiently large.

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