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SOME CLASSES OF COMPLETELY MONOTONIC FUNCTIONS

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Abstract. We prove: (i) Let

$$F_n(x) = P_n(x)\left[e - \left(1+\frac{1}{x}\right)^x\right] \quad \text{and} \quad G_n(x) = P_n(x)\left[\left(1+\frac{1}{x}\right)^{x+1} - e\right],$$

where $P_n(x) = x^n + \sum_{\nu=0}^{n-1} c_\nu x^\nu$ is a polynomial of degree $n \geq 1$ with real coefficients. F_n is completely monotonic if and only if $n=1$ and $c_0 \geq 11/12$; and G_n is completely monotonic if and only if $n=1$ and $c_0 \geq 1/12$.

(ii) The functions $x \mapsto e-(1+1/x)^x$ and $x \mapsto (1+1/x)^{x+1}-e$ are Stieltjes transforms and in particular they are completely monotonic.

(iii) Let $a>0$ and b be real numbers. The function $x \mapsto (1+a/x)^{x+b} - e^a$ is completely monotonic if and only if $a \leq 2b$.

Part (i) extends and complements a recently published result of Sándor and Debnath, while part (iii) generalizes a theorem of Schur.

1. Introduction

A function $f\colon (0,\infty) \to \mathbf{R}$ is said to be completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \qquad \text{for all } x>0 \text{ and } n = 0,1,2,\ldots.$$

This definition was introduced in 1921 by F. Hausdorff [23], who called such functions 'total monoton'. Bernstein's theorem, cf. [35, p. 161], states that f is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x>0$.

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Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [8], probability theory [13], [20], [26], physics [16], numerical and asymptotic analysis [22], [36], and combinatorics [4]. A detailed collection of the most important properties of completely monotonic functions can be found in [35, Chapter IV], and in an abstract setting in [7].

In the recent past, various authors showed that numerous functions, which are defined in terms of gamma, polygamma, and other special functions, are completely monotonic and used this fact to derive many interesting new inequalities; cf. [2], [3], [6], [9], [10], [14], [15], [24], [25], [27], [28], [31]. An elegant majorization theorem for completely monotonic functions is given in [21]. In [28] the author provides a counterpart of the classical theorem of Bohr–Mollerup by characterizing the gamma function via the notion of complete monotonicity. Hankel's determinant inequality for completely monotonic functions is proved in [35, p. 167], and in [19] it is shown that in connection with an interpolation problem there exists a close relation between completely monotonic functions and completely monotonic sequences.

Several related classes of functions are studied in [21], [30], [34], [35]. Historical remarks on this subject can be found in [11].

This work has been motivated by an article of J. Sándor and L. Debnath [33], who proved that the function

$$f(x) = (x+1)\left[e - \left(1 + \frac{1}{x}\right)^x\right]$$

is decreasing and convex on $(0, \infty)$. It is tempting to ask for an extension of this result: is f completely monotonic? We prove that the answer is 'yes'. Actually, we investigate a more general problem. We are looking for real polynomials $P_n(x) = x^n + \sum_{\nu=0}^{n-1} c_\nu x^\nu$ $(n \in \mathbf{N})$ such that

$$x \mapsto P_n(x)\left[e - \left(1 + \frac{1}{x}\right)^x\right] \tag{1.1}$$

is completely monotonic. In view of the well-known inequalities

$$\left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1} \qquad (x > 0),$$

it is natural also to study the monotonicity behaviour of

$$x \mapsto P_n(x)\left[\left(1 + \frac{1}{x}\right)^{x+1} - e\right] \tag{1.2}$$

and its derivatives. In Section 3 we present all P_n such that the functions defined in (1.1) and (1.2), respectively, are completely monotonic. Moreover, we prove that the functions $x \mapsto e - (1+1/x)^x$ and $x \mapsto (1+1/x)^{x+1} - e$ are Stieltjes transforms, which implies that they are completely monotonic.

A theorem of I. Schur [29, pp. 30 and 186] states that the sequence $n \mapsto (1+1/n)^{n+b}$ $(n = 1, 2, \ldots)$ is decreasing if and only if $b \geq \frac{1}{2}$. This result leads to the following question: for which real parameters $a \geq 0$ and b is the function

$$x \mapsto \left(1 + \frac{a}{x}\right)^{x+b} - e^a$$

completely monotonic? In Section 3 we answer this question.

2. Lemmas

In order to prove our main results we need some lemmas, which we collect in this section. In our first lemma we present an integral representation, which plays a crucial role in the proof of the theorems given in Section 3.

Lemma 1. *Let*

$$f(x) = (x+1)\left[e - \left(1 + \frac{1}{x}\right)^x\right] \qquad (x > 0). \tag{2.1}$$

Then we have

$$f(x) = \frac{e}{2} + \frac{1}{\pi}\int_0^1 \frac{s^s(1-s)^{1-s}\sin(\pi s)}{x+s}\, ds. \tag{2.2}$$

Proof. In the Addenda and Problems in [1, p. 127], it is stated that if a function f is holomorphic in the cut plane $\mathscr{A} = \mathbf{C} \setminus (-\infty, 0]$, and satisfies $\operatorname{Im} f(z) \leq 0$ for $\operatorname{Im} z > 0$ and $f(x) \geq 0$ for $x > 0$, then f is a Stieltjes transform, that is, it has the representation

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x+t} \qquad (x > 0),$$

where $a \geq 0$ and μ is a nonnegative measure on $[0, \infty)$ with $\int_0^\infty d\mu(t)/(1+t) < \infty$. A proof is written out in [5]. The constant a is given by $a = \lim_{x\to\infty} f(x)$, and μ is the limit in the vague topology of measures

$$d\mu(t) = \lim_{y\to 0^+} -\frac{1}{\pi}\operatorname{Im} f(-t+iy)\, dt,$$

i.e.

$$\int \varphi\, d\mu = \lim_{y\to 0^+} -\frac{1}{\pi}\int \operatorname{Im} f(-t+iy)\varphi(t)\, dt,$$

for all continuous functions φ on $\mathbf{R}$ with compact support. Let

$$f(z) = (1+z)\big[e - \exp\big(z\mathrm{Log}\,(1+1/z)\big)\big] \qquad (z\in\mathscr{A}),$$

where Log denotes the principal branch of the logarithm. We have

$$f(1/z) = e(1+1/z)\left[1-\exp\left(\frac{\mathrm{Log}\,(1+z)}{z}-1\right)\right],$$

so that the series representation

$$\frac{\mathrm{Log}\,(1+z)}{z} - 1 = -\frac{z}{2}+\frac{z^2}{3}-+\cdots \qquad (|z|<1)$$

implies

$$\lim_{|z|\to\infty} f(z) = \frac{e}{2} \qquad (z\in\mathscr{A}). \tag{2.3}$$

To prove that the harmonic function $\operatorname{Im} f$ satisfies $\operatorname{Im} f(z)\le 0$ for $\operatorname{Im} z>0$, we use the maximum principle for subharmonic functions, cf. [18, p. 20], and show that the lim sup of $\operatorname{Im} f$ at all boundary points including infinity is less than or equal to 0. From (2.3) we conclude that this is true at infinity.

Let $t\in\mathbf{R}$ and $z\in\mathbf{C}$ with $\operatorname{Im} z>0$. If z tends to t, then

$$f(z)\to\begin{cases} f(t), & \text{if } t>0,\\ e-1, & \text{if } t=0,\\ (1+t)\left[e-\exp\left(t\log\dfrac{1+t}{|t|}-i\pi t\right)\right], & \text{if } -1<t<0,\\ 1, & \text{if } t=-1,\\ (1+t)\left[e-\exp\left(t\log\dfrac{1+t}{t}\right)\right], & \text{if } t<-1.\end{cases}$$

In particular, if y tends to 0^+, then we obtain for $t\in\mathbf{R}$:

$$-\frac{1}{\pi}\operatorname{Im} f(-t+iy)\to\begin{cases} 0, & \text{if } t\le 0 \text{ or } t\ge 1,\\ \dfrac{1}{\pi}(1-t)\exp\left(-t\log\left(\dfrac{1-t}{t}\right)\right)\sin(\pi t), & \text{if } 0<t<1.\end{cases}$$

The limit above is uniform for t in compact subsets of the real axis, and therefore the (continuous) density on the right is the vague limit of the densities on the left. The proof of Lemma 1 is complete. $\square$

Remarks. (1) To obtain formula (2.2) we can alternatively use the residue theorem applied to the function $z \mapsto f(z)/(z-z_0)$ $(z_0 \in \mathscr{A})$, where the path of integration is a simple closed contour given by two circles $|z| = \varepsilon$ and $|z| = R$ (where $\varepsilon < |z_0| < R$) modified by inserting two horizontal segments with $x \leq 0$, $y = \pm\varepsilon$ from the small circle to the big one and removing the left part of the small circle and a small portion of the big circle close to $-R$. If we let ε tend to 0 and then R tend to ∞, then we get (2.2).

(2) The integral representation (2.2) implies that f is completely monotonic. This extends the monotonicity result of Sándor and Debnath mentioned in Section 1.

Lemma 2. *Let*

$$g(s) = \frac{1}{\pi} s^s (1-s)^{1-s} \sin(\pi s) \qquad (0 \leq s \leq 1). \tag{2.4}$$

Then we have

$$\begin{aligned} &\int_0^1 g(s)\,ds = \frac{e}{24}, \\ &\int_0^1 s g(s)\,ds = \frac{e}{48}, \qquad \textit{and} \\ &\int_0^1 \frac{1}{s} g(s)\,ds = \int_0^1 \frac{1}{1-s} g(s)\,ds = \frac{e}{2} - 1. \end{aligned} \tag{2.5}$$

Proof. Let f be the function given in (2.1). We set $x = 1/y$ and obtain

$$x\left[f(x) - \frac{e}{2}\right] = \frac{ey/2 + e - (1+y)^{1+1/y}}{y^2}. \tag{2.6}$$

The rule of l'Hospital yields

$$\lim_{y \to 0} \frac{ey/2 + e - (1+y)^{1+1/y}}{y^2} = \frac{e}{24}. \tag{2.7}$$

From (2.2), (2.6), and (2.7) we get

$$\int_0^1 g(s)\,ds = \lim_{x\to\infty} \int_0^1 \frac{x}{x+s} g(s)\,ds = \lim_{x\to\infty} x\left[f(x) - \frac{e}{2}\right] = \frac{e}{24}.$$

Using $g(s) = g(1-s)$ we obtain

$$\begin{aligned} \int_0^1 s g(s)\,ds &= \int_0^{1/2} s g(s)\,ds + \int_{1/2}^1 s g(s)\,ds = \int_0^{1/2} s g(s)\,ds + \int_0^{1/2} (1-s) g(s)\,ds \\ &= \int_0^{1/2} g(s)\,ds = \frac{1}{2}\int_0^1 g(s)\,ds = \frac{e}{48}. \end{aligned}$$

Applying Lemma 1 we get

$$\int_0^1 \frac{1}{1-s} g(s)\, ds = \int_0^1 \frac{1}{1-s} g(1-s)\, ds = \int_0^1 \frac{1}{s} g(s)\, ds = \lim_{x\to 0^+} f(x) - \frac{e}{2} = \frac{e}{2} - 1. \ \square$$

Lemma 3. *Let*

$$\Delta_1(x) = \frac{e-(1+1/x)^x}{(1+1/x)^x\big[\log(1+1/x) - 1/(x+1)\big]} - x \tag{2.8}$$

and

$$\Delta_2(x) = \frac{e-(1+1/x)^{x+1}}{(1+1/x)^{x+1}\big[\log(1+1/x) - 1/x\big]} - x. \tag{2.9}$$

Then

$$\lim_{x\to\infty} \Delta_1(x) = \frac{11}{12} \qquad \textit{and} \qquad \lim_{x\to\infty} \Delta_2(x) = \frac{1}{12}. \tag{2.10}$$

Proof. We have

$$\Delta_1(x) = \frac{b_1(x) + c_1(x)}{a_1(x)} \qquad \text{and} \qquad \Delta_2(x) = \frac{b_2(x) + c_2(x)}{a_2(x)},$$

where

$$a_1(x) = x^2\left[\log\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right],$$
$$b_1(x) = x\left[\frac{1}{2} - a_1(x)\right],$$
$$c_1(x) = x^2\left[\frac{e}{(1+1/x)^x} - 1\right] - \frac{x}{2},$$

and

$$a_2(x) = x^2\left[\log\left(1+\frac{1}{x}\right) - \frac{1}{x}\right],$$
$$b_2(x) = -x\left[\frac{1}{2} + a_2(x)\right],$$
$$c_2(x) = x^2\left[\frac{e}{(1+1/x)^{x+1}} - 1\right] + \frac{x}{2}.$$

The limit relations

$$\lim_{x\to\infty} a_1(x) = \frac{1}{2}, \quad \lim_{x\to\infty} b_1(x) = \frac{2}{3}, \quad \lim_{x\to\infty} c_1(x) = -\frac{5}{24},$$

and

$$\lim_{x\to\infty} a_2(x) = -\frac{1}{2}, \quad \lim_{x\to\infty} b_2(x) = -\frac{1}{3}, \quad \lim_{x\to\infty} c_2(x) = \frac{7}{24}$$

lead to (2.10). $\square$

A proof for the following lemma can be found in [26].

Lemma 4. *Let h: $[0,\infty)\to(0,1]$ be a continuous function. If h is completely monotonic on $(0,\infty)$, then we get for all $x,y\ge 0$:*

$$h(x)h(y)\le h(x+y).$$

The next lemma is given in [12, p. 83].

Lemma 5. *Let f and g be functions such that $f\big(g(x)\big)$ is defined for $x>0$. If f and g' are completely monotonic, then $x\mapsto f\big(g(x)\big)$ is also completely monotonic.*

Our final lemma is well known and is stated only for easy reference.

Lemma 6. *The sum and the product of completely monotonic functions are also completely monotonic.*

3. Main results

We are now in a position to determine all real polynomials P_n such that the functions given in (1.1) and (1.2), respectively, are completely monotonic.

Theorem 1. *Let $P_n(x)=x^n+\sum_{\nu=0}^{n-1}c_\nu x^\nu$ be a polynomial of degree $n\ge 1$ with real coefficients. The function*

$$x\mapsto P_n(x)\left[e-\left(1+\frac{1}{x}\right)^x\right]$$

is completely monotonic if and only if $n=1$ and $c_0\ge 11/12$.

Proof. Let

$$F_n(x)=P_n(x)\left[e-\left(1+\frac{1}{x}\right)^x\right]\qquad(x>0).$$

There exists a number $x_0>0$ such that P_n is positive on (x_0,∞). If F_n is completely monotonic, then we have

$$F_n'(x)=P_n'(x)\left[e-\left(1+\frac{1}{x}\right)^x\right]-P_n(x)\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right]\le 0,$$

and for $x>x_0$ this is equivalent to

$$x\frac{P_n'(x)}{P_n(x)}\le\left(1+\frac{\Delta_1(x)}{x}\right)^{-1}, \tag{3.1}$$

where Δ_1 is defined in (2.8). Applying Lemma 3 we get

$$n = \lim_{x\to\infty} x\frac{P_n'(x)}{P_n(x)} \le \lim_{x\to\infty}\left(1+\frac{\Delta_1(x)}{x}\right)^{-1} = 1.$$

This implies $n=1$. Hence, $P_n(x) = x + c_0$, so that (3.1) yields

$$\Delta_1(x) \le c_0 \qquad \text{for all sufficiently large } x. \tag{3.2}$$

From (2.10) and (3.2) we conclude that $c_0 \ge 11/12$.

Let

$$f_c(x) = (x+c)\left[e - \left(1+\frac{1}{x}\right)^x\right]. \tag{3.3}$$

We have

$$f_c(x) = f_{11/12}(x) + \frac{c-11/12}{x+11/12} f_{11/12}(x).$$

If $c \ge 11/12$, then $x \mapsto (c-11/12)/(x+11/12)$ is completely monotonic. Applying Lemma 6 we get: if $f_{11/12}$ is completely monotonic, then the same is true for f_c with $c \ge 11/12$.

We prove now that $f_{11/12}$ is completely monotonic. Let g be defined in (2.4). Applying

$$\frac{1}{(x+s)^n} = \frac{1}{(n-1)!}\int_0^\infty e^{-xt}e^{-st}t^{n-1}\,dt \qquad (x>0,\ s\ge 0,\ n=1,2,\ldots) \tag{3.4}$$

with $n=1$ we obtain

$$\int_0^1 \frac{g(s)}{x+s}\,ds = \int_0^\infty e^{-xt}h(t)\,dt, \tag{3.5}$$

where

$$h(t) = \int_0^1 e^{-ts}g(s)\,ds. \tag{3.6}$$

From (2.2) and (3.5) we get

$$f_1(x) = \frac{e}{2} + \int_0^\infty e^{-xt}h(t)\,dt. \tag{3.7}$$

Using (3.7), (3.4) with $s = n = 1$, and the convolution theorem for Laplace transforms we obtain

$$\begin{aligned} f_{11/12}(x) &= \frac{x+11/12}{x+1} f_1(x) = \frac{e}{2}\frac{x+11/12}{x+1} + \frac{x+11/12}{x+1}\int_0^\infty e^{-xt}h(t)\,dt \\ &= \frac{e}{2}\left[1-\frac{1}{12}\int_0^\infty e^{-xt}e^{-t}\,dt\right] + \left[1-\frac{1}{12}\int_0^\infty e^{-xt}e^{-t}\,dt\right]\int_0^\infty e^{-xt}h(t)\,dt \\ &= \frac{e}{2} - \frac{e}{24}\int_0^\infty e^{-xt}e^{-t}\,dt + \int_0^\infty e^{-xt}h(t)\,dt \\ &\quad - \frac{1}{12}\int_0^\infty e^{-xt}e^{-t}\,dt \int_0^\infty e^{-xt}h(t)\,dt \\ &= \frac{e}{2} + \int_0^\infty e^{-xt}u(t)\,dt, \end{aligned} \tag{3.8}$$

where

$$u(t) = h(t) - e^{-t}\left[\frac{e}{24} + \frac{1}{12}\int_0^t e^s h(s)\,ds\right].$$

In order to prove that u is positive on $(0,\infty)$ we set $v(t) = e^t h(t)$ and $w(t) = e^t u(t)$. Then we get

$$w(t) = v(t) - \frac{e}{24} - \frac{1}{12}\int_0^t v(s)\,ds.$$

Let $t > 0$. Differentiation gives

$$w'(t) = v'(t) - \frac{1}{12}v(t) = e^t\left[\frac{11}{12}h(t) + h'(t)\right]$$

and

$$\begin{aligned} w'(t)e^{-t} &= \int_0^{11/12} e^{-ts}[11/12 - s]g(s)\,ds + \int_{11/12}^1 e^{-ts}[11/12 - s]g(s)\,ds \\ &\geq e^{-(11/12)t}\int_0^{11/12}[11/12 - s]g(s)\,ds \\ &\quad + e^{-(11/12)t}\int_{11/12}^1 [11/12 - s]g(s)\,ds \\ &= e^{-(11/12)t}\int_0^1 [11/12 - s]g(s)\,ds. \end{aligned} \tag{3.9}$$

Applying (2.5) we obtain

$$\int_0^1 [11/12 - s]g(s)\,ds = \frac{5e}{288},$$

so that (3.9) reveals that w' is positive on $(0,\infty)$. Hence, we get for $t>0$:

$$w(t) > w(0) = u(0) = h(0) - \frac{e}{24} = \int_0^1 g(s)\,ds - \frac{e}{24} = 0.$$

This implies that w and u are positive on $(0,\infty)$. From the integral representation (3.8) we conclude that $f_{11/12}$ is completely monotonic. □

Sándor has proved the weaker result that the function f_c in (3.3) is convex for $c \ge 11/12$, cf. [32].

Our second result is a striking companion of Theorem 1.

Theorem 2. *Let $P_n(x) = x^n + \sum_{\nu=0}^{n-1} c_\nu x^\nu$ be a polynomial of degree $n \ge 1$ with real coefficients. The function*

$$x \mapsto P_n(x)\left[\left(1+\frac{1}{x}\right)^{x+1} - e\right]$$

is completely monotonic if and only if $n=1$ and $c_0 \ge 1/12$.

Proof. Let

$$G_n(x) = P_n(x)\left[\left(1+\frac{1}{x}\right)^{x+1} - e\right] \qquad (x>0).$$

There exists a number $x_1 > 0$ such that P_n is positive on (x_1,∞). If G_n is completely monotonic, then we have

$$G_n'(x) = P_n'(x)\left[\left(1+\frac{1}{x}\right)^{x+1} - e\right] + P_n(x)\left(1+\frac{1}{x}\right)^{x+1}\left[\log\left(1+\frac{1}{x}\right) - \frac{1}{x}\right] \le 0. \tag{3.10}$$

For $x > x_1$ the inequality (3.10) can be written as

$$x\frac{P_n'(x)}{P_n(x)} \le \left(1+\frac{\Delta_2(x)}{x}\right)^{-1}, \tag{3.11}$$

where Δ_2 is given in (2.9). We let x tend to ∞ and conclude from (2.10) that $n=1$. Hence, $P_n(x) = x + c_0$ and (3.11) gives for all sufficiently large x:

$$\Delta_2(x) \le c_0.$$

From (2.10) we obtain $c_0 \ge 1/12$.

It remains to show: if $c \ge 1/12$, then

$$g_c(x) = (x+c)\left[\left(1+\frac{1}{x}\right)^{x+1} - e\right] \tag{3.12}$$

is completely monotonic. We have

$$g_c(x) = g_{1/12}(x) + \frac{c-1/12}{x+1/12} g_{1/12}(x).$$

Since $x \mapsto (c-1/12)/(x+1/12)$ (with $c \geq 1/12$) is completely monotonic, we conclude from Lemma 6 that it suffices to prove that $g_{1/12}$ is completely monotonic. Let f_1 and h be the functions defined in (3.3) and (3.6), respectively. Using the identity

$$g_1(x) = \left(1+\frac{1}{x}\right)\left[e - f_1(x)\right], \tag{3.13}$$

(3.7), (3.4) with $s=0$, $n=1$, and the convolution theorem for Laplace transforms, we get

$$\begin{aligned} g_{1/12}(x) &= \frac{x+1/12}{x+1} g_1(x) = \left(1+\frac{1}{12x}\right)\left[e - f_1(x)\right] \\ &= \left(1+\frac{1}{12x}\right)\left[\frac{e}{2} - \int_0^\infty e^{-xt} h(t)\,dt\right] \\ &= \frac{e}{2} + \int_0^\infty e^{-xt}\sigma(t)\,dt, \end{aligned}$$

where

$$\sigma(t) = -h(t) + \frac{e}{24} - \frac{1}{12}\int_0^t h(s)\,ds.$$

We have to show that σ is nonnegative on $(0,\infty)$. Applying (2.5), (3.6), and

$$\begin{aligned} \int_0^t h(s)\,ds &= \int_0^t \int_0^1 e^{-su} g(u)\,du\,ds = \int_0^1 g(u) \int_0^t e^{-su}\,ds\,du \\ &= \int_0^1 \frac{1-e^{-tu}}{u} g(u)\,du, \end{aligned}$$

we obtain the integral representation

$$\sigma(t) = \int_0^1 \left(1-e^{-tu}\right)\left[1-\frac{1}{12u}\right] g(u)\,du.$$

Using (2.5) we get for $t>0$:

$$\begin{aligned} \sigma(t) &\geq \left(1-e^{-t/12}\right)\int_0^{1/12} \left[1-\frac{1}{12u}\right] g(u)\,du \\ &\quad + \left(1-e^{-t/12}\right)\int_{1/12}^1 \left[1-\frac{1}{12u}\right] g(u)\,du \\ &= \left(1-e^{-t/12}\right)\int_0^1 \left[1-\frac{1}{12u}\right] g(u)\,du = \frac{1}{12}\left(1-e^{-t/12}\right) > 0. \end{aligned}$$

This completes the proof of Theorem 2. □

Remark. In analogy with Lemma 1 we obtain that g_1 is a Stieltjes transform. Indeed, from (3.13), (2.2), and (2.5) we get

$$(x+1)\left[\left(1+\frac{1}{x}\right)^{x+1}-e\right]=\frac{e}{2}+\frac{1}{x}+\int_0^1\frac{(1-s)g(s)}{s}\frac{ds}{x+s},$$

where g is given in (2.4).

Theorem 3. *The functions* $p(x)=e-(1+1/x)^x$ *and* $q(x)=(1+1/x)^{x+1}-e$ *are Stieltjes transforms with the representations*

$$p(x)=\frac{1}{x+1}+\int_0^1\frac{g(s)}{1-s}\frac{ds}{x+s},\qquad q(x)=\frac{1}{x}+\int_0^1\frac{g(s)}{s}\frac{ds}{x+s},$$

and in particular they are completely monotonic. Here, g *denotes the function defined in* (2.4).

Proof. Using (2.2) and (2.5) we obtain

$$p(x)-\frac{1}{x+1}=\frac{1}{x+1}\left[\int_0^1\frac{g(s)\,ds}{1-s}+\int_0^1\frac{g(s)\,ds}{x+s}\right]=\int_0^1\frac{g(s)\,ds}{(1-s)(x+s)}.$$

Letting x tend to 0 we get

$$\int_0^1\frac{g(s)\,ds}{s(1-s)}=e-2,$$

which implies

$$\begin{aligned}q(x)-\frac{1}{x}&=\frac{e-1}{x}-\left(1+\frac{1}{x}\right)p(x)=\frac{e-2}{x}-\left(1+\frac{1}{x}\right)\int_0^1\frac{g(s)\,ds}{(1-s)(x+s)}\\&=\int_0^1\frac{g(s)\,ds}{s(x+s)}.\ \square\end{aligned}$$

Remarks. (1) Using (3.3) and (3.12) we obtain the identities

$$p(x)=\frac{1}{x+11/12}f_{11/12}(x)\qquad\text{and}\qquad q(x)=\frac{1}{x+1/12}g_{1/12}(x).$$

Lemma 6 and Theorems 1 and 2 imply that p and q are completely monotonic.

(2) Applying Lemma 4 and Theorem 3 we get the following functional inequalties involving p:

$$0<\frac{p(x)p(y)}{p(x+y)}\le e-1\qquad(x,y>0).$$

Moreover, since $\lim_{x\to 0} p(x) = e-1$ and $\lim_{x\to\infty} \big(p(x)\big)^2/p(2x) = 0$, we conclude that both bounds are best possible.

(3) A holomorphic function $f\colon \mathbf{C}\setminus\mathbf{R}\to\mathbf{C}$ is said to be a Pick function, if $f(\overline{z}) = \overline{f(z)}$ and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$; cf. [17]. It follows immediately from the first part of Theorem 3 that $\tilde{p}(z) = (1+1/z)^z$ is a Pick function. The function $z \mapsto (1+z)^{1/z}$ is holomorphic on $\mathbf{C}\setminus(-\infty,0]$ and has a negative imaginary part in the upper half-plane, because $\tilde{p}$ is a Pick function. Like in the proof of Lemma 1 it can be seen by [1, p. 127] that it is a Stieltjes transform with the representation

$$(1+x)^{1/x} = 1 + \frac{1}{x+1} + \frac{1}{\pi}\int_1^\infty \frac{\sin(\pi/t)}{(t-1)^{1/t}}\,\frac{dt}{x+t} \qquad (x>0).$$

This formula follows also from the representation of p in Theorem 3 by a change of variable.

(4) A function $f\colon (0,\infty)\to[0,\infty)$ is called a Bernstein function, if f has derivatives of all orders and f' is completely monotonic; cf. [8]. Since $\tilde{p}' = -p'$ we conclude that $\tilde{p}$ is a Bernstein function. Theorem 3 immediately implies that $x\mapsto(1+1/x)^{x+1}$ is a Stieltjes transform. Using that if f is a Stieltjes transform, then so is $x\mapsto 1/f(1/x)$, and if f is a nonzero Stieltjes transform, then $1/f$ is a Bernstein function, cf. [5], we obtain that $x\mapsto(1+x)^{1+1/x}$ is a Bernstein function.

Our next theorem extends in part the results on complete monotonicity given in Theorem 3 and provides a generalization of Schur's monotonicity theorem mentioned in Section 1.

Theorem 4. *Let $a>0$ and b be real numbers. The function*

$$x\mapsto\left(1+\frac{a}{x}\right)^{x+b} - e^a$$

is completely monotonic if and only if $a\leq 2b$.

Proof. Let

$$q_{a,b}(x) = \left(1+\frac{a}{x}\right)^{x+b} - e^a.$$

If $q_{a,b}$ is completely monotonic, then we get

$$q'_{a,b}(x) = \left(1+\frac{a}{x}\right)^{x+b}\left[\log\left(1+\frac{a}{x}\right) - \frac{a(x+b)}{x(x+a)}\right] \leq 0. \tag{3.14}$$

We set $x=at$. Then (3.14) yields

$$t\big[(t+1)\log(1+1/t) - 1\big] \leq b/a \qquad (t>0).$$

Letting t tend to ∞, we get $1/2 \le b/a$.

Next, we define for $x > 0$ and $b \ge a/2 > 0$:

$$y_{a,b}(x) = -(x+b)\log(1+a/x).$$

Differentiation gives

$$y'_{a,b}(x) = \frac{a(x+b)}{x(x+a)} - \log\left(1+\frac{a}{x}\right) \quad \text{and} \quad -y''_{a,b}(x) = \frac{a}{(x+a)^2}\left[\frac{2b-a}{x} + \frac{ab}{x^2}\right].$$

Applying (3.4) and Lemma 6 we obtain that $-y''_{a,b}$ is completely monotonic. Since $\lim_{x\to\infty} y'_{a,b}(x) = 0$, it follows that $y'_{a,b}$ is also completely monotonic. Using Lemma 5 (with $g = y_{a,b}$ and $f = 1/\exp$) and the limit relation $\lim_{x\to\infty} q_{a,b}(x) = 0$, we conclude that $\exp(-y_{a,b}) = q_{a,b} + e^a$ and $q_{a,b}$ are completely monotonic. □

Remark. It remains an *open problem* to determine all real parameters $\alpha > 0$ and β such that

$$p_{\alpha,\beta}(x) = e^\alpha - \left(1+\frac{\alpha}{x}\right)^{x+\beta}$$

is completely monotonic. Theorem 3 implies that $p_{1,0}$ is completely monotonic. This result can be extended easily: if $0 < \alpha \le 1$, then $p_{\alpha,0}$ is completely monotonic. Let $0 < \alpha \le 1$ and $z_\alpha(x) = p_{\alpha,0}(\alpha x)$. Then we have

$$-z'_\alpha(x) = \alpha[e - p_{1,0}(x)]^{\alpha-1}\big(-p'_{1,0}(x)\big).$$

Using Theorem 3, Lemma 5 (with $g(x) = e - p_{1,0}(x)$ and $f(x) = x^{\alpha-1}$), and Lemma 6 we conclude that $-z'_\alpha$ is completely monotonic. Since $\lim_{x\to\infty} z_\alpha(x) = 0$, we obtain that z_α and $p_{\alpha,0}$ are also completely monotonic.

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