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HIGHER INTEGRABILITY FOR MAXIMAL OSCILLATORY FOURIER INTEGRALS

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Abstract. For a real number t, for ξ in $\mathbf{R}^n$ and for a real positive number a we define S^a by

$$(S^a)(t)\hat{}(\xi) = e^{it|\xi|^a}\hat{f}(\xi), \qquad f \in \mathscr{S}(\mathbf{R}^n).$$

The results in this paper concern the case $0 < a \neq 1$. For $0 < a < 1$, $n = 1$ we improve on the local integrability of the maximal function $x \mapsto \|(S^a f)[x]\|_{L^\infty(-1,1)}$. In higher dimensions we give a result for radial testfunctions. For $a > 1$, $n = 1$ we prove a weighted global estimate of which a known $L^4(\mathbf{R})$-estimate is a special case.

The methods include asymptotics for the kernel of the Fourier multiplier $\xi \mapsto \exp(i|\xi|^a a)|\xi|^{-2s}$ and Pitt's inequality.

1. Introduction

1.1. Let $u(x,t)$ denote the solution to the free time-dependent Schrödinger equation $\Delta_x u = i\partial_t u$ with initial data f, $(x,t) \in \mathbf{R}^n \times \mathbf{R}_+$. At least for f in the Schwartz class $\mathscr{S}(\mathbf{R}^n)$, u is represented by an oscillatory integral with quadratic phase. We are interested in the behaviour of $u(x,t)$ as t tends to 0. Cf. Carleson [4], [5]. For rougher initial data this requires a method of making the values of u precise. See e.g. Sjögren, Sjölin [19, p. 14–15].

In this as in many other papers the stated convergence problem is viewed as a summability problem for Fourier integrals corresponding to the multiplier $m_2, m_a(\xi) = \exp(i|\xi|^a)$. Accordingly we define $(S^a f)(t)$ as in the abstract and observe that $u(x,t) = (S^2 f)[x](t)$. However, the kernel of m_a does not belong to $L^1(\mathbf{R}^n)$ but we do have the weak unity condition $m_a(0) = 1$.

Intimately connected with the convergence result described here are $L^q_{\mathrm{loc}}(L^\infty)$-estimates, i.e. L^q_{loc}-estimates for maximal functions. For the multiplier m_a it is known that there is regarding such estimates a significant difference between the cases $0 < a < 1$ and $a > 1$ when $q = 2$. See [31, Section 2.5, p. 488]. The principle of duality of phases (see Stein [24, Chapter VIII, Sections 5.3 and 5.4, pp. 357–358]) offers one way of understanding this difference.

The main purpose of this paper is to improve known one-dimensional L^q_{loc}-results for maximal functions in the case $0 < a < 1$. We have the following theorem.

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Theorem A. *Let* $q = (4-4a)/(2-4s-a)$. *Then there is a number* C *independent of* f *in the Schwartz class* $\mathscr{S}(\mathbf{R})$ *such that the inequality*

$$\left(\int_{-1}^{1} \|(S^a f)[x]\|^q_{L^\infty(-1,1)}\,dx\right)^{1/q} \le C\,\|f\|_{\dot H^s(\mathbf{R})}$$

holds if s *is greater than and close enough to* $\frac14 a$ *where* $0<a<1$.

For the definition of the spaces $\dot H^s(\mathbf{R}^n)$ and $H^s(\mathbf{R}^n)$ used in this introduction we refer to Section 2.2 below.

We also have the following theorems.

Theorem B. *Let* $2\le q\le 4$. *Then there is a number* C *independent of* f *in the Schwartz class* $\mathscr{S}(\mathbf{R})$ *such that the inequality*

$$\left(\int_{\mathbf{R}} \|(S^a f)[x]\|^q_{L^\infty(\mathbf{R})}\,|x|^{q/4-1}\,dx\right)^{1/q} \le C\,\|f\|_{\dot H^{1/4}(\mathbf{R})}$$

holds if $a>1$.

Theorem C. *Let* $n>1$. *Then there is a number* C *independent of* f *in the Schwartz subclass of radial functions such that the inequality*

$$\left(\int_{|x|\le 1} \|(S^a f)[x]\|^q_{L^\infty(-1,1)}\,dx\right)^{1/q} \le C\,\|f\|_{\dot H^s(\mathbf{R}^n)}, \qquad q=\frac{4n(1-a)}{2n(1-a)+a-4s}$$

holds if s *is greater than and close enough to* $\frac14 a$ *where* $0<a<1$.

Note that q in Theorem A is greater than 2 if the stated conditions on s and a are fulfilled. Theorem A therefore improves our $L^2_{\mathrm{loc}}(\mathbf{R})$-result in [31, Theorem 1.2(a), p. 486].

Theorem A and C are corollaries to Theorem 2.6 and 2.7 respectively. Theorem B will be proved in Section 4.10.

1.2. Remark. The case $q=4$ in Theorem B is in accordance with the special case $\varphi(\xi)=|\xi|^a$, $a>1$ of Kenig, Ponce, Vega [10, Theorem 2.5, p. 41].

1.3. Earlier results. The problem sketched above was introduced in Carleson [5] and has been studied by many authors during recent years. We will give a brief description of earlier results. Among other papers and reports we will mention those which contain results of the category *best known*. With some exceptions we will restrict ourselves to $L^q_{\mathrm{loc}}(\mathbf{R}^n)$-results.

1.3.1. The case $n=1$. As already mentioned results for $0<a<1$ may be found in [31]. For the case $a>1$ Sjölin [20, Theorem 3 and 4, p. 700] has shown that $f\in H^s(\mathbf{R})$, $s\ge\frac14$ is necessary and sufficient for the local integrability of $x\longmapsto \|(S^a f)[x]\|_{L^\infty(-1,1)}$. The best known integrability property may be found in Kenig, Ponce, Vega [10, Theorem 2.5, p. 41]. Cf. Remark 1.2 above.

Results reminiscent of Theorem B may be found in Gülkan [7] and in Sjölin [22].

1.3.2. The case $n=2$. For the cases $a=2$ and $a>1$ Bourgain [3], Moyua, Vargas, Vega [15], [16], Tao, Vargas [26] and Tao, Vargas, Vega [27] give sufficient conditions on $f\in H^s(\mathbf{R}^2)$ for the local integrability of the maximal function. The conditions are of the type $s=\frac{1}{2}-\varepsilon$ for some small positive number ε.

1.3.3. The case $n\geq 3$. For $a>1$ Sjölin [20] proved using local smoothing that $f\in H^s(\mathbf{R}^n)$, $s>\frac{1}{2}$ is sufficient for the local integrability of the maximal function, $n\geq 2$. Also cf. Vega [28], Si Lei Wang [35] and [33].

The relationship between (local) smoothing and maximal estimates is explained e.g. in [20, p. 704–706] and [31, Section 2.3, p. 487–488]. For results of which $m(t,x,\rho)=\exp(it\rho^a)$ is a special case see Vega [29] and [32, Theorem 14.3, p. 219]. Those results are derived without any smoothness assumptions on m.

Smoothing results in accordance with [20], [32] and [33] may be found e.g. in Ben-Artzi, Devinatz [1], Ben-Artzi, Klainerman [2] and Kato, Yajima [9].

1.3.4. Other references in the case $a>1$. In the work of Vega [28] already mentioned in Section 1.3.3 it is shown that $f\in H^s(\mathbf{R}^n)$, $s\geq\frac{1}{4}$ is a necessary condition for the local integrability of the maximal function, $n>1$. This also follows from the work of Sjölin in [20], [21] by translating one-dimensional counterexamples to higher dimension using the oscillation of Bessel functions at infinity. For radial testfunctions there are results by Fukuma [6], Prestini [18], Sjölin [21], [22] and Sichun Wang [34]. Weighted estimates for general dispersive equations including the case $a>1$ are treated in Heinig, Wang [8]. Other interesting results on oscillatory Fourier integral operators may be found in Kolasa [11], [12].

1.4. The plan of this paper. In Section 2 we introduce notation used in this paper and state our theorems. In Section 3 we collect some auxiliary results which are classical. In Section 4 we prove our theorems in the case $n=1$ and in Section 5 in the case $n>1$.

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2. Notation and statement of theorems

2.1. Oscillatory integrals. For x and ξ in $\mathbf{R}^n$ we let $x\xi=x_1\xi_1+\dots+x_n\xi_n$. If a is a real positive number and if f is in the Schwartz class $\mathscr{S}(\mathbf{R}^n)$ we define

$$(S^af)(t)[x]=(S^af)[x](t)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}e^{i(x\xi+|t\xi|^a)}\hat{f}(\xi)\,d\xi,\qquad t\in\mathbf{R}.$$

Here $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(\xi)=\int_{\mathbf{R}^n}e^{-ix\xi}f(x)\,dx.$$

Observe that we have redefined S^a slightly compared with the abstract and Section 1.1. We have replaced the summability parameter $t^{1/a}$ in Section 1.1 by t. Therefore, according to our redefinition of S^a, $u(x,t) = (S^2 f)[x](t^{1/2})$.

2.2. Sobolev spaces. We introduce homogeneous and inhomogeneous fractional Sobolev spaces

$$\dot{H}^s(\mathbf{R}^n) = \left\{ f \in \mathscr{S}'(\mathbf{R}^n) : \|f\|^2_{\dot{H}^s(\mathbf{R}^n)} = \int_{\mathbf{R}^n} |\xi|^{2s}\, |\hat{f}(\xi)|^2 \, d\xi < \infty \right\},$$
$$H^s(\mathbf{R}^n) = \left\{ f \in \mathscr{S}'(\mathbf{R}^n) : \|f\|^2_{H^s(\mathbf{R}^n)} = \int_{\mathbf{R}^n} (1+|\xi|^2)^s\, |\hat{f}(\xi)|^2 \, d\xi < \infty \right\}.$$

2.3. Auxiliary notation. B denotes the open unit ball in $\mathbf{R}$, $\dot{\mathbf{R}}$ denotes the punctured real line $\mathbf{R} \setminus \{0\}$.

Throughout this paper we will use auxiliary functions χ and ψ such that $\chi \in \mathscr{C}_0^\infty(\mathbf{R})$ is even,

$$\chi(\mathbf{R} \setminus 2B) = \{0\}, \qquad \chi(\mathbf{R}) \subseteq [0,1], \qquad \chi(B) = \{1\}$$

and $\psi = 1 - \chi$. From these functions we obtain two families of functions as follows: for each positive number N and M set $\chi_N(\xi) = \chi(\xi/N)$ and $\psi_M(\xi) = \psi(M\xi)$. Associated with our auxiliary functions are certain exponents

$$c(\chi) = 1 - 2s \qquad \text{and} \qquad c = c(\psi) = \frac{4s - 2 + a}{2a - 2} \tag{2.1}$$

which will play an important rôle in Section 4.9.2 when we use the Riesz potential I_{1-c}. In our theorems we will assume that

$$\text{(i)} \quad \tfrac{1}{4}a < s \le \tfrac{1}{4},\ s < \tfrac{1}{2}a,\ 0 < a < 1 \qquad \text{or} \qquad \text{(ii)} \quad s = \tfrac{1}{4},\ a > 1. \tag{2.2}$$

We will also use power weights $x \longmapsto |x|^{\delta(q)}$ where

$$\delta(q) = \frac{c-1}{2} + n\left(\frac{1}{2} - \frac{1}{q}\right).$$

2.4. Numbers denoted by C (sometimes with subscripts) may be different at each occurrence even within the same chain of (in-)equalities. The letter R (with subscripts) will denote various (weighted) linearisations of maximal operators. There is no definition of such linearisation which is fixed throughout the paper.

Unless otherwise explicitly stated all functions f and g are supposed to belong to $\mathscr{S}(\mathbf{R}^n)$.

2.5. Remark. The conditions on a and s in (2.2) give $0 < c(\chi) \leq c < 1$ in case (i) and $c(\chi) = c = \frac{1}{2}$ in case (ii).

2.6. Theorem. *Let $2 \leq q \leq 2/c$. Then there is a number C independent of f such that the inequality*

$$\left(\int_B \|(S^a f)[x]\|^q_{L^\infty(B)} \, |x|^{cq/2-1} \, dx \right)^{1/q} \leq C \, \|f\|_{\dot{H}^s(\mathbf{R})}$$

holds if (2.2) *is satisfied.*

Case (2.2i) in this theorem is—as already pointed out—an improvement of our result in [31, Theorem 1.2(a), p. 486] and case (2.2ii) is an improvement of Sjölin [20, Theorem 3, p. 700] in the case $n = 1$. We use Pitt's inequality as stated in Muckenhoupt [17] instead of the inequality of Hardy, Littlewood and Sobolev to achieve these improvements and carry out the proof of the two cases in (2.2) simultaneously.

Since $2/c = (4a-4)/(4s-2+a)$ Theorem A in Section 1.1 follows directly from the case $q = 2/c$ in Theorem 2.6.

2.7. Theorem. *Let $2 \leq q \leq 2/c$ and $n > 1$. Then there is a number C independent of f in the Schwartz subclass of radial functions such that the inequality*

$$\left(\int_{B^n} \|(S^a f)[x]\|^q_{L^\infty(B)} |x|^{q\delta(q)} \, dx \right)^{1/q} \leq C \, \|f\|_{\dot{H}^s(\mathbf{R}^n)}$$

holds if (2.2) *is satisfied.*

It is straightforward to verify that

$$\delta\left(\frac{4n(1-a)}{2n(1-a)+a-4s} \right) = 0.$$

Also, if the conditions on a and s of Theorem C in Section 1.1 are fulfilled the inequality

$$\tilde{q} := \frac{4n(1-a)}{2n(1-a)+a-4s} \leq \frac{2}{c}$$

holds. Theorem C in Section 1.1 now follows directly from the case $q = \tilde{q}$ in Theorem 2.7.

3. Some preparation

3.1. In this section we introduce some notation and collect some well-known results which will be used in the proofs of our theorems. Standard references are given.

3.2. Notation. We define the Riesz potential I_β as

$$[I_\beta f](x) = c_\beta \int_{\mathbf{R}^n} |x-x'|^{-n+\beta} f(x')\,dx', \qquad n>\beta>0,\ f\in\mathscr{S}(\mathbf{R}^n). \tag{3.1}$$

See Stein [23, p. 117]. Only the finiteness of the number c_β will be used in this paper.

3.3. Theorem (cf. Stein [23, Lemma 1(b), p. 117]). *The identity* $[I_\beta f]\hat{}(\xi) = |\xi|^{-\beta}\hat f(\xi)$ *holds in the sense that*

$$\int_{\mathbf{R}^n} |\xi|^{-\beta}\hat f(\xi)\,g(\xi)\,d\xi = \int_{\mathbf{R}^n} [I_\beta f](x)\,\hat g(x)\,dx.$$

3.4. Theorem (Pitt's inequality, Muckenhoupt [17, p. 729]). *Assume that* $q\ge p$, $0\le\alpha<1-1/p$, $0\le\gamma<1/q$ *and* $\gamma=\alpha+1/p+1/q-1$. *Then there exists a number* C *independent of* f *such that*

$$\left(\int_{\mathbf{R}} |\hat f(\xi)|^q\,|\xi|^{-\gamma q}\,d\xi\right)^{1/q} \le C\left(\int_{\mathbf{R}} |f(x)|^p\,|x|^{\alpha p}\,dx\right)^{1/p}.$$

3.5. Theorem (Stein, Weiss [25, Theorem 3.10, p. 158]). *Let* f *be radial. Then*

$$\hat f(\xi) = (2\pi)^{n/2}\,|\xi|^{-n/2+1}\int_0^\infty f_0(r)\,J_{n/2-1}(r|\xi|)\,r^{n/2}\,dr$$

where J_λ *is the Bessel function of the first kind of order* λ ([25, p. 154]).

3.6. Theorem (Asymptotics of the Bessel function, [25, Lemma 3.11, p. 158]). *If* $\lambda>-\frac12$, *then there is a number* C_λ *independent of* $\rho>1$ *such that*

$$\left|J_\lambda(\rho) - \left(\frac{2}{\pi\rho}\right)^{1/2}\cos\left(\rho-\frac{\lambda\pi}{2}-\frac{\pi}{4}\right)\right| \le C_\lambda\rho^{-3/2}.$$

4. Proofs for $n = 1$

4.1. Discussion. Let E be a measurable subset of $\mathbf{R}$ and let $t\colon \mathbf{R} \longrightarrow E$ be measurable. Define

$$[R_t f](x) = \int_{\mathbf{R}} \chi(x)\,|x|^{c/2-1/q}\, e^{i(x\xi+t(x)|\xi|^a)}|\xi|^{-s}\hat{f}(\xi)\,d\xi. \tag{4.1}$$

R_t can be extended to a t-uniformly bounded mapping $L^2(\mathbf{R}) \longrightarrow L^q(\mathbf{R})$ if and only if there is a number C independent of f such that

$$\left(\int_B \|(S^a f)[x]\|^q_{L^\infty(E)}\,|x|^{cq/2-1}\,dx\right)^{1/q} \le C\|f\|_{\dot{H}^s(\mathbf{R})}$$

holds. Theorem 2.6 therefore follows by proving such boundedness for R_t when $2 \le q \le 2/c$ and $E = B$. To derive it we need estimates for the inverse Fourier transform of

$$m(\xi) = \exp(\pm i|\xi|^a)|\xi|^{-2s}, \qquad \xi \in \mathbf{R},$$

where a and s satisfy the conditions in (2.2). Write $m = \chi m + \psi m$ and let K_χ and K_ψ be the inverse Fourier transforms of χm and ψm respectively.

4.2. Lemma. *K_χ is bounded and there is a number C independent of x such that*

$$|K_\chi(x)| \le C\,|x|^{-c(\chi)}, \qquad |x| \ge 1.$$

4.3. Lemma (Miyachi [14, Proposition 5.1, p. 289]; also cf. Wainger [30, p. 41] and Miyachi [13, Lemma 4, p. 174]).

(a) *K_ψ decreases rapidly and there is a number C independent of x such that*

$$|K_\psi(x)| \le C\,|x|^{-c(\psi)}, \qquad |x| < 1,\ 0 < a < 1.$$

(b) *K_ψ is smooth and there is a number C independent of x such that*

$$|K_\psi(x)| \le C\,|x|^{-c(\psi)}, \qquad |x| \ge 1,\ a > 1.$$

4.4. Lemma. *Let $f(x) = |x|^{-\alpha}$, $x \in \mathbf{R}^n$, $0 < \alpha < n$, and let $g \in \mathscr{C}(\mathbf{R}^n)$ decrease rapidly. Then there is a number C independent of x such that*

$$|(f * g)(x)| \le C\,|x|^{-\alpha}, \qquad |x| \ge 1.$$

Proof. Make the splitting

$$\int_{\mathbf{R}^n} |y|^{-\alpha} g(x-y)\,dy = \int_{|y|\le|x|/2} + \int_{|y|\ge|x|/2}.$$

The first integral can be majorised by a number C independent of x times

$$\sup_{|x-y|\ge|x|/2} |g(x-y)|\,|x|^{n-\alpha},$$

which decreases rapidly in x. The second integral can be majorised by

$$C\,\|g\|_{L^1(\mathbf{R}^n)}\,|x|^{-\alpha}, \qquad |x| \ge 1,$$

where C may be chosen to be independent of x.

4.5. Proof of Lemma 4.2. Since $-2s > -1$, the integral

$$\int_{\mathbf{R}} e^{i(x\xi \pm |\xi|^a)}\, |\xi|^{-2s}\, \chi(\xi)\, d\xi$$

is absolutely convergent. Hence K_χ is bounded (and continuous). To derive the asymptotic estimate we write

$$2\pi K_\chi(x) = \lim_{M\to\infty} G_M(x) + H(x)$$

where

$$G_M(x) = \int_{\mathbf{R}} e^{ix\xi}(e^{\pm i|\xi|^a} - 1)|\xi|^{-2s}[\,\chi\psi_M](\xi)\, d\xi$$

and

$$H(x) = \int_{\mathbf{R}} e^{ix\xi}\, |\xi|^{-2s}\, \chi(\xi)\, d\xi.$$

By Taylor's formula and integration by parts

$$|G_M(x)| \le \frac{C}{|x|}\bigg(\int_{\mathbf{R}} |\xi|^{a-2s-1}\,[\,\chi\psi_M](\xi)\, d\xi + \int_{\mathbf{R}} |\xi|^{a-2s}\, |[\,\chi\psi_M]'(\xi)|\, d\xi\bigg),$$

where C may be chosen to be independent of x and M. The first integral remains bounded as M tends to infinity. To bound the second integral independently of M we notice again that $a-2s>0$ and also that $|[\,\chi\psi_M]'|$ is like two approximative units whose supports approach 0 as M tends to infinity.

To handle H we use Theorem 3.3 and Lemma 4.4. We get that there is a number C independent of x such that

$$|H(x)| \le C\,|x|^{-1+2s}, \qquad |x| \ge 1.$$

We can now conclude that there is a number C independent of x such that

$$|K_\chi(x)| \le C\,(|x|^{-1} + |x|^{-1+2s}) \le C\,|x|^{-c(\chi)}, \qquad |x| \le 1.$$

4.6. Lemma. *Let a and s satisfy* (2.2) *and let $c(\chi)$ and $c(\psi)$ be as in* (2.1). *Then there is a number C_A independent of $\varepsilon \in [0, A]$, N and x such that*

$$\left|\int_{\mathbf{R}} e^{i(x\xi + |\varepsilon\xi|^a)}|\xi|^{-2s}\, \chi(\xi/N)\ d\xi\right| \le C_A\,(|x|^{-c(\chi)} + |x|^{-c(\psi)}).$$

Proof. For $\varepsilon > 0$ we set

$$\eta = \varepsilon\xi, \qquad v = \frac{x}{\varepsilon}, \qquad \text{and} \qquad L = \varepsilon N.$$

By a change of variables

$$\int_{\mathbf{R}} e^{i(x\xi+|\varepsilon\xi|^a)}\,|\xi|^{-2s}\,\chi(\xi/N)\;d\xi=\varepsilon^{2s-1}\int_{\mathbf{R}} e^{iv\eta}\,m(\eta)\,\chi(\eta/L)\,d\eta.$$

Let ζ denote either ψ or χ. By Lemmas 4.2 and 4.3 and a change of variables we get

$$\begin{aligned}\varepsilon^{2s-1}\left|\int_{\mathbf{R}} e^{iv\eta}\zeta(\eta)m(\eta)\chi(\eta/L)\,d\eta\right| &= \varepsilon^{2s-1}\left|\int_{\mathbf{R}} K_\zeta(u)L\widehat{\chi}(Lu-Lv)\,du\right| \\ &\le C\varepsilon^{2s-1}\int_{\mathbf{R}} |Lu|^{-c(\zeta)}\,|\widehat{\chi}(Lu-Lv)|\,L\,du\,L^{c(\zeta)} \qquad (4.2)\\ &\le C\varepsilon^{2s-1}\,|Lv|^{-c(\zeta)}L^{c(\zeta)}=C\varepsilon^{2s-1+c(\zeta)}\,|x|^{-c(\zeta)}.\end{aligned}$$

To get the last inequality we have also used Lemma 4.4. If $\zeta=\chi$ the exponent of ε is 0. If $\zeta=\psi$ it is $a(1-4s)/(2-2a)$, which is *non-negative in both of the cases* (2.2i) *and* (2.2ii).

Now we have proved the estimate in the lemma in the case $\varepsilon>0$. Since the integral

$$\int_{\mathbf{R}} e^{i(x\xi+|\varepsilon\xi|^a)}|\xi|^{-2s}\,\chi(\xi/N)\;d\xi$$

is continuous with respect to ε this estimate is valid also in the case $\varepsilon=0$.

4.7. Corollary. *Let a and s satisfy* (2.2) *and let c be as in* (2.1). *Then there is a number C_A independent of $\varepsilon\in[0,A]$, N and x such that*

$$\left|\int_{\mathbf{R}} e^{i(x\xi+|\varepsilon\xi|^a)}|\xi|^{-2s}\chi(\xi/N)\;d\xi\right|\le C_A\,|x|^{-c},\qquad |x|\le 1.$$

Proof. Cf. Remark 2.5.

4.8. Corollary. *Let a and s satisfy* (2.2ii) *and let c be as in* (2.1). *Then there is a number C independent of $\varepsilon\in\mathbf{R}, N$ and x such that*

$$\left|\int_{\mathbf{R}} e^{i(x\xi+|\varepsilon\xi|^a)}|\xi|^{-2s}\chi(\xi/N)\;d\xi\right|\le C|x|^{-c}.$$

Proof. According to Remark 2.5 $c(\zeta)=\frac{1}{2}$ in both of the cases of ζ. Hence the exponent $2s-1+c(\zeta)$ of ε in (4.2) is 0 in both of the cases of ζ.

4.9. Theorem. *R_t defined by the formula* (4.1) *can be extended to a t-uniformly bounded mapping $L^2(\mathbf{R})\longrightarrow L^q(\mathbf{R})$ if* (2.2) *is satisfied.*

Proof. Functions f and g appearing in this proof are assumed to belong to $\mathscr{C}_0(\mathbf{R})$ and to have support in $\dot{\mathbf{R}}$. We temporarily change notation and replace q by p^*, the conjugate exponent of some exponent p.

In the proof we follow Sjölin [20] and [31] with some modifications.

4.9.1. Reduction to a kernel estimate. We can replace $\hat{f}$ by f in the definition of R_t, since the Fourier transformation (apart from a multiple) is an isometry of $L^2(\mathbf{R}^n)$.

Set

$$[R_N f](x) = \int_{\mathbf{R}} \chi(x)\,|x|^{c/2-1/p^*}\, e^{i(x\xi+t(x)|\xi|^a)}\,|\xi|^{-s}\,\chi_N(\xi)\,f(\xi)\,d\xi.$$

Here the integration is performed over a compact set and for R_N the boundedness $L^2(\mathbf{R}) \longrightarrow L^{p^*}(\mathbf{R})$ can easily be verified. A computation of the adjoint shows that

$$[R_N^* g](\xi) = \int_{\mathbf{R}} \chi(x)\,|x|^{c/2-1/p^*}\, e^{-i(x\xi+t(x)|\xi|^a)}\,|\xi|^{-s}\,\chi_N(\xi)\,g(x)\,dx.$$

We will prove that the mapping R_N^* is bounded $L^p(\mathbf{R}) \longrightarrow L^2(\mathbf{R})$ uniformly with respect to t and N. Then R_N will be bounded $L^2(\mathbf{R}) \longrightarrow L^{p^*}(\mathbf{R})$ uniformly with respect to t and N. Since

$$[R_t f](x) = \lim_{N\to\infty} [R_N f](x),$$

we can by Fatou's lemma conclude that R_t is bounded $L^2(\mathbf{R}) \longrightarrow L^{p^*}(\mathbf{R})$ and that the bound is independent of t.

A computation involving Fubini's theorem shows that

$$\int_{\mathbf{R}} |[R_N^* g](\xi)|^2\,d\xi \le \iint_{\mathbf{R}^2} |K_N(x,x')|\,|g(x)g(x')|\,dx\,dx', \tag{4.3}$$

where

$$K_N(x,x') = \chi(x)\chi(x')|xx'|^{c/2-1/p^*} \int_{\mathbf{R}} e^{-i((x-x')\xi+(t(x)-t(x'))|\xi|^a)}|\xi|^{-2s}\chi_N(\xi)^2\,d\xi.$$

We shall prove the following *kernel estimate*: There is a number C independent of g, t and N such that

$$\iint_{\mathbf{R}^2} |K_N(x,x')|g(x)g(x')\,dx\,dx' \le C\|g\|^2_{L^p(\mathbf{R})}, \qquad g \ge 0. \tag{4.4}$$

Once this kernel estimate is proved the desired uniform boundedness follows by combining the kernel estimate with (4.3).

4.9.2. Proof of the kernel estimate. In proving (4.4) we first assume that $g \in \mathscr{C}_0^\infty(\mathbf{R})$ and that $\operatorname{supp} g \subseteq \dot{\mathbf{R}}$. There is a number C independent of t, N, x and x' such that

$$|K_N(x,x')| \le C\,\widetilde{\chi}(x)\,\widetilde{\chi}(x')\,|x-x'|^{-c} \tag{4.5}$$

where $\widetilde{\chi}(x) = \chi(x)\,|x|^{c/2-1/p^*}$. This estimate follows from Corollary 4.7. After replacing $K_N(x,x')$ in (4.4) by the right-hand side of (4.5) we apply (3.1) and get that there is a number C independent of g such that

$$\begin{aligned} \iint_{\mathbf{R}^2} \widetilde{\chi}(x)\widetilde{\chi}(x')\,|x-x'|^{-c}\,g(x)g(x')\;dx\,dx' &= C\int_{\mathbf{R}} [I_{1-c}(\widetilde{\chi}g)](x)\,\widetilde{\chi}(x)g(x)\,dx \\ &= C\int_{\mathbf{R}} |\xi|^{c-1}\,|\widehat{\widetilde{\chi}g}(\xi)|^2\;d\xi. \end{aligned} \tag{4.6}$$

Here we would like to apply Theorem 3.4 with $q=2$ and $-\gamma q = c-1$. Since

$$2 \le p^* \le \frac{2}{c}$$

the inequalities

$$q \ge p, \qquad 0 \le \alpha < 1 - \frac{1}{p} \qquad \text{and} \qquad 0 \le \gamma < \frac{1}{q}$$

are satisfied, where $\gamma = \alpha + 1/p + 1/q - 1$, i.e. $\alpha = \gamma - 1/p - 1/q + 1$. Since $(c/2 - 1/p^*)p + \alpha p = 0$ we get that there is a number C independent of f such that

$$\begin{aligned} \int_{\mathbf{R}} |\xi|^{c-1}|\widehat{\widetilde{\chi}g}(\xi)|^2\,d\xi &\le C\Big(\int_{\mathbf{R}} |\widetilde{\chi}(x)g(x)|^p\,|x|^{\alpha p}\,dx\Big)^{2/p} \\ &= C\Big(\int_{\mathbf{R}} |\chi(x)g(x)|^p\,dx\Big)^{2/p} \le C\|g\|^2_{L^p(\mathbf{R})}, \end{aligned} \tag{4.7}$$

Combining (4.5)–(4.7) now proves (4.4) in the case $g \in \mathscr{C}_0^\infty(\mathbf{R})$.

The equation (4.4) may now be proved in the case $g \in \mathscr{C}_0(\mathbf{R})$ by approximating g by positive functions in $\mathscr{C}_0^\infty(\mathbf{R})$.

4.10. Proof of Theorem B in Section 1.1. We repeat the proof of Theorem 2.6 with $E = B$ replaced by $E = \mathbf{R}$. See Section 4.1. We also replace $\chi(x)$ by $\chi_M(x)$ and observe that the number C in (4.5) will be independent of M. According to Corollary 4.8 that number C will also be independent of $\varepsilon = t(x) - t(x')$.

5. Proof for $n > 1$

5.1. Notation. For a measurable function $t\colon \mathbf{R}_+ \longrightarrow B$ and $f_0 \in \mathscr{C}_0(\mathbf{R}_+)$ we define

$$(5.1) \qquad [R_t f_0](r) = \int_0^\infty \chi(r)\, r^{c/2-1/q}\, r^{1/2}\, J_{n/2-1}(r\rho)\, \rho^{1/2}\, e^{it(r)\rho^a}\, \rho^{-s}\, f_0(\rho)\, d\rho.$$

5.2. Lemma. *Let $2 \le q \le 2/c$ and let R_t be defined by the formula* (5.1). *Then R_t can be extended to a t-uniformly bounded mapping $L^2(\mathbf{R}_+) \longrightarrow L^q(\mathbf{R}_+)$, if* (2.2) *is satisfied.*

Proof. Write $R_t = R_{t,1} + R_{t,2}$, where

$$[R_{t,1} f_0](r) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \chi(r) r^{c/2-1/p^*} \cos\!\left(r\rho - \frac{n}{4} + \frac{\pi}{4}\right) e^{it(r)\rho^a}\, \rho^{-s} f_0(\rho)\, d\rho.$$

As in the proof of Theorem 4.9 we temporarily change notation and replace q by p^*, the conjugate exponent of some exponent p.

It follows from Theorem 3.6 that

$$|[R_{t,2}^* g_0](\rho)| \le C \int_0^\infty \chi(r) r^{c/2-1/p^*} \frac{1}{1+r\rho} \rho^{-s} |g_0(r)|\, dr, \qquad g_0 \in \mathscr{C}_0(\mathbf{R}_+),$$

where C is independent of g_0. According to Theorem 4.9 $R_{t,1}$ can be extended to a t-uniformly bounded mapping $L^2(\mathbf{R}_+) \longrightarrow L^{p^*}(\mathbf{R}_+)$. Hence the theorem will be proved if we can show that the remainder $R_{t,2}^*$ can be extended to a t-uniformly bounded mapping $L^p(\mathbf{R}_+) \longrightarrow L^2(\mathbf{R}_+)$.

Write $R_{t,2}^* g_0 = \chi R_{t,2}^* g_0 + \psi R_{t,2}^* g_0$. Without loss of generality we can assume that $g_0 \ge 0$.

5.2.1. Estimate for $\chi R_{t,2}^*$. By Hölder's inequality there is a number C independent of g_0 such that

$$|[R_{t,2}^* g_0](\rho)| \le C \rho^{-s} \int_0^\infty \chi(r)\, r^{c/2-1/p^*}\, g_0(r)\, dr \le C \rho^{-s} \|g_0\|_{L^p(\mathbf{R}_+)}.$$

Upon squaring and integrating,

$$\|\chi R_{t,2}^* g_0\|_{L^2(\mathbf{R}_+)}^2 \le C \|g_0\|_{L^p(\mathbf{R}_+)}^2,$$

where C is independent of g_0.

5.2.2. Estimates for $\psi R^*_{t,2}$. We shall use the fact that there is a number C independent of g_0 such that

$$\int_0^1 [I_s(\widetilde{\chi} g_0)](t)^2\,dt \le C\,\|g_0\|^2_{L^p(\mathbf{R}_+)}, \tag{5.2}$$

where $\widetilde{\chi}(r) = \chi(r)\,r^{c/2-1/p^*}$. The proof of this is postponed to Section 5.2.3. We have to estimate

$$\int_1^\infty \left(\int_0^{1/\rho} \chi(r)\,r^{c/2-1/p^*}\ \rho^{-s}\,g_0(r)\,dr\right)^2 d\rho, \tag{5.3}$$

and

$$\int_1^\infty \left(\int_{1/\rho}^\infty \chi(r)\,r^{-1+c/2-1/p^*}\ \rho^{-1-s}\,g_0(r)\,dr\right)^2 d\rho. \tag{5.4}$$

Let us deal with (5.4) first. We make the change of variables $\rho \mapsto t(\rho) = 1/\rho$ and therefore consider the integral

$$\int_0^1 t^{2s}\left(\int_t^\infty \chi(r)\,r^{-1+c/2-1/p^*}\ g_0(r)\,dr\right)^2 dt.$$

Using $\max\{t, r-t\} \le r$ we get

$$\begin{aligned} t^s \int_t^\infty \chi(r)\,r^{-1+c/2-1/p^*}\ g_0(r)\,dr &= t^s \int_t^\infty r^{-1}\,\widetilde{\chi}(r)\,g_0(r)\,dr \\ \le \int_t^\infty r^{s-1}\,\widetilde{\chi}(r)\,g_0(r)\,dr &\le \int_t^\infty |t-r|^{s-1}\,\widetilde{\chi}(r)\,g_0(r)\,dr \le C[I_s(\widetilde{\chi} g_0)](t). \end{aligned}$$

After squaring and integrating with respect to t (5.2) yields

$$\int_1^\infty \left(\int_{1/\rho}^\infty \chi(r)\,r^{-1+c/2-1/p^*}\ \rho^{-1-s}\,g_0(r)\,dr\right)^2 d\rho \le C\|g_0\|^2_{L^p(\mathbf{R}_+)}$$

where C is independent of g_0.

The equation (5.3) is dealt with in a similar way to get

$$\begin{aligned} &\int_1^\infty \left(\int_0^{1/\rho} \chi(r)\,r^{c/2-1/p^*}\ \rho^{-s}\,g_0(r)\,dr\right)^2 d\rho \\ &\quad = \int_0^1 \left(t^{s-1}\int_0^t \chi(r)\,r^{c/2-1/p^*}\ g_0(r)\,dr\right)^2 dt \\ &\quad \le \int_0^1 \left(\int_0^t |t-r|^{s-1}\,\widetilde{\chi}(r)\,g_0(r)\,dr\right)^2 dt \\ &\quad \le C\int_0^1 [I_s(\widetilde{\chi} g_0)](t)^2\,dt \le C\,\|g_0\|^2_{L^p(\mathbf{R}_+)}. \end{aligned}$$

We have proved that

$$\|\psi R^*_{t,2} g_0\|^2_{L^2(\mathbf{R}_+)} \le C\|g_0\|^2_{L^p(\mathbf{R}_+)},$$

where C is independent of g_0.

5.2.3. Proof of the estimate (5.2). The function g_0 may be extended with 0 so as to be continuous on $\mathbf{R}$ with compact support in $\mathbf{R}_+$. Hence $I_s(\widetilde{\chi} g_0)$ as well as its Fourier transform belongs to $L^2(\mathbf{R})$. Parseval's formula and Theorem 3.3 give

$$\int_0^1 [I_s(\widetilde{\chi} g_0)](t)^2\,dt \le \|I_s(\widetilde{\chi} g_0)\|^2_{L^2(\mathbf{R})} = \frac{1}{2\pi}\int_{\mathbf{R}} |\tau|^{-2s} |(\widetilde{\chi} g_0)\widehat{\ }(\tau)|^2\,dt.$$

It is easy to verify that

$$\tfrac{1}{2}c > \tfrac{1}{2}c - \tfrac{1}{2} + s \ge 0$$

with equality on the right if and only if $s = \frac{1}{4}$. Therefore, if we choose $\tilde{p}$ such that

$$\frac{1}{\tilde{p}} - \frac{1}{p} = \frac{c}{2} - \frac{1}{2} + s,$$

then $2 \ge \tilde{p}$ and $0 \le 1/p^* - c/2 < 1 - 1/\tilde{p}$. We can now apply Theorem 3.4 with $\gamma = s$ and $\alpha = 1/p^* - c/2$ and Hölder' s inequality with $\tilde{p}$ and $\tilde{p}^*$ $(\tilde{p} \le p)$ to get that there is a number C independent of g_0 such that

$$\begin{aligned}\int_{\mathbf{R}} |\tau|^{-2s}\, |(\widetilde{\chi} g_0)\widehat{\ }(\tau)|^2\,dt &\le C\left(\int_{\mathbf{R}} |r|^{\alpha\tilde{p}} |\widetilde{\chi}(r)\, g_0(r)|^{\tilde{p}}\,dr\right)^{2/\tilde{p}}\\ &= C\|\chi g_0\|^2_{L^{\tilde{p}}(\mathbf{R}_+)} \le C\|g_0\|^2_{L^p(\mathbf{R}_+)}.\end{aligned}$$

5.3. Remark. The method for estimating $R^*_{t,2}$ is the same as in Sjölin [21, p. 139–140]. Here we have generalised it to other values of the involved parameters.

5.4. Remark. One might suggest to majorise the integral in e.g. (5.4) using Minkowski's and Hölder's inequality. It is straightforward to show that a necessary condition for such a majorisation is that the integral

$$\int_0^\infty |\chi(r)\, r^{-1/2+s+c/2-1/p^*}|^{p^*}\,dr$$

is convergent which happens if and only if

$$\tfrac{1}{2}c - \tfrac{1}{2} + s > 0. \tag{5.5}$$

However, as we have seen in Section 5.2.3, (5.5) is fulfilled in the case (2.2i) but not fulfilled in the case (2.2ii).

5.5. Proof of Theorem 2.7. We define

$$(\widetilde{S}^a f)[x](t) = |x|^{\delta(q)} \int_{\mathbf{R}^n} e^{i(x\xi+|t\xi|^a)}\, |\xi|^{-s}\, f(\xi)\,d\xi, \qquad t \in \mathbf{R}.$$

Let B^n we denote the open unit ball in $\mathbf{R}^n$. The theorem says that there is a number C independent of f in the Schwartz subclass of radial functions such that

$$\|\widetilde{S}^a f\|_{L^q(B^n,L^\infty(B))} \le C\|f_0\|_{L^2(\mathbf{R}_+)}, \tag{5.6}$$

where $f(\xi) = f_0(|\xi|)\,|\xi|^{-n/2+1/2}$.

According to Theorem 3.5

$$(\widetilde{S}^a f)[x](t) = (2\pi)^{n/2}\,|x|^{\delta(q)-n/2+1}\int_0^\infty J_{n/2-1}(|x|\rho)\,e^{it^a\rho^a}\,\rho^{-s}f_0(\rho)\,\rho^{1/2}\,d\rho.$$

Using polar coordinates we get that there is a number C independent of f such that

$$\|\widetilde{S}^a f\|_{L^q(L^\infty)} = C\left(\int_0^1 \sup_{t\in B}\left|\int_0^\infty r^{(1+c)/2-1/q}J_{n/2-1}(r\rho)\rho^{1/2}e^{it^a\rho^a}\rho^{-s}f_0(\rho)\,d\rho\right|^q dr\right)^{1/q}.$$

Here the right-hand side can be majorized by $C\|f_0\|_{L^2(\mathbf{R}_+)}$ where C is independent of f by Lemma 5.2. We have proved (5.6).

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