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FINELY HOLOMORPHIC AND FINELY SUBHARMONIC FUNCTIONS IN CONTOUR-SOLID PROBLEMS

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Abstract. We establish the purely fine contour-solid theory for finely holomorphic and finely hypoharmonic functions containing refined, strengthened and extended theorems for these classes of functions in finely open sets of the complex plane with preservable majorants (from the maximal classes of such majorants for these function classes). The work is based on various new arguments and on a new, unified approach common for finely hypoharmonic and finely holomorphic functions. We give also strengthened and extended results on cluster properties of holomorphic and finely holomorphic functions.

1. Introduction

Initial particular results in the contour-solid problems for usual holomorphic functions were given by G.H. Hardy, J.E. Littlewood, S. Warschawski, J.L. Walsh, W.E. Sewell and inspired the formulation of several open problems on the topic in the Sewell's monograph [Sew, p. 31–32] published in 1942.

In 1971 the author [T1] has completely solved these open problems (see also [T2], [T3] where strengthened and more general results were established as well). The obtained results found various applications in geometric and constructive function theory, in theory of singular integral operators, in boundary problems and in other topics. Further developments and related investigations were fulfilled by a number of scientists.

In 1983 the author had established [T4]–[T6] extended and refined contour-solid theorems for holomorphic (see also [T8], [T11]) and subharmonic (see also [T7], [T9], [T10], [T12]) functions in open sets of the complex plane $\mathbf{C}$ with arbitrary bilogarithmically concave majorants and their logarithms, respectively, and these classes of majorants are maximal classes of majorants preservable in the theorems for the above-mentioned function classes, respectively. In the papers mentioned above one can find references to earlier and other publications on the topics.

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For instance, we mention the following very particular case of one of local contour-solid results for holomorphic functions from [T4]–[T6], [T8], [T11] using notations from Section 4, see below. Let μ: $(0,+\infty)\to[0,+\infty)$ be a function for which the set $I^\mu := \{x : \mu(x) > 0\}$ is connected and the restriction of the function $\log\mu(x)$ to I^μ is concave with respect to $\log x$. For μ, let parameters μ_0, μ_∞, m_0 and m_∞ be defined as in Section 4.

Theorem. *Let $G\subset\mathbf{C}$ be an open set, $a\in\partial G$ be a fixed point; h be a holomorphic function in G, continuous up to the boundary subset $(\partial G)\setminus\{a\}$ with*

$$|h(z)|\le\mu(|z-a|)\quad \text{for all } z\in(\partial G)\setminus\{a\}.$$

Assume that $|h|$ is majorized by some rational functions in neighbourhoods of the point a and (if G is unbounded) of the point ∞. If a (respectively ∞) is an isolated boundary point of G, assume in addition that

$$\mu_0<+\infty,\qquad h(\zeta)=o(|\zeta-a|^{m_0-1})\qquad(\zeta\to a,\ \zeta\in G),$$

respectively

$$\mu_\infty>-\infty,\qquad h(\zeta)=o(|\zeta|^{m_\infty+1})\qquad(\zeta\to\infty,\ \zeta\in G).$$

Then one and only one of the following two possibilities holds: either the inequality

$$|h(\zeta)|\le\mu(|\zeta-a|)\qquad \text{for all } \zeta\in G,$$

or the following exceptional case:

$$G=\mathbf{C}\setminus\{a\},\quad \mu(x)=\beta x^m \text{ for all } x>0,\quad h(\zeta)=c(\zeta-a)^m \text{ for all } \zeta\in G,$$

m is an integer and $\beta\ge 0$, $c\in\mathbf{C}$ are constants with $|c|>\beta$.

In [TS1], [TS2] certain analogues of results from [T4], [T7], [T9], [T10], [T12] were proved for finely hypoharmonic functions, and in [TS3] some analogues of results from [T4]–[T6], [T8], [T11] were given for finely holomorphic functions in finely open sets on $\mathbf{C}$.

In the present work we establish refined, strengthened and extended contour-solid theorems for finely holomorphic and finely hypoharmonic functions in finely open sets of the complex plane with arbitrary bilogarithmically concave majorants and their logarithms, respectively. We give also strengthened and extended results on cluster properties of holomorphic and finely holomorphic functions.

While in contour-solid results of [TS1]–[TS3] some requirements relate to the standard, Euclidean topology, in results of this paper all requirements relate to the fine topology. So, we get purely fine contour-solid theory for finely holomorphic and finely hypoharmonic functions. One more difference is that in the present paper restrictions on functions concerning boundedness requirements and majorization are either avoided or essentially weakened. We have completely avoided the upper boundedness and finely inner majorization restrictions on functions (usual for related problems and assumed on certain parts of sets in earlier publications) and essentially relaxed the majorization requirements to their boundary limit values.

Concerning cluster properties of analytic functions, we mention classical results by F. Iversen and M. Tsuji (see [Tsu, p. 331–339]). On the basis of our contour-solid results of [T4]–[T6], we have established the substantial extension and generalization of the Iversen–Tsuji theorem for holomorphic functions (see [T13], [T14] and Theorem 9 below), and this has consequences for further cluster properties of meromorphic functions. We have also obtained fine analogues of these theorems.

The results of this work contain new assertions even for usual subharmonic and holomorphic functions in (standard) open sets.

The work is based on various new arguments and on a new, unified approach common for finely hypoharmonic and finely holomorphic functions, despite essential differences between results valid for these classes of functions. One of the tools is a new extended maximum principle for finely hypoharmonic functions free of any (upper) boundedness restrictions on the functions and of any global majorization requirements to them.

An essential part of this investigation was carried out at the Linköping University (see preprint [T15]) where I was on a kind invitation and under warm and encouraging hospitality of Professor Lars-Inge Hedberg, to whom I am greatly thankful.

Some results of this paper were announced in the author's talks at the 7-th International Colloquium on Finite or Infinite Dimensional Complex Analysis and at the Second ISAAC Congress (both held in August 1999 in Fukuoka, Japan) and are published without proofs in [T16] and [T17].

2. Some notions and notation

Let $\overbracket{\mathbf{C}}$ be the compact Riemann sphere.

We refer to [B], [F1]—[F7] concerning the fine topology and related notions such as thinness, the fine boundary and the fine closure of a set, fine limits of functions, fine superior and fine inferior limits of functions, finely hypoharmonic, finely hyperharmonic, finely subharmonic, finely harmonic, finely holomorphic functions, the Green's function for a fine domain and so on.

Let $E \subset \overbracket{\mathbf{C}}$. Denote by $\overbracket{\mathbf{E}}$ the standard closure of E in $\overbracket{\mathbf{C}}$, and by $\overline{E}$ the standard closure of a set $E \subset \mathbf{C}$ in $\mathbf{C}$. The set of all points $x \in \overbracket{\mathbf{C}}$ in which E is not thin is called *the base of the set E in $\overbracket{\mathbf{C}}$* and is denoted by $b(E)$. The set $\widetilde{E} := E \cup b(E)$ is called *the fine closure of the set E in $\overbracket{\mathbf{C}}$*. Clearly, $\widetilde{E} \subset \overbracket{\mathbf{E}}$. Denote by $\overbracket{\partial_f E}$ the fine boundary of E in $\overbracket{\mathbf{C}}$. Let $\partial_f E := \mathbf{C} \cap \overbracket{\partial_f E}$,

$(E)_i := E \setminus b(E)$, $(E)_r := E \setminus (E)_i$. Points $x \in (E)_r$ and $x \in (E)_i$ are called *regular* and *irregular* points, respectively, of the set E.

For a set $E \subset \overline{\mathbf{C}}$ let us denote $\overline{\mathbf{C}} \setminus E =: FE$, and for a set $E \subset \mathbf{C}$, denote also $\mathbf{C} \setminus E =: CE$.

Let $G \subset \overline{\mathbf{C}}$ be a finely open set.

Let $z \in G$. If the set FG is non-polar, then we denote by ω_z^G the generalized harmonic measure relative to G and z $(\in G)$, see [F1, p. 1]. If FG is polar, then we introduce the harmonic measure relative to G and $z \in G$ by the equality

$$\omega_z^G = 0.$$

Notice that if FG is non-polar, then $\omega_z^G \neq 0$. In any case for ω_z^G we use also the term *the harmonic measure relative to G at the point $z \in G$.*

If z belongs to a finely connected component T of G, Y is a subset of the set FT and $\omega_z^T(Y) = 0$, then $\omega_\zeta^T(Y) = 0$ at every point $\zeta \in T$ (see [F1, pp. 150–151]).

A set $Q \subset FG$ will be called *nearly negligible relative to G* if for every finely connected component T of G the set $Q \cap \partial_f T$ contains no compact subset K of the harmonic measure $\omega_z^T(K) > 0$ at some (and therefore at any) point $z \in T$. This requirement is equivalent to the following alternative: either FG is polar (and then Q is also polar), or for every finely connected component T of G both $\partial_f T$ is non-polar and Q is a set of inner harmonic measure zero relative to T and any point $z \in T$.

If a set $E \subset FG$ is such that for every finely connected component T of G it contains no compact subset $K \subset \partial_f T$ of logarithmic capacity $\operatorname{Cap} K > 0$, then E is nearly negligible relative to G.

In particular, any set $E \subset FG$ of inner logarithmic capacity zero is nearly negligible relative to G.

If T is a finely connected component of G and $z \in T$, then $\omega_z^G = \omega_z^T$.

Let D be a fine domain in $\overline{\mathbf{C}}$, i.e. a finely open, finely connected set. Then $\widetilde{D} = \overline{D}$. In particular, then D is finely separable from a point $z \in \overline{\mathbf{C}}$ if and only if it is separable from z in the standard topology. So in such a situation we may speak of separability not specifying in what sense.

Let G be a finely open set in $\overline{\mathbf{C}}$, and $z \in \overline{\partial_f G}$. Given any functions $u\colon G \to [-\infty, +\infty]$ and $h\colon G \to \overline{\mathbf{C}}$, we introduce the following notation for fine superior limits of functions:

$$\text{fine}\limsup_{\zeta\to z,\, \zeta\in G} u(\zeta) =: (u)\check{}_{G,f}(z) =: \check{u}_{G,f}(z), \tag{2.1}$$

$$\text{fine}\limsup_{\zeta\to z,\, \zeta\in G} |h(\zeta)| =: \bar{h}_{G,f}(z). \tag{2.2}$$

Remark 1. Let $X \subset \overline{\mathbf{C}}$ and $Y \subset \overline{\mathbf{C}}$ be non-empty sets. For a function $v\colon X \to [-\infty, +\infty)$ let us consider two requirements:

(a) v is upper bounded on every part of X finely separable from Y;

(b) v is upper bounded on every part of X separable (in standard topology) from Y.

The requirement (a) is more restrictive than (b). It is the reason why in our results we use (when possible) restrictions of the type (b) rather than of the type (a). Such results are valid with any of these types of restrictions, but with (b) they are more general.

3. Maximum principle for finely hypoharmonic functions

In this work we prove and use the following maximum principle for finely hypoharmonic functions that is free of any (upper) boundedness restrictions on them, as well as of any global majorization requirements.

Lemma 1. *Let $D \subset \overbracket{\mathbf{C}}$ be a finely open set, $E \subset FD$ be a set nearly negligible relative to $D, (\overbracket{\partial_f D}) \setminus E \neq \emptyset; u$ be a function finely hypoharmonic in D with*

$$\check{u}_{D,f}(z) < +\infty \quad \text{for all } z \in \overbracket{\partial_f D}, \tag{3.1}$$

$$\check{u}_{D,f}(z) \leq 0 \quad \text{for all } z \in (\overbracket{\partial_f D}) \setminus E. \tag{3.2}$$

Then $u \leq 0$ in D.

In R^n for $n \geq 3$ such a result is not true.

Remark 2. Under the additional requirement that E is a (polar) set of irregular fine boundary points of D, a particular case of Lemma 1 was established in [F2, p. 82] (c.f. results of [F3] where u in D is assumed to be upper bounded or majorized by certain potential).

4. Contour-solid theorems for finely holomorphic functions

In [T4], [T5] (see also [T6], [T8], [T11]) we had introduced the following notions. Let $\mathfrak{M}$ be the class of all functions μ: $(0, +\infty) \to [0, +\infty)$ for each of which the set $I^\mu := \{x : \mu(x) > 0\}$ is connected and the restriction of the function $\log \mu(x)$ to I^μ is concave with respect to $\log x$. Let $\mathfrak{M}^*$ be the class of all $\mu \in \mathfrak{M}$ for which I^μ is non-empty.

For $\mu \in \mathfrak{M}^*$ let us denote by x_-^μ and x_+^μ the left and the right ends of the interval I^μ, respectively. Obviously, $0 \leq x_-^\mu \leq x_+^\mu \leq +\infty$. When $x_-^\mu < x_+^\mu$, the concavity condition is equivalent to the combination of the following conditions: the function $\log \mu(x)$ is concave with respect to $\log x$ (and therefore continuous) in the interval (x_-^μ, x_+^μ) and lower semicontinuous on I^μ. For $\mu \in \mathfrak{M}$ the limits

$$\mu_0 := \lim_{x \to 0} \frac{\log \mu(x)}{\log x}, \qquad \mu_\infty := \lim_{x \to +\infty} \frac{\log \mu(x)}{\log x} \tag{4.1}$$

exist, and we have

$$\mu_0 \geq \mu_\infty, \qquad \mu_0 > -\infty, \qquad \mu_\infty < +\infty.$$

In particular, if $x_-^\mu > 0$ (analogously, if $x_+^\mu < +\infty$), then $\mu_0 = +\infty$ ($\mu_\infty = -\infty$, respectively). When $\mu_0 < +\infty$, define the integer m_0 by the conditions $m_0 - 1 < \mu_0 \leq m_0$, and when $\mu_\infty > -\infty$, define the integer m_∞ by the conditions $m_\infty \leq \mu_\infty < m_\infty + 1$.

For every fixed $\alpha \in \mathbf{R}$, $\beta \in (0, +\infty)$, the function $\mu(x) := \beta x^\alpha$ belongs to $\mathfrak{M}^*$, and for it we have $\mu_0 = \mu_\infty = \alpha$. If, moreover, α is an integer, then $m_0 = m_\infty = \alpha$.

Now we introduce fine analogues of some notions from [T4], [T5], [T8], [T11].

Let G be a finely open set in $\mathbf{C}$ and $a \in CG$ be a fixed point. For a function $h: G \to \mathbf{C}$ let us denote:

$$h_{a,G,f} := \begin{cases} \operatorname{fine} \limsup_{\zeta \to a,\, \zeta \in G} \dfrac{\log|h(\zeta)|}{|\log|\zeta - a||} & \text{when } a \in \partial_f G, \\ 0 & \text{when } a \notin \partial_f G, \end{cases} \tag{4.2}$$

$$h_{\infty,G,f} := \begin{cases} \operatorname{fine} \limsup_{\zeta \to \infty,\, \zeta \in G} \dfrac{\log|h(\zeta)|}{\log|\zeta|} & \text{when } \infty \in \overline{\partial_f} G, \\ 0 & \text{when } \infty \notin \overline{\partial_f} G. \end{cases} \tag{4.3}$$

Given functions $p: X \to \overline{\mathbf{C}}$ and $q: X \to \mathbf{C}$ on a set $X \subset \overline{\mathbf{C}}$ and a finely limit point w for X, we use the following notation. If there exist a finite number $l \geq 0$ and a fine neighbourhood U of w for which $|p(z)| \leq l|q(z)|$ for all $z \in X \cap U$, we write

$$p(z) = \operatorname{fine} O\big(q(z)\big) \qquad (z \to w,\ z \in X),$$

and if for every $\varepsilon > 0$ there exists a fine neighbourhood U of w for which $|p(z)| \leq \varepsilon|q(z)|$ for all $z \in X \cap U$, then we write

$$p(z) = \operatorname{fine} o\big(q(z)\big) \qquad (z \to w,\ z \in X).$$

4.1. Local results. Let $G \subset \mathbf{C}$ be a finely open set, $h: G \to \mathbf{C}$ be a finely holomorphic function, and $\mu \in \mathfrak{M}$. Consider the following conditions:

(A, ∞) $\infty \in b(CG)$ and for every finely connected component T of G with $\infty \in b(T)$ there holds $h_{\infty,T,f} < +\infty$;

(B, ∞) $\infty \notin b(CG)$, $\mu_\infty > -\infty$ and

$$h(\zeta) = \operatorname{fine} o(|\zeta|^{m_\infty + 1}) \qquad (\zeta \to \infty,\ \zeta \in G); \tag{4.1.1}$$

(B_0, ∞) $\infty \notin b(CG)$, $\mu_\infty \geq 0$ and (4.1.1) is true.

If $z \in \mathbf{C}$ is a fixed point, then we consider also the following conditions:
(A, z) $z \in b(CG)$ and for every finely connected component T of G with $z \in b(T)$ there holds $h_{z,T,f} < +\infty$;
(B, z) $z \notin b(CG)$, $\mu_0 < +\infty$ and

$$h(\zeta) = \operatorname{fine} o(|\zeta - z|^{m_0 - 1}) \qquad (\zeta \to z,\ \zeta \in G); \tag{4.1.2}$$

$(\mathrm{B}_1, \mathrm{z})$ $z \notin b(CG)$, $\mu_0 \le 1$ and (4.1.2) is true.

Using this notation and (2.2), we get the following statement.

Theorem 1. *Let $a \in \mathbf{C}$ be a fixed point; $G \subset \mathbf{C} \setminus \{a\}$ be a finely open set; $\mu \in \mathfrak{M}$; $h\colon G \to \mathbf{C}$ be a finely holomorphic function for which*

$$\bar{h}_{G,f}(z) \le \mu(|z - a|) \qquad \text{for all } z \in (\partial_f G) \setminus \{a\}. \tag{4.1.3}$$

Denote $z_1 := a$, $z_2 := \infty$ and suppose that for each $s = 1, 2$ (independently from each other) one of the conditions (A, z_s) or (B, z_s) is satisfied. Then one and only one of the following two possibilities holds: either the inequality

$$|h(\zeta)| \le \mu(|\zeta - a|) \qquad \text{for all } \zeta \in G, \tag{4.1.4}$$

or the following exceptional case:

$$G = \mathbf{C} \setminus \{a\}, \quad \mu(x) = \beta x^m \text{ for all } x > 0, \quad h(\zeta) = c(\zeta - a)^m \text{ for all } \zeta \in G,$$

m is an integer and $\beta \ge 0$, $c \in \mathbf{C}$ are constants with $|c| > \beta$.

Remark 3. For every fixed $s = 1, 2$ the following statements are true.

(a) If there exists a finely connected component T of G with $z_s \in b(T)$, then the conditions $z_s \in b(CT)$ and $z_s \in b(CG)$ are equivalent.

(b) $z_s \in b(CG)$ if and only if for every finely connected component T of G we have $z_s \in b(CT)$.

(c) If $z_s \notin b(CG)$, then there exists one and only one finely connected component T of G with $z_s \in b(T)$, and for this T we have $z_s \notin b(CT)$.

(d) The condition $(\mathrm{B}, \mathrm{z_s})$ implies the inequality $h_{z_s,G,f} < +\infty$.

(e) If $z_s \notin b(CG)$, then the condition $(\mathrm{B}, \mathrm{z_s})$ in Theorem 1 may not be omitted, and $\operatorname{fine} o(\,\cdot\,)$ in it may not be replaced by $\operatorname{fine} O(\,\cdot\,)$.

Theorem 1 is a particular case of the following extended statement.

Theorem 2. *Let $a \in \mathbf{C}$ be a fixed point; $G \subset \mathbf{C} \setminus \{a\}$ be a finely open set; Q be a set contained in $\overbracket{\partial_f G}$ and containing the points a and ∞; $\mu \in \mathfrak{M}$; $h\colon G \to \mathbf{C}$ be a finely holomorphic function. Suppose that for every finely connected component T of G the following conditions are satisfied:*

$$\bar{h}_{T,f}(z) < +\infty \qquad \text{for all } z \in (\partial_f T) \setminus \{a\}; \tag{a2}$$

(b2) $$\bar{h}_{T,f}(z) \le \mu(|z-a|) \qquad \text{for all } z \in (\partial_f T) \setminus Q;$$

(c2) *Q is nearly negligible relative to T;*

(d2) $$h_{a,T,f} < +\infty, \qquad h_{\infty,T,f} < +\infty.$$

Then the function $h(\zeta)$ is bounded both on every part of G separable from a and ∞, and on every finely connected part of G finely separable from a and ∞.

Denote $z_1 := a$, $z_2 := \infty$ and suppose that for each $s = 1, 2$ (independently from each other) one of the conditions (A, z_s) or (B, z_s) is satisfied.

Under these assumptions one and only one of the following two possibilities holds—either (4.1.4), or the following exceptional case: $Q = \overline{\mathbf{C}} \setminus G, Q$ is polar,

$$\mu(x) = \beta x^m \quad \text{for all } x > 0, \quad h(\zeta) = c(\zeta - a)^m \quad \text{for all } \zeta \in G,$$

m is an integer and $\beta \ge 0$, $c \in \mathbf{C}$ are constants with $|c| > \beta$.

4.2. Global results.

Theorem 3. *Let $G \subset \mathbf{C}$ be a finely open set; $\mu \in \mathfrak{M}$; h: $\widetilde{G} \cap \mathbf{C} \to \mathbf{C}$ be a function finely holomorphic in G and satisfying the condition*

(4.2.1) $$|h(\zeta) - h(z)| \le \mu(|\zeta - z|) \qquad \text{for all } z, \zeta \in \partial_f G,\ z \ne \zeta.$$

Let one of the conditions (A, ∞) or (B_0, ∞) be satisfied for the restriction of h onto G (instead of h). Suppose also that for every finely connected component T of G the restriction of h onto $\widetilde{T} \cap \mathbf{C}$ is finely continuous.

Under these assumptions we have

(4.2.2) $$|h(\zeta) - h(z)| \le \mu(|\zeta - z|) \qquad \text{for all } z \in (\partial_f G)_r, \text{ for all } \zeta \in \widetilde{G} \cap \mathbf{C},\ z \ne \zeta.$$

Remark 4. Theorem 3 remains true if we require the inequality in (4.2.1) to be valid only for $z \in (\partial_f G)_r$ (not for all $z \in \partial_f G$).

Theorem 4. *Let G, μ, h satisfy all assumptions of Theorem 3. Additionally suppose that $z_0 \in (CG)_i \cup G$ is a fixed point, $\mu_0 < +\infty$ and*

(4.2.3) $$|h(\zeta) - h(z_0)| = \operatorname{fine} o(|\zeta - z_0|^{m_0 - 1}) \qquad (\zeta \to z_0,\ \zeta \in G).$$

Then one and only one of the following two possibilities holds: either the inequality

(4.2.4) $$|h(\zeta) - h(z_0)| \le \mu(|\zeta - z_0|) \quad \text{for all } \zeta \in \widetilde{G} \setminus \{z_0, \infty\},$$

or the following exceptional case:

$$(\partial_f G)_r = \emptyset, \quad \mu(x) = \beta x^m \text{ for all } x > 0, \quad h(\zeta) = c(\zeta - z_0)^m + b \text{ for all } \zeta \in \mathbf{C},$$

$m \ge 1$ is an integer and $\beta \ge 0$, $c \in \mathbf{C}$, $b \in \mathbf{C}$ are constants with $|c| > \beta$; when $m = 1$ or $z_0 \in (CG)_i$, then the set CG contains at most one point and

$$|h(\zeta) - h(z_0)| > \mu(|\zeta - z_0|) \quad \text{for all } \zeta \in \mathbf{C}.$$

From Theorems 3 and 4 the following global contour-solid result follows.

Theorem 5. *Let G, μ, h satisfy all assumptions of Theorem* 3. *Additionally suppose that $\mu_0 \le 1$. Then one and only one of the following two possibilities holds*: *either the inequality*

$$|h(\zeta) - h(z)| \le \mu(|\zeta - z|) \qquad \text{for all } z, \zeta \in \widetilde{G} \cap \mathbf{C},\ z \ne \zeta, \tag{4.2.5}$$

or the following exceptional case: *CG contains at most one point,*

$$\mu(x) = \beta x \qquad \text{for all } x > 0,\ h(\zeta) = c\,\zeta + b \quad \text{for all } \zeta \in \mathbf{C}$$

and $\beta \ge 0$, $c \in \mathbf{C}$, $b \in \mathbf{C}$ are constants with $|c| > \beta$.

In the next theorem the requirement $\mu_0 \le 1$ is omitted, but instead a certain fine continuity condition for the function $h(\zeta)$ is assumed at ∞.

Theorem 6. *Let G, μ, h satisfy all assumptions of Theorem* 3. *If $\infty \in \widetilde{G}$, additionally suppose that h can be extended to ∞ in such a way that the extended function $\tilde{h}$ satisfies the following hypothesis*: *the restriction $\tilde{h}|_{\widetilde{T}}$ of $\tilde{h}$ to $\widetilde{T}$ is finely continuous* (*and finite*) *at ∞ for every finely connected component T of G with $\infty \in \widetilde{T}$. Under these assumptions the inequality* (4.2.5) *is true.*

4.3. Difference maximum principle for finely holomorphic functions. We establish and use the following maximum principle for the difference of values of a finely holomorphic function which is a purely fine analogue and refinement of the corresponding results from [T5], [T11], [TS3].

Lemma 2. *Let $D \subset \overline{\mathbf{C}}$ be a finely open set*; *p: $\widetilde{D} \to \mathbf{C}$ be a non-constant function, finely holomorphic in D and satisfying the following assumption*: *the restriction $p|_{\widetilde{T}}$ of p to $\widetilde{T}$ is finely continuous for every finely connected component T of D. Then the set $(\partial_f D)_r$ is non-empty and for every $\delta > 0$ we have*

$$\sup_{z \in (\partial_f D)_r} \ \sup_{\zeta \in \widetilde{D},\, |\zeta - z| = \delta} |p(\zeta) - p(z)| = \sup_{\zeta, z \in \widetilde{D},\, |\zeta - z| = \delta} |p(\zeta) - p(z)|. \tag{4.3.1}$$

5. Contour-solid theorems for finely hypoharmonic functions

5.1. In [T7] (see also [T9], [T10], [T12]) we had introduced the following notions. Let L be the class of all functions λ: $(0, +\infty) \to [-\infty, +\infty)$ for each of which the set $I_\lambda := \{x : \lambda(x) > -\infty\}$ is connected and the restriction of λ to I_λ is concave with respect to $\log x$. Let L^* be the class of all $\lambda \in L$ for which I_λ is non-empty. For $\lambda \in L^*$ let us denote by x_λ^- and x_λ^+, respectively, the left and the right ends of the interval I_λ. Obviously, $0 \le x_\lambda^- \le x_\lambda^+ \le +\infty$. When $\lambda(\,\cdot\,)$ runs through the classes L or L^*, the function $\exp \lambda(\,\cdot\,)$ runs through the classes $\mathfrak{M}$ or $\mathfrak{M}^*$, respectively. When $x_\lambda^- < x_\lambda^+$, the concavity condition is equivalent to the combination of the following conditions: the function $\lambda(x)$ is concave with respect to $\log x$ (and therefore continuous) in the interval $(x_\lambda^-, x_\lambda^+)$ and lower semicontinuous on I_λ. For $\lambda \in L$ the limits

$$\lambda^0 := \lim_{x\to 0} \frac{\lambda(x)}{\log x}, \qquad \lambda^\infty := \lim_{x\to +\infty} \frac{\lambda(x)}{\log x} \tag{5.1}$$

exist, and we have

$$\lambda^0 \geq \lambda^\infty, \qquad \lambda^0 > -\infty, \qquad \lambda^\infty < +\infty.$$

For every fixed $\alpha \in \mathbf{R}, \theta \in \mathbf{R}$ the function $\lambda(x) := \alpha \log x + \theta$ belongs to L^*, and for it we have $\lambda_0 = \lambda_\infty = \alpha$.

Let G be a finely open set in $\mathbf{C}$, and u be a function finely hypoharmonic in G. Then for $\zeta \in G$ we denote

$$\gamma_G(u,\zeta) := \inf_v \{v(\zeta) : v \text{ finely hyperharmonic in } G,\ v \geq u \text{ in } G\}.$$

Clearly, $u(\cdot) \leq \gamma_G(u,\cdot)$. If $\gamma_G(u,\cdot) < +\infty$, then on the basis of Lemma 10.3 and Theorems 9.14 and 12.9 from [F1, pp. 103–104, 96, 158] one can show that $\gamma_G(u,\zeta)$ is finely hypoharmonic in G, and for every finely connected component T of G the following alternative holds: either $\gamma_G(u,\zeta) = u(\zeta) = -\infty$ in T, or the functions $\gamma_G(u,\zeta)$ and $u(\zeta)$ are finely subharmonic in T and $\gamma_G(u,\zeta)$ is finely harmonic on the set $\{\zeta \in T : \gamma_G(u,\zeta) \neq -\infty\}$.

Let $a \in CG$ be a fixed point.

Now we are going to introduce the following fine analogues of some notions from [T4], [T7], [T9], [T10], [T12].

Let us denote

$$u^a_{G,f} := \begin{cases} \operatorname{fine\,lim\,sup}_{\zeta\to a,\,\zeta\in G} \dfrac{u(\zeta)}{|\log|\zeta-a||} & \text{when } a \in \partial_f G, \\ 0 & \text{when } a \notin \partial_f G, \end{cases} \tag{5.2}$$

$$u^\infty_{G,f} := \begin{cases} \operatorname{fine\,lim\,sup}_{\zeta\to \infty,\,\zeta\in G} \dfrac{u(\zeta)}{\log|\zeta|} & \text{when } \infty \in \overline{\partial_f} G, \\ 0 & \text{when } \infty \notin \overline{\partial_f} G, \end{cases} \tag{5.3}$$

$$M_{G,a}(u,r) := \inf_{p,q} \{p\log r + q :\ p \in (-\infty,+\infty),\ q \in (-\infty,+\infty],$$
$$u(\zeta) \leq p \log|\zeta - a| + q \ \text{ for all } \zeta \in G\} \qquad (0 < r < +\infty).$$

Obviously, the following alternative holds: either

$$M_{G,a}(u,r) = +\infty \qquad \text{for all } r > 0,$$

or

$$M_{G,a}(u,r) < +\infty \quad \text{for all } r > 0.$$

If the latter case holds true, the function $M_{G,a}(u,r)$ of $r > 0$ belongs to the class L. Suppose that $M_{G,a}(u,\cdot\,) \not\equiv -\infty$, and denote by r^- and r^+ the left and the right ends of the maximal interval where this function is $> -\infty$. Then the function $\zeta \mapsto M_{G,a}(u,|\zeta - a|)$ is superharmonic under $r^- < |\zeta - a| < r^+$, and $M_{G,a}(u,|\zeta - a|) = -\infty$ under $|\zeta - a| < r^-$ and under $|\zeta - a| > r^+$. Furthermore, in this case u has in G a finely harmonic majorant and

$$u(\zeta) \le \gamma_G(u,\zeta) \le M_{G,a}(u,|\zeta - a|) \qquad \text{for all } \zeta \in G. \tag{5.4}$$

These inequalities are valid also in the case when $M_{G,a}(u,|\zeta - a|) \equiv -\infty$.

Denote

$$\varrho^- := \inf_{\zeta\in G} |\zeta|, \qquad \varrho^+ := \sup_{\zeta\in G} |\zeta|.$$

Then

$$\varrho^- \le r^- \le r^+ \le \varrho^+.$$

Let $\lambda \in L$. Consider the following conditions:

(A',∞) $\infty \in b(CG)$ and for every finely connected component T of G with $\infty \in b(T)$ there holds $u^{\infty}_{T,f} < +\infty$;

(B',∞) $\infty \notin b(CG)$ and there exist a constant $t \in \mathbf{R}$ and a fine neighbourhood V of ∞ for which

$$u(\zeta) \le \lambda(|\zeta - a|) + t \qquad \text{for all } \zeta \in G \cap V; \tag{5.5}$$

(A',a) $a \in b(CG)$ and for every finely connected component T of G with $a \in b(T)$ there holds $u^{a}_{T,f} < +\infty$;

(B',a) $a \notin b(CG)$ and there exist a constant $t \in \mathbf{R}$ and a fine neighbourhood V of a for which

$$u(\zeta) \le \lambda(|\zeta - a|) + t \qquad \text{for all } \zeta \in G \cap V. \tag{5.6}$$

5.2. Using this notation and (2.1), we get the following result.

Theorem 7. *Let $a \in \mathbf{C}$ be a fixed point; $G \subset \mathbf{C} \setminus \{a\}$ be a finely open set; $\lambda \in L$; $u\colon G \to [-\infty,+\infty)$ be a finely hypoharmonic function for which*

$$\check{u}_{G,f}(z) \le \lambda(|z - a|) \quad \text{for all } z \in (\partial_f G) \setminus \{a\}.$$

Denote $z_1 := a$, $z_2 := \infty$ and suppose that for each $s = 1,2$ (independently from each other) one of the conditions (A',z_s) or (B',z_s) is satisfied. Then u has in G

a finely harmonic majorant, and one and only one of the following two possibilities holds true: either the estimates

$$u(\zeta) \le \gamma_G(u,\zeta) \le \lambda(|\zeta - a|) \qquad \text{for all } \zeta \in G, \tag{5.7}$$

or the following exceptional case: $G = \mathbf{C} \setminus \{a\}$ *and*

$$u(\zeta) = \gamma_G(u,\zeta) = M_{G,a}(u, |\zeta - a|) = \nu \log|\zeta - a| + t \qquad \text{for all } \zeta \in G, \tag{5.8}$$

$$\lambda(x) = \nu \log x + l \qquad \text{for all } x > 0 \tag{5.9}$$

with constants $\nu, t \in \mathbf{R}$, $l \in [-\infty, t)$.

Moreover, if the exceptional case is not valid, then one of the following two possibilities holds true: either

$$M_{G,a}(u,r) = -\infty \qquad \text{for all } r > 0, \tag{5.10}$$

or $\lambda \in L^*$, $\quad x_\lambda^- < x_\lambda^+$ *and*

$$u(\zeta) = \gamma_G(u,\zeta) = -\infty \qquad \text{for all } \zeta \in G : |\zeta - a| \notin (x_\lambda^-, x_\lambda^+), \tag{5.11}$$

$$M_{G,a}(u,r) \le \lambda(r) \qquad \text{for all } r : x_\lambda^- \ne r \ne x_\lambda^+, \tag{5.12}$$

$$M_{G,a}(u,r) = -\infty \qquad \text{for all } r \notin [x_\lambda^-, x_\lambda^+]. \tag{5.13}$$

Remark 5. For every fixed $s = 1, 2$ the following statements are true.

(a)–(c) The same as (a)–(c) in Remark 3.

(d) The condition (B', z_s) implies the inequality $u_{G,f}^{z_s} < +\infty$.

(e) If $z_s \notin b(CG)$, then the condition (B', z_s) in Theorem 7 may not be omitted.

(f) Now suppose that there exist more than one finely connected components T of G with $z_s \in b(T)$. Then for every such T we have $z_s \in b(CT)$, and in Theorem 7 in this case we may assume the condition $u_{T,f}^{z_s} < +\infty$ for all such T instead of the alternative containing the conditions (A', z_s) and (B', z_s).

Theorem 7 is contained in the following more general and sharper theorems.

Theorem 8.1. *Let* $a \in \mathbf{C}$ *be a fixed point;* $G \subset \mathbf{C} \setminus \{a\}$ *be a finely open set;* Q *be a set contained in* $\overset{\frown}{\partial_f} G$ *and containing the points* a *and* ∞; $\lambda \in L$; $u \colon G \to [-\infty, +\infty)$ *be a finely hypoharmonic function. Suppose that for every finely connected component* T *of* G *the following conditions are satisfied:*

(a8) $\qquad \check{u}_{T,f}(z) < +\infty \qquad$ *for all* $z \in (\partial_f T) \setminus \{a\}$;

(b8) $\qquad \check{u}_{T,f}(z) \le \lambda(|z - a|) \qquad$ *for all* $z \in (\partial_f T) \setminus Q$;

(c8) Q *is nearly negligible relative to* T;

(d8) $\qquad u_{T,f}^{a} < +\infty, \qquad u_{T,f}^{\infty} < +\infty.$

Then $u(\zeta)$ *is upper bounded both on every part of* G *separable from* a *and* ∞, *and on every finely connected part of* G *finely separable from* a *and* ∞.

For G, a, u, λ under consideration and $s=1,2$ we introduce the quantities $\sigma_f^s=\sigma_f^s(G,a,u,\lambda)$ defined by the following conditions. If $\lambda\in L^*$, then we denote

$$\sigma_f^1 := \begin{cases} \big(u(\,\cdot\,)-\lambda(|\cdot-a|)\big)^a_{G,f} & \text{when } x_\lambda^- = 0, \\ 0 & \text{when } x_\lambda^- > 0, \end{cases} \tag{5.14}$$

$$\sigma_f^2 := \begin{cases} \big(u(\,\cdot\,)-\lambda(|\cdot-a|)\big)^\infty_{G,f} & \text{when } x_\lambda^+ = +\infty, \\ 0 & \text{when } x_\lambda^+ < +\infty. \end{cases} \tag{5.15}$$

If $\lambda\equiv-\infty$, then we assume

$$\sigma_f^1=\sigma_f^2=0. \tag{5.16}$$

For $\lambda\in L^*$ the following equalities are valid: if $\lambda^0\neq+\infty$, then $\sigma_f^1=u^a_{G,f}+\lambda^0$, and if $\lambda^\infty\neq-\infty$, then $\sigma_f^2=u^\infty_{G,f}-\lambda^\infty$.

Theorem 8.2. *Let $a\in\mathbf{C}$ be a fixed point; $G\subset\mathbf{C}\setminus\{a\}$ be a finely open set; Q be a set contained in $\overline{\partial_f G}$ and containing the points a and ∞; $\lambda\in L$; $u\colon G\to[-\infty,+\infty)$ be a finely hypoharmonic function. Suppose that for every finely connected component T of G the conditions* (a8)–(c8) *of Theorem* 8.1 *are satisfied. Denote $z_1:=a$, $z_2:=\infty$ and suppose that for each $s=1,2$ (independently from each other) one of the conditions (A,z_s) or (B,z_s) is valid. Then u has in G a finely harmonic majorant, and one and only one of the following two possibilities holds true: either the estimates* (5.7) *and*

$$-\infty\le\sigma_f^1\le 0,\qquad -\infty\le\sigma_f^2\le 0, \tag{5.17}$$

or the following exceptional case: $Q=\overline{\mathbf{C}}\setminus G$, Q is polar and (5.8), (5.9) *hold with constants $\nu,t\in\mathbf{R}$, $l\in[-\infty,t)$.*

Theorem 8.3. *Let all assumptions of Theorem* 8.2 *be satisfied, but its exceptional case and* (5.10) *be not valid. Then $\lambda\in L^*$, $x_\lambda^-<x_\lambda^+$ and* (5.11)–(5.13) *are true. Moreover, for every finely connected component T of G we have either*

$$u(\zeta)=\gamma_G(u,\zeta)=-\infty\qquad \textit{for all } \zeta\in T, \tag{5.18}$$

or

$$\lambda(|\zeta-a|)>-\infty\qquad \textit{for all } \zeta\in T. \tag{5.19}$$

For any finely open set $G\subset\mathbf{C}$ with a non-polar complement CG, and $w\in\overline{\mathbf{C}}$, $\zeta\in G$, $w\neq\zeta$, there exists the Green's function $g_G(w,\zeta)$, and it satisfies the equality

$$g_G(w,\zeta)-g_G(\infty,\zeta)=\int\log\left|\frac{w-z}{w-\zeta}\right|\,d\omega^G_\zeta(z).$$

Let us make the following agreements concerning possible indefinite commutative expressions:

$$\infty \cdot 0 = 0, \qquad -\infty + \infty = -\infty. \tag{5.20}$$

Theorem 8.4. *Let the assumptions and notation of Theorem 8.2 be valid. Suppose that CG is non-polar. Then for every finely connected component T of G we have*

$$\gamma_G(u,\zeta) \le \lambda(|\zeta - a|) + \sum_{s=1}^{2} \sigma_f^s g_G(z_s,\zeta) \qquad \text{for all } \zeta \in T, \tag{5.21}$$

$$M_{G,a}\Big(u(\,\cdot\,) - \sum_{s=1}^{2} \sigma_f^s g_G(z_s,\,\cdot\,), |\zeta - a|\Big) \le \lambda(|\zeta - a|) \qquad \text{for all } \zeta \in T. \tag{5.22}$$

If there is a point z_s not in $b(CT)$ and $\sigma_f^s = -\infty$ for it, then (5.18) *is true.*

Theorems 1–8.4 are purely fine refinements and extensions of results from [T4]–[T12], [TS1]–[TS3]. In particular, in our present notation, results given in [TS1]–[TS3] were established under additional requirements among which there were the following: the restrictions onto a finely hypoharmonic function u to be upper bounded on each part of G separable from the points ∞ and a, see [TS1], [TS2]; the restriction onto a finely holomorphic function h to be bounded on each part of G separable from the points ∞ and a, see Theorem 2 of [TS3] which is the most general result of that work (the latter restriction concern also Theorem 1 of [TS3] which is a particular case of Theorem 2 of that paper, but the formulation of Theorem 1 of [TS3] missed out the restriction mentioned above).

We use the term *solid inequalities* for naming the estimates for functions in G and on $\widetilde{G}$ given by theorems of the type under consideration. The question about the equality sign in solid inequalities at a finely inner point of G is treated in [TS2], [TS3] (see also [T7], [T9]–[T12] concerning the same questions for holomorphic and hypoharmonic functions in standard open sets).

6. On assumptions of contour-solid theorems and examples

Up to what extent are the assumptions of the above theorems essential and needed? To get an idea on the answer to this question, we may restrict ourselves by the following statements concerning holomorphic functions in open sets (see [T4]–[T6], [T8], [T11]).

Let μ: $(0,+\infty) \to [0,+\infty)$ be an arbitrary function. If for such a μ the inequality (4.1.4) of Theorem 1 holds with every G, a and h under consideration, then $\mu \in \mathfrak{M}$ (see [T11] and a reference given there). It is the case even if we consider only disks G or points $a \in \partial G$.

This shows that the assumption $\mu \in \mathfrak{M}$ is not only sufficient in Theorems 1 and 2, but it is also necessary for them to be valid. Hence $\mathfrak{M}$ is the natural, maximal class of majorants for the problems in question.

Denote $G_0 := \{\zeta : 0 < |\zeta| < 1\}$, $G_0' := \{\zeta \in G_0,\ \arg\zeta \neq \pi\}$, $G_\infty := \{\zeta : 1 < |\zeta| < +\infty\}$, $G_\infty' := \{\zeta \in G_\infty,\ \arg\zeta \neq \pi\}$. Let us consider the following examples in which $\beta > 0$ is a constant.

Example 1. $G := G_0'$, $\mu(x) := \beta x$.

Example 2. $G := G_\infty'$, $\mu(x) := \beta x$.

Example 3. $G := G_0$, $\mu(x) := \beta x^m$, m is an integer.

Example 4. $G := G_\infty$, $\mu(x) := \beta x^m$, m is an integer.

For Example 1 (similarly, Example 2) with $\beta := e$, $a := 0$ and the function $h(\zeta) := e^{1/\zeta}(h(\zeta) := e^\zeta$, respectively) the assertion of Theorem 1 fails because of the equality $h_{a,G,f} = +\infty$ ($h_{\infty,G,f} = +\infty$, respectively), while all other assumptions of this theorem are satisfied.

For Example 3 (similarly, Example 4) with $\beta := 1$, $a := 0$ and the function $h(\zeta) := \zeta^{m-1}$ ($h(\zeta) := \zeta^{m+1}$, respectively) the assertion of Theorem 1 fails because of the lack of the assumption (4.1.2) for $z = 0$ ((4.1.1) for $z = \infty$, respectively), while all other assumptions of this theorem are fulfilled and instead of the condition (4.1.2) ((4.1.1), respectively) a similar condition with the replacement of fine $o(\,\cdot\,)$ by $O(\,\cdot\,)$ holds true.

Concerning Theorem 3 we state the following. The assumption (4.1.1) may not be omitted from the condition (B,∞). This is seen from the following example.

Example 5. Let ∂G consist of the points $z = 0$ and $z = 2^{-k}$, when k runs over all positive integers, and $G = \mathbf{C} \setminus \partial G$. Let $m > 0$ be an integer and $\mu(x) = x^{m/2}$, $h(\zeta) = \zeta^m$.

In (4.2.2) we may not replace the string "for all $z \in (\partial_f G)_r$" by the string "for all $z \in \widetilde{G}$" or even by "for all $z \in \partial_f G$". This is seen from the following example.

Example 6. Let ∂G consist of the points $z = 0$ and $z = 2^k$, when k runs over all integers, and $G = \mathbf{C} \setminus \partial G$. Let $m \geq 2$ be an integer and $\mu(x) = (2x)^m$, $h(\zeta) = \zeta^m$. Then $\mu_\infty = m$ and all assumptions of Theorem 3, including (4.2.1), are satisfied. But for any $z \in \overline{G} \setminus \{0\}$ the analogue of (4.2.2) fails.

Concerning Theorem 4, one may state the following. In its exceptional case under $m > 1$, the cardinal number of the set $\mathbf{C} \setminus G$ can take any of the values $0, 1, \ldots, m$ (for instance, $G = \mathbf{C} \setminus \{z_0 + \exp(2\pi i k/m)\}_{k \in K}$ where K is an arbitrary subset of the sequence $0, 1, \ldots, m$) and even greater values.

If $\mu_0 \leq 1$, the condition (4.2.3) is automatically satisfied.

Example 6 shows that in Theorem 6 the fine continuity assumption (including the finiteness requirement) for h at $z = \infty$ may not be omitted.

From the same example we see that in Theorem 5 the condition $\mu_0 \leq 1$ may not be omitted.

In this respect, in the case $\mu_0 > 1$, it is interesting to know what conditions guarantee that (4.2.1) implies (4.2.2) at least for z, ζ close to each other.

Concerning the answer to this question see [T11].

7. On cluster properties of holomorphic and finely holomorphic functions

Using our preceding contour-solid theorems for (usual) holomorphic functions, earlier we had established the following extension of the Iversen–Tsuji theorem.

Theorem 9. *Let $D \subset \mathbf{C}$ be an open set, $Q \subset \partial D$ be a set nearly negligible relative to D, $a \in Q$ be a fixed limit point for the set $(\partial D) \setminus Q$. Let ϕ: $D \to \mathbf{C}$ be a holomorphic function. Denote*

$$\bar{\phi}_D(z) := \limsup_{\zeta \to z,\, \zeta \in D} |\phi(\zeta)| \qquad (z \in \partial D),$$

$$\bar{\phi}_D\big(a, (\partial D) \setminus Q\big) := \limsup_{z \to a,\, z \in (\partial D) \setminus Q} \bar{\phi}_D(z).$$

Suppose that

$$\bar{\phi}_D(z) < +\infty \qquad \text{for all } z \in (\partial D) \setminus \{a\}$$

and for every connected component T of D with $a \in \partial T$ there exist a neighbourhood V of the point a and a rational function majorizing $|\phi|$ in $T \cap V$. Then

$$\bar{\phi}_D(a) = \bar{\phi}_D\big(a, (\partial D) \setminus Q\big).$$

In [T13] and [T14] this result was given under the traditional assumption that the function ϕ is bounded in a neighbourhood of the point a, and there it was also remarked that such an assumption was assumed in the whole paper only for simplicity of formulations. As a matter of fact, we had avoided this assumption on the basis of our Theorems 3 and 3_* from [T5] (see also [T8], [T11]).

Now we have established the following fine analogue of this result on the basis of Theorem 2 of the present work, using the notation (2.2).

Theorem 10. *Let $D \subset \mathbf{C}$ be a finely open set, $Q \subset \partial_f D$ be a set nearly negligible relative to D, $a \in Q$ be a fixed finely limit point for the set $(\partial_f D) \setminus Q$. Let ϕ: $D \to \mathbf{C}$ be a finely holomorphic function. Denote*

$$\bar{\phi}_{D,f}\big(a, (\partial_f D) \setminus Q\big) := \inf_U \Big\{ \sup_{z \in U \cap (\partial_f D) \setminus Q} \bar{\phi}_{D,f}(z),\ U \text{ a fine neighbourhood of } a \Big\}.$$

Suppose that

$$\bar{\phi}_{D,f}(z) < +\infty \qquad \text{for all } z \in (\partial_f D) \setminus \{a\}$$

and for every finely connected component T of D with $a \in \partial_f T$ there exist a fine neighbourhood V of the point a and a rational function majorizing $|\phi|$ in $T \cap V$. Then

$$\bar{\phi}_{D,f}(a) = \bar{\phi}_{D,f}\big(a, (\partial_f D) \setminus Q\big).$$

Theorems 9 and 10 have various consequences for cluster sets of meromorphic and finely meromorphic functions.

8. More lemmas and proofs of lemmas

Let D be a finely open set in $\overline{\mathbf{C}}$ and $z \in \overline{\partial}_f D$. For any function $v\colon D \to [-\infty, +\infty]$ denote

$$\text{fine lim inf}_{\zeta\to z,\, \zeta\in D} v(\zeta) =: (v)\hat{}_{D,f}(z) =: \hat{v}_{D,f}(z). \tag{8.1}$$

We establish the following statement.

Lemma 3. *Let T be a fine domain in $\overline{\mathbf{C}}$ with a non-polar complement and Z a non-empty subset of the fine boundary $\overline{\partial}_f T$ such that the harmonic measure $\omega_x^T(Z)$ is zero at some fixed point $x \in T$. Then there exists a finely superharmonic function $v\colon T \to (0, +\infty]$ for which $v(x) = 1$ and*

$$\hat{v}_{T,f}(z) = +\infty \qquad \textit{for all } z \in Z. \tag{8.2}$$

Proof. Denote $\overline{\partial}_f T =: X$ and let $\chi\colon X \to [-\infty, +\infty]$ be any numerical function. Denote by $\chi^*(\chi_*)$ the pointwise infimum (supremum) of all fine superfunctions (fine subfunctions) for χ relative to T, see [F1, pp. 173–177]. Then $\chi_* \le \chi^*$ in T (this follows from Theorem 14.6 of [F1], but this can be shown also on the basis of Theorem 9.1 of [F1]).

Let now χ be the characteristic function of the set Z. Then $\chi^* \ge 0$ because the function identically equal to zero in T is a subfunction for χ relative to T, and hence $\chi_* \ge 0$.

Since $\omega_x^T(Z) = 0$, for any $k \in \mathbf{N}$ there exists a fine superfunction w_k for χ relative to T with

$$w_k(x) < 2^{-k}. \tag{8.3}$$

From the definition of fine superfunctions there follows that $w_k(x)$ is finely hyperharmonic in T and in the notation (8.1) the following inequality holds:

$$(w_k)\hat{}_{T,f}(z) \ge 1 \qquad \text{for all } z \in Z. \tag{8.4}$$

We have also $w_k \ge \chi^* \ge 0$ and from (8.4) we see that $w_k > 0$. Then the function

$$w(\zeta) := \sum_{k\in\mathbf{N}} w_k(\zeta) \tag{8.5}$$

is finely hyperharmonic in T because it is a pointwise limit of an increasing sequence of finely hyperharmonic functions (see Corollary 2 from [F1, p. 84]). From (8.3)–(8.5) we get $w > 0$, $w(x) < 1$ and (8.2). Therefore (see Theorem 12.9 from [F1, p. 158]) w is finely superharmonic in T.

From the above assertions we see that the function $v := w/w(x)$ possesses all properties being stated in the lemma. Lemma 3 is proved.

It has the following extension for higher dimensions.

Lemma 3^n. *Let T be a fine domain in $\mathbf{R}^n$ with a non-polar complement, let $Z \subset \partial_f T$ be a non-empty set whose (generalized) harmonic measure $\omega_x^T(Z)$ is zero at some fixed point $x \in T$. Then there exists a finely superharmonic function $v\colon T \to (0, +\infty]$ for which $v(x) = 1$ and*

$$\hat{v}_{T,f}(z) := \text{fine lim inf}_{\zeta \to z,\, \zeta \in T}\, v(\zeta) = +\infty \qquad \textit{for all } z \in Z.$$

The proof is completely analogous.

Lemma 3 is used for proving Lemma 1.

Proof of Lemma 1. Irregular points of $\overline{\partial}_f D$ form a polar set $\left(\overline{\partial}_f D\right)_i =: S$, and because of (3.1) the function u has a finely hypoharmonic extension (see [F1, p. 96]) to a regular finely open set $D \cup S =: D^r$ (see [F1, pp. 34, 149]). So without loss of generality we may assume that u is defined and finely hypoharmonic in the regular finely open set D^r and $E \cap D^r = \emptyset$. Then from (3.2) there follows that under this new setting we have either $(\overline{\partial}_f D^r) \setminus E \neq \emptyset$, or $u(z_0) \le 0$ at some point $z_0 \in D^r$.

Now let us use the specific properties of the 2-dimensional fine potential theory.

If $D^r = \overline{\mathbf{C}}$, then because of [F2, Theorem 2.2] the function u is hypoharmonic in $\overline{\mathbf{C}}$, therefore it is constant, and the above inequality $u(z_0) \le 0$ holds at some point z_0 and implies the inequality $u \le 0$ in D^r.

Let now $D^r \neq \overline{\mathbf{C}}$. Then the sets $\overline{\mathbf{C}} \setminus D^r =: F^r$ and $\widetilde{D}^r$ are bases (see [F1, pp. 34, 149]). We have also $\overline{\partial}_f D^r = \widetilde{D}^r \cap \widetilde{F}^r$; this set is a base and therefore it is non-polar (see [F1, p. 149]). Moreover, D^r, $\widetilde{D}^r$, F^r, $\overline{\mathbf{C}} \setminus \widetilde{D}^r$ and $\overline{\partial}_f D^r$ are Borel sets (see [B, Chapter VII.3], [F1, p. 27]).

Fix any finely connected component T of the set D^r and any point $x \in T$. Clearly the set E is nearly negligible relative to T as well. Denote $\overline{\partial}_f T =: Y$.

If the set $E \cap Y$ is finely closed and non-empty, then $\omega_x^T(E \cap Y) = 0$ and we consider the finely superharmonic function $v\colon T \to (0, +\infty]$ from Lemma 3 with $v(x) = 1$ and

$$\hat{v}_{T,f}(z) = +\infty \qquad \text{for all } z \in E \cap Y.$$

If $E \cap Y$ is empty, then we assume $v \equiv 0$.

Fix any $\varepsilon > 0$. In any of these cases the function $w_\varepsilon := u - \varepsilon v$ is finely hypoharmonic in T, and in the notation of (2.1) and (8.1) applied to T and the functions w_ε, u, v, one can show that

$$(w_\varepsilon)\check{}_{T,f}(z) \le u\check{}_{T,f}(z) - \varepsilon \hat{v}_{T,f}(z) \le 0 \qquad \text{for all } z \in Y.$$

Now we use the known result mentioned in Remark 2 (in [F2, p. 82] it is given under the additional restriction that a finely open set under consideration is bounded;

but in fact this extra restriction is unnecessary). On the basis of this deep result we get $w_\varepsilon \le 0, u(x) \le \varepsilon v(x)$. Letting $\varepsilon \to 0$ in the last inequality, we prove that $u(x) \le 0$.

Since x is an arbitrary point of T, we have also $u(\zeta) \le 0$ for all $\zeta \in T$.

If the set $E \cap Y$ is not finely closed, we fix any $\varepsilon > 0$ and consider the hypoharmonic function $u_\varepsilon(\zeta) := u(\zeta) - \varepsilon$ in T and the set $T_\varepsilon := \{\zeta \in T : u(\zeta) \ge \varepsilon\}$ finely closed in T. Suppose that $T_\varepsilon \ne \emptyset$. Denote $\overline{\partial}_f T_\varepsilon =: Y_\varepsilon$. We have $Y_\varepsilon \subset T \cup Y$. If $z_1 \in T \cap Y_\varepsilon$, then both $z_1 \in T_\varepsilon$ and $u(z_1) \le \varepsilon$ (the latter because otherwise the finely continuous function u is strictly greater than ε in a fine neighbourhood of z_1, and then z_1 is a finely inner point of T_ε, which contradicts our assumption). Hence $u(z_1) = \varepsilon$. Denote $Y \cap Y_\varepsilon =: E_\varepsilon$. Let now $z_2 \in E_\varepsilon$. Then $z_2 \in E$. Consequently, $E_\varepsilon \subset E$. The set E_ε is finely closed. Therefore $\omega_x^T(E_\varepsilon) = 0$. Obviously,

$$(u_\varepsilon)\check{}_{T,f}(z) < +\infty \qquad \text{for all } z \in Y$$

and

$$(u_\varepsilon)\check{}_{T,f}(z) \le 0 \qquad \text{for all } z \in Y \setminus E_\varepsilon.$$

Therefore from the assertion proved above we get $u_\varepsilon \le 0$ in T, $u \le \varepsilon$ in T. Letting $\varepsilon \to 0$, we get $u \le 0$ in T.

Since T is an arbitrary finely connected component of D^r, $u \le 0$ in D. So Lemma 1 is proved.

Proof of Lemma 2. The set $\big(\overline{\partial}_f D\big)_i$ is polar, and p is finely holomorphic in the set $D \cup \big(\overline{\partial}_f D\big)_i$ because of [F6, p. 62]. Therefore without loss of generality we assume that $\big(\overline{\partial}_f D\big)_i = \emptyset$.

Let us suppose that $(\partial_f D)_r = \emptyset$. Then $\widetilde{D} = \overline{\mathbf{C}}$ and p is finite and finely holomorphic in $\overline{\mathbf{C}}$. Hence p is holomorphic in $\overline{\mathbf{C}}$ (see [F6, p. 63]) which contradicts the condition $p \not\equiv \mathrm{const}$. Thus $(\partial_f D)_r \ne \emptyset$.

Fix any $\delta > 0$. Let us denote by $A(\delta)$ and $B(\delta)$ the left-hand and the right-hand sides, respectively, of the equality (4.3.1). Obviously $A(\delta) \le B(\delta)$. Let us suppose that $A(\delta) < B(\delta)$. Then there exist points $\zeta_1, \zeta_2 \in \widetilde{D}$ for which

$$|p(\zeta_1) - p(\zeta_2)| > A(\delta), \qquad |\zeta_1 - \zeta_2| = \delta.$$

If at least one of the points ζ_1, ζ_2 belongs to $\partial_f D$, then it belongs to $(\partial_f D)_r$ as well according to the above assumption. Hence the last inequality contradicts the definition of $A(\delta)$. It means that $\zeta_1, \zeta_2 \in D$.

Let us denote by D^1 and D^2 finely connected components of D containing the points ζ_1 and ζ_2, respectively. For every $n = 1, 2$ let us denote $D_n := \{\zeta \in \overline{\mathbf{C}} : \zeta + \zeta_n \in D^n\}$. The function $\phi(\zeta) := |p(\zeta + \zeta_1) - p(\zeta + \zeta_2)|$ is well-defined, finite and finely continuous on the set $F := \widetilde{D}_1 \cap \widetilde{D}_2$. Clearly $\phi(0) > A(\delta)$. Set $\frac{1}{2}\big(A(\delta) + \phi(0)\big) =: t$. It follows that $\phi(0) > t > A(\delta) \geq 0$.

If F contains the point ∞, then $\phi(\infty) = 0$ and there is a fine neighbourhood V of ∞ such that

$$\phi(\zeta) < \tfrac{1}{2}t \qquad \text{for all } \zeta \in F \cap V. \tag{8.6}$$

The finely open set $D_0 := D_1 \cap D_2$ contains the point $\zeta = 0$, $D_0 \subset F$ and $\phi(\zeta)$ is finely subharmonic in D_0 and finely continuous in $\widetilde{D}_0$ $(\subset \widetilde{F})$.

Let D_* be that finely connected component of the finely open set $\{\zeta \in D_0 : \phi(\zeta) > t\}$ which contains the point $\zeta = 0$. We have $\widetilde{D}_* \subset \widetilde{F}$ and

$$\phi(\zeta) \geq t \qquad \text{for all } \zeta \in \widetilde{D}_*.$$

Notice also that $\widetilde{D}_* = \overline{D_*}$, and hence $\widetilde{D}_*$ is a closed connected set.

If D_* is unbounded, then $\widetilde{D}_*$ contains the point ∞, and $\phi(\infty) = 0$. In this case $\widetilde{D}_*$ intersects V and because of (8.6) the inequality $\phi(\zeta) \leq \frac{1}{2}t$ $(< t)$ must be valid on the non-empty set $D_* \cap V$ which contradicts the above oppposite estimate. Hence, the fine domain D_* is bounded.

Let $z \in \partial_f D_*$. If for the fixed z at least one of two points $z + \zeta_1$ or $z + \zeta_2$ belongs to $(\partial_f D)_r$, then $\phi(z) \leq A(\delta) < t$. Otherwise $z \in D_0 \cap \partial_f D_*$, and then $\phi(z) = t$. Because of Lemma 1, it follows that $\phi(\zeta) \leq t$ in D^*. But this is in contradiction with the above inequality $\phi(0) > t$. This proves the equality (4.3.1). Lemma 2 is proved.

Let $B \subset \mathbf{C}$ be a bounded finely open set, $t \in (0,1)$, $w, z \in \mathbf{C}$, $l_t(w,z) := \log\max\{t, |w - z|\}$ and

$$H_B\big(l_t(w,\cdot),\zeta\big) := \int l_t(w,x)\, d\omega_\zeta^B(x). \tag{8.7}$$

Since B is bounded, the Green's function satisfies the equalities

$$\begin{aligned} &g_B(\infty,\zeta) = 0 \qquad \text{for all } \zeta \in B,\\ &g_B(w,\zeta) = \int \log\left|\frac{w-z}{w-\zeta}\right| d\omega_\zeta^B(z) \qquad \text{for all } \zeta \in B. \end{aligned} \tag{8.8}$$

We use the following lemma (cf. [T7], [T9], [T10], [T12], [TS1]–[TS3]).

Lemma 4. *For every $\zeta, w \in \mathbf{C}$ the limit*

$$\lim_{t\to 0} H_B\big(l_t(w,\cdot),\zeta\big) =: h_B(w,\zeta) \tag{8.9}$$

exists, is (finely) subharmonic with respect to $w \in \mathbf{C}$, finely subharmonic with respect to $w \in B$ and satisfies the equation

$$h_B(w,\zeta) - g_B(w,\zeta) = \log|w - \zeta| \qquad \textit{for all } \zeta \in B,\ \textit{for all } w \in \mathbf{C}. \tag{8.10}$$

Proof. For every fixed $\zeta \in \mathbf{C}$ the function $l_t(w,\zeta)$ is subharmonic with respect to $w \in \mathbf{C}$, monotonically decreases and tends to $\log|w-\zeta|$ when $t \downarrow 0$. Because of (8.7), the limit (8.9) exists, is hypoharmonic with respect to $w \in \mathbf{C}$ and satisfies the equality

$$h_B(w,\zeta) = \int \log|w-z|\, d\omega_\zeta^B(z) \qquad \text{for all } \zeta \in B, \text{ for all } w \in \mathbf{C} : w \neq \zeta.$$

From here and (8.8) we get (8.10). Therefore the function $h_B(w,\zeta)$ is subharmonic with respect to $w \in \mathbf{C}$ and finely subharmonic with respect to $w \in B$. Lemma 4 is proved.

9. Proof of Theorems 8.1–8.4

Let us use the notation and assumptions of Section 5.1.

From the definition of $\lambda \in L$ we see that there are $\sigma, \nu \in \mathbf{R}$ such that

$$\lambda(x) \leq \nu \log x + \sigma \qquad \text{for all } x > 0. \tag{9.1}$$

Fix any $\sigma, \nu \in \mathbf{R}$ for which (9.1) holds. Then from (5.1) we see that

$$\lambda^0 \geq \nu \geq \lambda^\infty. \tag{9.2}$$

Denote $a =: z_1$, $\infty =: z_2$. Taking into account (5.2), (5.3), (5.5), (5.6) and (9.1), we see that any of the conditions (A', z_s) and (B', z_s) implies the inequality $u_{T,f}^{z_s} < +\infty$ for any finely connected component T of G. So, the assumptions of Theorem 8.2 imply the assumptions of Theorem 8.1.

On the other hand, if for every $s = 1, 2$ there holds $z_s \notin b(CG)$, then the assumptions of Theorem 8.1 imply the assumptions of Theorem 8.2 and are equivalent to them.

For every $s = 1, 2$ there is at most one finely connected component T of G with $z_s \notin b(CT)$, and we denote this component by T_s. If $z_s \in b(CG)$, then we assume

$$T_s = \emptyset. \tag{9.3}$$

The condition (9.3) is equivalent to the condition $z_s \in b(CG)$.

The fine domains T_1 and T_2 may coincide, but they may differ as well.

Remark 6. With this notation the assumptions of Theorem 8.1 for given G and u imply that the set $G \setminus (T_1 \cup T_2) =: G_*$ (as well as any finely connected component of G_*) and the restriction of u to this set satisfy also the assumptions of both Theorem 8.1 and Theorem 8.2.

9.1. Let the assumptions of Theorem 8.1 be valid. From (5.2), (5.3) and the condition (d8) of Theorem 8.1 we conclude that for any finely connected component T of G and for each $s = 1, 2$ (independently from each other) one of the following two possibilities is valid:

(P1) $z_s \in b(T)$ and $u^{z_s}_{T,f} < +\infty$;

(P2) $z_s \notin b(T)$.

Fix any finely connected component T of G. In case (P1) there exist a fine neighbourhood V_s of z_s and a constant $A_s > |\nu|$ for which

$$u(\zeta) \le A_s \big|\log|\zeta - a|\big| \qquad \text{for all } \zeta \in V_s \cap T.$$

Then in any neighbourhood of z_s there exists a circle $K_s \subset V_s$ of the radius $r_s \in (0, +\infty)$ centered at a for which

$$u(\zeta) \le A_s|\log r_s| \qquad \text{for all } \zeta \in K_s \cap T.$$

The same is true also in case (P2) because then the fine domain T is separable from z_s in the standard topology, and K_s under consideration can be chosen with the condition $K_s \cap T = \emptyset$.

Let U_s be the closed neighbourhood of z_s with the boundary K_s. We may assume that $r_1 < 1 < r_2$. So we have

$$u(\zeta) \le A_s \big|\log|\zeta - a|\big| \qquad \text{for all } \zeta \in U_s \cap T, \text{ for all } s = 1, 2. \tag{9.1.1}$$

Fix any constant $\sigma_0 > \sigma$ such that

$$\sigma_0 > \max_{s=1,2} A_s|\log r_s|.$$

From here and (9.1) there follows that the function

$$v(\zeta) := u(\zeta) - \nu \log|\zeta - a|$$

(assumed as u) satisfies in $T \setminus (U_1 \cup U_2)$ (assumed as D) all assumptions of Lemma 1 with Q as E. Using this lemma, we consecutively deduce

$$\begin{aligned} &v(\zeta) \le \sigma_0 && \text{for all } \zeta \in T \setminus (U_1 \cup U_2), \\ &u(\zeta) \le \nu \log|\zeta - a| + \sigma_0 && \text{for all } \zeta \in T \setminus (U_1 \cup U_2). \end{aligned} \tag{9.1.2}$$

Because of (9.1.1) the circle $K_s \subset V_s$ may be chosen in an arbitrary neighbourhood of z_s, $s = 1, 2$, and from (9.1.2) we see that u is upper bounded on every part of T separable (in the standard topology) from $\{z_1, z_2\}$. It easily follows that u is also upper bounded on each finely connected part of T finely separable from $\{z_1, z_2\}$, because the part mentioned is separable from $\{z_1, z_2\}$.

Remark 7. We have thus proved that Theorem 8.1 is true under the additional requirement that G is finely connected.

If the statement of Theorem 8.1 is valid for fixed G and u, then there is $q \in \mathbf{R}$ such that

$$u(\zeta) \le q \qquad \text{for all } \zeta \in G \cap \{\zeta : |\zeta - a| = 1\}. \tag{9.1.3}$$

Thus we have proved that the statement of Theorem 8.1 is true with respect to every finely connected component of G, including T_1 and T_2. Hence it remains to confirm the statement of Theorem 8.1 with respect to G_* altogether.

9.2. Additionally assume that for each $s = 1, 2$ (independently from each other) one of the conditions (A', z_s) or (B', z_s) is valid (which coincides with assumptions of Theorem 8.2).

We shall prove that

$$u(\zeta) \le \nu \log |\zeta - a| + \sigma \quad \text{for all } \zeta \in G. \tag{9.2.1}$$

9.3. In this section we suppose for a while that for the fixed G and u the statement of Theorem 8.1 is true (according to Section 9.1, it is the case at least for finely connected G).

As we concluded in Section 9.1, in this case there is $q \in \mathbf{R}$ such that (9.1.3) holds. We may take

(9.3.1). $$q > |\sigma|.$$

Denote

$$G \cap \{\zeta : |\zeta - a| < 1\} =: G_a, \qquad G \cap \{\zeta : |\zeta - a| > 1\} =: G^a. \tag{9.3.2}$$

First prove that under the condition (9.3.1) there holds

$$u(\zeta) \le \nu \log |\zeta - a| + q \qquad \text{for all } \zeta \in G_a. \tag{9.3.3}$$

We have

$$\check{u}_{G_a, f}(z) - \nu \log |z - a| - q \le 0 \qquad \text{for all } z \in (\partial_f G_a) \setminus Q. \tag{9.3.4}$$

Let $s = 1$, $z_s = a$, and T be any finely connected component of G_a. Then one of the conditions (P1) or (P2) of Section 9.1 holds true. Therefore u is upper bounded on every part of T separable from a, and under the condition (P1) there exists a neighbourhood V of a and a constant $A > |\nu|$ for which

$$u(\zeta) \le A \big| \log |\zeta - a| \big| \qquad \text{for all } \zeta \in V \cap T.$$

First assume that $a \notin b(T)$. Then u is upper bounded in T and we consider the function

$$v(\zeta) := u(\zeta) - \nu \log|\zeta - a| - q \qquad (\zeta \in T) \tag{9.3.5}$$

which is also upper bounded in T. Because of (9.3.4) we may apply Lemma 1 to the function v in T with the set Q as E. As a result we get $v \le 0$ in T and

$$u(\zeta) \le \nu \log|\zeta - a| + q \qquad \text{for all } \zeta \in T. \tag{9.3.6}$$

Now assume that $a \in b(T)$. If $a \notin b(CT)$, then $a \notin b(CG)$ and the condition (B', a) is valid. Therefore (5.6) is valid and may be combined with (9.1). Since u is upper bounded on every part of T separable from a, it follows that the function (9.3.5) in this case is also upper bounded in T because of the above assertions and (5.6), (9.1). Therefore this function satisfies all assumptions of Lemma 1 with Q as E, and in this T we again get the inequalities $v \le 0$ and (9.3.6).

Let now $a \in b(CT)$. Then

$$g_T(a,\zeta) = 0 \qquad \text{for all } \zeta \in T$$

and we shall show that (9.3.6) again is valid.

Consider the function

$$v(\zeta) := u(\zeta) + A \log|\zeta - a| - q \qquad (\zeta \in T)$$

which is upper bounded both in $V \cap T$ and in $T \setminus V$, and hence it is upper bounded in T.

Using (9.1), (9.1.3) and the condition (b8) of Theorem 8.1, we deduce

$$\check{v}_{T,f}(z) \le c \log|z - a| \quad \text{for all } z \in (\partial_f T) \setminus Q \tag{9.3.7}$$

with the constant $c = \nu + A$.

Let $t \in (0,1)$. From the definition of l_t and (9.3.7) we get

$$\check{v}_{T,f}(z) \le c l_t(a,z) \quad \text{for all } z \in (\partial_f T) \setminus Q. \tag{9.3.8}$$

In T the function v is upper bounded and finely hypoharmonic, the function $H_T\big(l_t(a,\cdot),\zeta\big)$ is lower bounded and finely harmonic with respect to ζ for every $t \in (0,1)$. Therefore in T the function $v - cH_T\big(l_t(a,\cdot),\zeta\big)$ satisfies all assumptions of Lemma 1 with Q as E. Using this lemma, we get

$$v(\zeta) \le cH_T\big(l_t(a,\cdot),\zeta\big) \quad \text{for all } \zeta \in T, \text{ for all } t \in (0,1).$$

Letting $t \downarrow 0$, from Lemma 4 we derive

$$v(\zeta) \le ch_T(a,\zeta) = c\big(\log|\zeta - a| + g_T(a,\zeta)\big) \quad \text{for all } \zeta \in T.$$

Since $g_T(a,\zeta) = 0$, we get

$$v(\zeta) \leq (\nu + A)\log|\zeta - a| \qquad \text{for all } \zeta \in T,$$

and therefore we again obtain (9.3.6).

Thus we have proved that (9.3.6) is valid in every finely connected component of G_a, and therefore (9.3.3) is true.

In the same manner one can prove the analogue of the estimate (9.3.3) for the set G^a defined in (9.3.2). It may be done also by considering the images of the sets G^a, Q and G under the mapping $\zeta_1 = 1/(\zeta - a)$ and using the above arguments to the point $a_1 := 0$, the functions $u_1(\zeta_1) := u(\zeta)$, $\lambda_1(x) := \lambda(-x)$ and the numbers $\nu_1 := -\nu$, $\sigma_1 := \sigma$, $q_1 := q$ (which is eligible because the assumptions of Theorem 8.2 are invariant under such substitutions). As a result, we get the analogue of (9.3.3) from which the inverse substitutions lead to the estimate

$$u(\zeta) \leq \nu \log|\zeta - a| + q \qquad \text{for all } \zeta \in G^a. \tag{9.3.9}$$

From (9.1.3), (9.3.3) and (9.3.9) there follows

$$u(\zeta) \leq \nu \log|\zeta - a| + q \qquad \text{for all } \zeta \in G.$$

The function

$$v(\zeta) := u(\zeta) - \nu \log|\zeta - a| - \sigma$$

in G satisfies the inequality

$$\check{v}_{G,f}(z) \leq 0 \qquad \text{for all } z \in (\partial_f G) \setminus Q$$

and is upper bounded.

First suppose that $(\partial_f G)\setminus Q \neq \emptyset$. Then we may apply Lemma 1 to G (as D), v (as u), Q (as E). It gives $v \leq 0$ in G and (9.2.1).

Now suppose that (9.2.1) fails. Then it must be $(\partial_f G) \setminus Q = \emptyset$, $\partial_f G \subset Q = \overset{\sqcap}{\partial_f G}$, and since the set Q in this case is finely closed and nearly negligible relative to G, it must be polar. Therefore the upper bounded, finely hypoharmonic function v is extendable (see [F1, p. 96]) onto $\overset{\sqcap}{\mathbf{C}}$ to a finely hypoharmonic function v_* which is also upper bounded. Therefore (see [F2, Theorem 2.2]) v_* is hypoharmonic in $\overset{\sqcap}{\mathbf{C}}$. Hence v_* and v are constants. Since (9.2.1) fails, we have $v \equiv \text{const} \in (0,+\infty)$ and

$$u(\zeta) - \nu \log|\zeta - a| - \sigma \equiv \text{const} =: \delta \in (0,+\infty) \quad \text{for all } \zeta \in G. \tag{9.3.10}$$

Thus u is bounded on each bounded part of G separable from the point a. We have also $G \cup \partial_f G = \mathbf{C}$, G is a fine domain and the points a and ∞ belong to $b(G)$ and do not belong to $b(CG)$. Let us use the following information and arguments: the points a and ∞ belong to $b(G)$; u is bounded (from both sides!) on every bounded part of G separable from the point a; the conditions (B', z_s) are satisfied for $s = 1, 2$, which implies the inequalities (5.5), (5.6); hence λ is bounded (from both sides!) on every closed subinterval of the interval $(0, +\infty)$, the function $\zeta \mapsto \lambda(|\zeta - a|)$ is bounded on every bounded part of $\mathbf{C}$ separable from the point a, and the function

$$w(\zeta) := u(\zeta) - \lambda(|\zeta - a|) \qquad (\zeta \in G)$$

is upper bounded in fine neighbourhoods of the points a and ∞. From here we deduce that $\lambda \in L^*$, $I_\lambda = (0, +\infty)$, and the function $\zeta \mapsto \lambda(|\zeta - a|)$ is superharmonic in $\mathbf{C} \setminus \{a\}$. Therefore the function w is finite and finely subharmonic in G and bounded on every bounded part of G separable from the point a. Hence it has the finely subharmonic extension onto $\mathbf{C}$ which is bounded on each bounded part of $\mathbf{C}$ separable from a. Moreover, it is also upper bounded in fine neighbourhoods of the points a and ∞. It means that $w \equiv \text{const}$ in G. From here and (9.3.10) we obtain

$$\lambda(x) \equiv \nu \log x + p \qquad \text{for all } x \in (0, +\infty) \tag{9.3.11}$$

with some constant $p \in \mathbf{R}$. On the basis of (9.1) and (9.3.11) we see that $\sigma \geq p$. Comparing (9.3.10) and (9.3.11), we get

$$u(\zeta) - \lambda(|\zeta - a|) \equiv \sigma + \delta - p \geq \delta > 0 \qquad \text{for all } \zeta \in G, \tag{9.3.12}$$

and we have the exceptional case of the statement of Theorem 8.2.

Thus, we have proved that if the estimate (9.2.1) fails, then $\partial_f G$ is polar, $\partial_f G \subset Q$ and we have (9.3.12) and the exceptional case of Theorem 8.2.

Remark 8. Consequently, if in Theorem 8.2 the exceptional case is not valid, then (9.2.1) is true for any $\sigma, \nu \in \mathbf{R}$ satisfying the condition (9.1).

Recall that the last assertions are established under the extra requirement that for fixed G and u the statement of Theorem 8.1 is true. In particular, this requirement is justified in the case when G is finely connected (see Section 9.1).

9.4. Now we drop the extra assumption of Section 9.3 and make use of the last assertion of that section. Then according to Remarks 6–8 for every finely connected component T of the set G_* (if it is non-empty) and the restriction of u to this component the assumptions of Theorems 8.1 and 8.2 are satisfied and the statement of Theorem 8.1 is true, therefore the estimate (9.2.1) is valid (because for T in question the exceptional case of Theorem 8.2 is impossible). Since the right-hand side in this estimate (9.2.1) is common for all components of G_*, we get

$$u(\zeta) \leq \nu \log |\zeta - a| + \sigma \qquad \text{for all } \zeta \in G_*. \tag{9.4.1}$$

Now from (9.4.1) and the last assertion of Section 9.1 we see that Theorem 8.1 is true in the general case.

Hence on the basis of the result of Section 9.3 the estimate (9.2.1) also is true in the general case mentioned in Section 9.2.

9.5. Now suppose that (9.2.1) is true for any $\sigma, \nu \in \mathbf{R}$ satisfying the condition (9.1) (and consequently, the exceptional case of Theorem 8.2 does not hold). We shall show that under this assumption the estimates (5.7) are valid, and if (5.10) fails, then $\lambda \in L^*$, $x_\lambda^- < x_\lambda^+$ and (5.11)–(5.13) hold true.

Fix $r \in (0, +\infty)$. If $\lambda(x) = -\infty$ in a neighbourhood of r, then one may choose numbers $\sigma, \nu \in \mathbf{R}$ with the inequality (9.1) in such a way that the quantity $\nu \log r + \sigma$ becomes less than any pregiven number $M > -\infty$. Therefore we get $M_{G,a}(u, r) = -\infty$. Hence (5.13) is true. Moreover, if $\lambda \equiv -\infty$, then (5.10) is valid.

Fix $\zeta_0 \in G$, and let $G(\zeta_0)$ be that finely connected component of G which contains ζ_0.

Let now $\lambda \in L^*$. If $|\zeta_0 - a|$ does not belong to the closed interval $[x_\lambda^-, x_\lambda^+]$, then from (5.13) and (5.4) we deduce

$$u(\zeta_0) = \gamma_G(u, \zeta_0) = M_{G,a}(u, |\zeta_0 - a|) = \lambda(|\zeta_0 - a|) = -\infty. \tag{9.5.1}$$

From (9.5.1) we see that the equality

$$u(\zeta) = -\infty \tag{9.5.2}$$

is valid for every $\zeta \in G$ for which $|\zeta - a| \notin [x_\lambda^-, x_\lambda^+]$.

If $|\zeta_0 - a| \leq x_\lambda^-$ or $|\zeta_0 - a| \geq x_\lambda^+$, then in some non-empty, finely open subdomain of $G(\zeta_0)$ the equality (9.5.2) is valid, and therefore

$$u(\zeta) \equiv \gamma_G(u, \zeta) \equiv -\infty \qquad \text{for all } \zeta \in G(\zeta_0). \tag{9.5.3}$$

So (9.5.2) holds true for all $\zeta \in G$ for which either $|\zeta - a| \leq x_\lambda^-$, or $|\zeta - a| \geq x_\lambda^+$. From here and (9.5.3) we deduce (5.11).

In particular, if $x_\lambda^- = x_\lambda^+$, then from (5.11) and (5.13) we obtain

$$u \equiv -\infty, \quad \gamma_G(u, \cdot\,) \equiv -\infty, \quad M_{G,a}(u, r) = -\infty \quad \text{for all } r > 0.$$

Hence, if (5.10) fails, then $\lambda \in L^*$ and $x_\lambda^- < x_\lambda^+$.

If $x_\lambda^- < |\zeta_0 - a| < x_\lambda^+$, then one may choose $\sigma, \nu \in \mathbf{R}$ in such a way that (9.1) and the equality $\lambda(|\zeta_0 - a|) = \nu \log |\zeta_0 - a| + \sigma$ are valid. From here and (9.2.1) we get

$$u(\zeta_0) \leq \nu \log |\zeta_0 - a| + \sigma = \lambda(|\zeta_0 - a|).$$

Therefore, taking into consideration the definition of $M_{G,a}(u, r)$, we get

$$M_{G,a}(u, |\zeta_0 - a|) \leq \lambda(|\zeta_0 - a|).$$

It follows

$$M_{G,a}(u, r) \leq \lambda(r) \qquad \text{for all } r \in (x_\lambda^-, x_\lambda^+). \tag{9.5.4}$$

From (9.5.4) and (5.13) we get (5.12). From (5.4), (5.12) and (5.11) we deduce (5.7). The inequalities (5.17) follow from (5.7), (5.14)–(5.16). This completes the proof of Theorem 8.2.

Let us consider the situation of Theorem 8.3. Since (5.10) is excluded, it must be $\lambda \in L^*$, $x_\lambda^- < x_\lambda^+$. Let T be a finely connected component of G. Suppose that there is a point $\zeta_0 \in T$ with $\lambda(\zeta_0) = -\infty$. Then $|\zeta_0 - a| \notin (x_\lambda^-, x_\lambda^+)$. Using (9.5.3) with T as $G(\zeta_0)$, we get (5.18). Hence if (5.18) fails, then (5.19) is true. Theorem 8.3 is proved.

Thus, we have proved Theorems 8.1–8.3.

Theorem 7 is contained in Theorems 8.1–8.3.

9.6. Proof of Theorem 8.4. Since the set CG is non-polar, the exceptional case of Theorem 8.2 is impossible and we have (5.7).

Fix any numbers τ_1, τ_2 such that for each $s = 1, 2$ (independently from each other) we have either $\tau_s = \sigma_f^s = 0$, or $\sigma_f^s < \tau_s \le 0$. Let $\nu, \sigma \in \mathbf{R}$ be such that (9.1) is valid. In view of (5.7), then (9.2.1) holds true as well.

For $r > 0$ and any function $v\colon G \to [-\infty, +\infty)$ let us denote

$$\sup_{\zeta \in G,\, |\zeta - a| = r} v(\zeta) := m_{G,a}(v, r).$$

Introduce in G the function

$$V_{\tau_1,\tau_2}(\zeta) := u(\zeta) - (\nu \log|\zeta - a| + \sigma) - \tau_1 g_G(a, \zeta) - \tau_2 g_G(\infty, \zeta).$$

The function $V_{\tau_1,\tau_2}(\zeta)$ is finely hypoharmonic in G and upper bounded in every fine subdomain of G finely separable from a and ∞. Moreover, at every finite point $z \in b(\partial_f G) \setminus \{a\}$ there holds

$$\text{fine lim sup}_{\zeta \to z,\, \zeta \in G} V_{\tau_1,\tau_2}(\zeta) \le 0.$$

There exist finite constants c_1, c_2 such that under $|\zeta - a| \ge 1$ we have $g_G(a, \zeta) \le c_1$, $g_G(\infty, \zeta) \le c_2 + \log|\zeta - a|$.

Let first $\tau_2 = 0$. Then

$$m_{G,a}(V_{\tau_1,\tau_2}, r) \le -\tau_1 c_1 \qquad \text{for all } r \ge 1.$$

Let now $\sigma_f^2 < \tau_2 < 0$. If $u_{G,f}^\infty = -\infty$, then there is a fine neighbourhood X of ∞ with the property

$$u(\zeta) < (\nu + \tau_2) \log|\zeta - a| \qquad \text{for all } \zeta \in X \setminus \{\infty\}.$$

If $u_{G,f}^\infty \ne -\infty$, then we have also $\lambda^\infty \ne -\infty$, and then there is a fine neighbourhood X of ∞ with the property

$$u(\zeta) < (u_{G,f}^\infty + \tau_2 - \sigma_f^2) \log|\zeta - a| \quad \text{for all } \zeta \in X \setminus \{\infty\},$$

and then in view of (9.2) we have

$$\begin{aligned} u(\zeta) - (\nu + \tau_2) \log|\zeta - a| &< (u^{\infty}_{G,f} - \nu - \sigma^2_f) \log|\zeta - a| \\ &= (\lambda^{\infty} - \nu) \log|\zeta - a| \le 0 \quad \text{for all } \zeta \in X \setminus \{\infty\}. \end{aligned}$$

In both of these cases we have

$$\begin{aligned} V_{\tau_1,\tau_2}(\zeta) &\le u(\zeta) - (\nu \log|\zeta - a| + \sigma) - \tau_1 c_1 - \tau_2(\log|\zeta - a| + c_2) \\ &\le \text{const} < +\infty \quad \text{for all } \zeta \in X \setminus \{\infty\}. \end{aligned}$$

Hence, in any case the function V_{τ_1,τ_2} is upper bounded in some fine neighbourhood of ∞.

There exist finite constants c_3, c_4 such that under $0 < |\zeta - a| \le 1$ we have $g_G(\infty, \zeta) \le c_3$, $g_G(a, \zeta) \le c_4 - \log|\zeta - a|$.

First assume that $\tau_1 = 0$. Then

$$m_{G,a}(V_{\tau_1,\tau_2}, r) \le -\tau_2 c_3 \qquad \text{for all } r \in (0, 1].$$

Let now $\sigma^1_f < \tau_1 < 0$. If $u^a_{G,f} = -\infty$, then there is a fine neighbourhood Y of the point a with the property

$$u(\zeta) < (\nu - \tau_1) \log|\zeta - a| \qquad \text{for all } \zeta \in Y \setminus \{a\}.$$

If $u^a_{G,f} \neq -\infty$, then also $\lambda^0 \neq +\infty$ and there is a fine neighbourhood Y of the point a with the property

$$u(\zeta) < (-u^a_{G,f} - \tau_1 + \sigma^1_f) \log|\zeta - a|,$$

and then in view of (9.2) we have

$$\begin{aligned} u(\zeta) - (\nu - \tau_1) \log|\zeta - a| &< (-u^a_{G,f} - \nu + \sigma^1_f) \log|\zeta - a| \\ &= (\lambda^0 - \nu) \log|\zeta - a| \le 0 \quad \text{for all } \zeta \in Y \setminus \{a\}. \end{aligned}$$

In both of these cases we have

$$\begin{aligned} V_{\tau_1,\tau_2}(\zeta) &\le u(\zeta) - (\nu \log|\zeta - a| + \sigma) - \tau_1(-\log|\zeta - a| + c_4) - \tau_2 c_3 \\ &\le \text{const} < +\infty \quad \text{for all } \zeta \in Y \setminus \{a\}. \end{aligned}$$

Thus, the function V_{τ_1,τ_2} is upper bounded in some fine neighbourhood of the point a as well. It follows that

$$(V_{\tau_1,\tau_2})\check{}_{G,f}(z) < +\infty \quad \text{for all } z \in \overparen{\partial}_f G.$$

The set $E := \left(\overline{\partial}_f G\right)_i$ is polar, and because of (9.2.1) we have

$$(V_{\tau_1,\tau_2})\check{}_{G,f}(z) \le 0 \qquad \text{for all } z \in \left(\overline{\partial}_f G\right) \setminus E.$$

Hence we may apply Lemma 1 to the function V_{τ_1,τ_2} (considered as u) in G (considered as D). As a result we get

$$V_{\tau_1,\tau_2}(\zeta) \le 0 \qquad \text{for all } \zeta \in G,$$

which means

$$u(\zeta) \le \nu \log|\zeta - a| + \sigma + \tau_1 g_G(a,\zeta) + \tau_2 g_G(\infty,\zeta) \qquad \text{for all } \zeta \in G.$$

Letting $\tau_s \to \sigma_f^s$ for $s = 1, 2$, we derive

$$u(\zeta) \le \nu \log|\zeta - a| + \sigma + \sigma_f^1 g_G(a,\zeta) + \sigma_f^2 g_G(\infty,\zeta) \quad \text{for all } \zeta \in G, \tag{9.6.1}$$

where under every $\zeta \in G$ the following agreements are adopted. If $g_G(a,\zeta) = 0$, then $\sigma_f^1 g_G(a,\zeta) = 0$ (even under $\sigma_f^1 = -\infty$), and if $g_G(\infty,\zeta) = 0$, then $\sigma_f^2 g_G(\infty,\zeta) = 0$ (even under $\sigma_f^2 = -\infty$).

Fix any $\zeta_0 \in G$. If the condition

$$\sigma_f^1 g_G(a,\zeta_0) + \sigma_f^2 g_G(\infty,\zeta_0) = -\infty \tag{9.6.2}$$

holds (which is equivalent to the assumptions $\sigma_f^s = -\infty$, $z_s \notin b\big(CG(\zeta_0)\big)$ for at least one $s = 1, 2$), then we have

$$u(\zeta) = \gamma_G(u,\zeta) = -\infty \qquad \text{for all } \zeta \in G(\zeta_0).$$

Otherwise for each $s = 1, 2$ with $\sigma_f^s = -\infty$ we have $z_s \in b\big(CG(\zeta_0)\big)$ and $g_G(z_s,\zeta) = 0$ in $G(\zeta_0)$. In this case the inequality (9.6.1) and fine harmonicity of all summands in its right-hand side imply the following inequalities for every $\zeta \in G(\zeta_0)$:

$$\gamma_G(u,\zeta) \le \nu \log|\zeta - a| + \sigma + \sum_{s=1}^{2} \sigma_f^s g_G(z_s,\zeta), \tag{9.6.3}$$

$$u(\zeta) - \sum_{s=1}^{2} \sigma_f^s g_G(z_s,\zeta) \le \nu \log|\zeta - a| + \sigma. \tag{9.6.4}$$

The estimate (9.6.3) for $\zeta \in G(\zeta_0)$ is valid also in the case when the equality (9.6.2) holds, and this follows from the consequences of this equality. In this case the same is true for the estimate (9.6.4) as well (under the agreement (5.20)). From here we get

$$(9.6.5)\quad M_{G,a}\Big(u(\,\cdot\,)-\sum_{s=1}^{2}\sigma_f^s g_G(z_s,\cdot),|\zeta-a|\Big)\le \nu\log|\zeta-a|+\sigma \quad \text{for all } \zeta\in G(\zeta_0).$$

Let T be a finely connected component of G and $\zeta_0\in T$. If (5.18) is valid, then (5.21) and (5.22) are evidently true.

Let now (5.18) fail. Then because of Theorem 8.3 the following is true: $\lambda\in L^*, x_\lambda^- < x_\lambda^+$, (5.19) holds, and for every $\zeta\in T$ we have $x_\lambda^- < |\zeta-a| < x_\lambda^+$. Then $\sigma,\nu\in\mathbf{R}$ may be chosen in such a way that (9.1) and the equality

$$\lambda(|\zeta_0-a|)=\nu\log|\zeta_0-a|+\sigma$$

are valid. Therefore on the basis of (9.6.3), (9.6.5) we obtain

$$\gamma_G(u,\zeta_0)\le\lambda(|\zeta_0-a|)+\sum_{s=1}^{2}\sigma_f^s g_G(z_s,\zeta_0),$$

$$M_{G,a}\Big(u(\,\cdot\,)-\sum_{s=1}^{2}\sigma_f^s g_G(z_s,\cdot),|\zeta_0-a|\Big)\le\lambda(|\zeta_0-a|).$$

These estimates are valid for every $\zeta_0\in T$, and hence we get (5.21) and (5.22) also in the case when (5.18) is supposed to fail. Thus estimates (5.21) and (5.22) are true in any case.

Now let us suppose that there is s for which both $z_s\notin b(CT)$ and $\sigma_f^s=-\infty$. Then from (5.21) we see that (5.18) is true. Theorem 8.4 is proved.

10. Proof of Theorem 2

Let h, μ be functions from Theorem 2 satisfying the properties (a2)–(d2) for every finely connected component T of G with $z_s\in b(T)$.

Denote

$$(10.1)\qquad \log|h(\zeta)|=:u(\zeta)\quad(\zeta\in G),\qquad \log\mu(x)=:\lambda(x)\quad(0<x<+\infty).$$

Clearly, u is finely hypoharmonic in $G,\lambda\in L$. Then in the notations (2.1), (2.2), (4.1)–(4.3), (5.1)–(5.3) we have:

$$\mu_0=\lambda^0,\qquad \mu_\infty=\lambda^\infty$$

and for every $s=1,2$ and any finely connected component T of G there hold

$$\log\bar h_{T,f}=\check u_{T,f};\qquad h_{z_s,T,f}=u^{z_s}{}_{T,f}.$$

Therefore the conditions (a8)–(d8) of Theorem 8.1 are true as well, and we may apply Theorem 8.1 to u. Thus we derive that u is upper bounded on every bounded part of G separable from a. In particular, there exists a constant $q \in \mathbf{R}$ such that the inequality (9.1.3) is true.

Obviously for every $s = 1, 2$ the condition (A, z_s) (see Section 4.1) implies the condition (A', z_s) (see Section 5).

We shall show that the condition (B, z_s) from Section 4.1 implies the condition (B', z_s) from Section 5 ($s = 1, 2$). Evidently it is the case if $h \equiv 0$ in a fine neighbourhood of z_s. So it remains to investigate the opposite situation.

Consider the case when $s = 1$, $z_s = a \in b(G)$ and the condition (B, a) is valid. Let T be that (unique, see Remark 3(c)) finely connected component of G for which $z_s \in b(T)$. Then from (4.1), (4.2), (d8), (4.1.2) and the definition of m_0 we get

$$h_{a,T,f} = h_{a,G,f} \leq 1 - m_0 < +\infty.$$

We have $a \notin b(CG)$. Introduce the function

$$\Psi(\zeta) := h(\zeta)(\zeta - a)^{1-m_0} \qquad (\zeta \in G).$$

This function is finely holomorphic in G. From (4.1.2) we see that it is bounded in a fine neighbourhood of a and there exists

$$\text{fine } \lim_{\zeta \to a,\, \zeta \in G} \Psi(\zeta) = 0.$$

Now we make use of known properties of a finely holomorphic function concerning its structure and elimination of singularities (see [F7], [F6]). In this way we establish that the function Ψ extended to the point a by the value $\Psi(a) = 0$ is finely holomorphic in $G \cup \{a\}$, and since $\Psi \not\equiv 0$ in any fine neighbourhood of a, it has a zero of a finite order at a. From here we see that there exist an integer $k \geq m_0$ and a function Φ finely holomorphic in $G \cup \{a\}$ such that $\Phi(a) \neq 0$ and

$$h(\zeta) = \Phi(\zeta)(\zeta - a)^k \qquad \text{for all } \zeta \in G. \tag{10.2}$$

Using (4.2) and (10.2), we get $h_{a,G,f} = -k$. Thus $h_{a,G,f}$ is an integer and

$$\Phi_{a,G,f} = 0.$$

Let us use the notation (9.3.2). Then there is (the unique) finely connected component D of the set G_a for which $a \in (\partial_f D)_i$. We have also $a \in b(D) \setminus b(CD)$ and

$$u^a_{D,f} = u^a_{G,f} = h_{a,G,f} < +\infty.$$

Clearly the analogue of (4.1.2) is valid for D as well.

Denote $\max\{q, \log|\Phi(a)|\} =: q^*$ and consider the function

$$\lambda^*(x) := k \log x + q^* \qquad (0 < x < +\infty).$$

Then the function hypoharmonic in D,

$$u^*(\zeta) := u(\zeta) - \lambda^*(|\zeta - a|) = \log|\Phi(\zeta)| - q^*,$$

is upper bounded in a fine neighbourhood of the point a.

Let $0 < x \le 1$, $t := \log x$, $p(t) := \lambda(\log t)$. Then $p'(t) \le \lambda_0$ and

$$\lambda(1) - \lambda(x) = p(0) - p(t) = \int_t^0 p'(\tau)d\tau \le -\lambda_0 t = -\lambda_0 \log x \quad (0 < x < 1).$$

Therefore

$$\begin{aligned}\lambda(x) &\ge \lambda(1) + \lambda_0 \log x \ge \lambda(1) + m_0 \log x \ge \lambda(1) + k \log x \\ &= \lambda(1) - q^* + \lambda^*(x) \quad (0 < x < 1).\end{aligned}$$

It yields

$$\lambda^*(x) - \lambda(x) \le q^* - \lambda(1) \qquad \text{for all } x \in (0,1). \tag{10.3}$$

Using the upper boundedness of u^* in a fine neighbourhood of the point a and the estimate (10.3), we establish the upper boundedness of the function

$$u(\zeta) - \lambda(|\zeta - a|) = u^*(\zeta) + \lambda^*(|\zeta - a|) - \lambda(|\zeta - a|)$$

in a fine neighbourhood of the point a.

So we have proved that the condition (B, a) for G, h, μ implies the condition (B', a) for G, u, λ.

In a similar way one can show that the condition (B, ∞) for G, h, μ implies the condition (B', ∞) for G, u, λ (it can be done by means of the transformations and substitutions used in Section 9.3 for proving the estimate (9.3.9)). Hence, the assumptions of Theorem 2 imply the validity of all assumptions of Theorem 8.2 in the notation (10.1). Therefore all statements of Theorem 8.2 can be reformulated with respect to notions and notation of Theorem 2.

Suppose that the exceptional case of Theorem 8.2 holds. Then Q is polar, $G = \mathbf{C} \setminus Q$, $a \in b(G) \setminus b(CG)$ and

$$u(\zeta) = \nu \log|\zeta - a| + t \qquad \text{for all } \zeta \in G$$

with some constants $\nu, t \in \mathbf{R}$. Hence $u^a_{G,f} = -\nu$, $h_{a,G,f} = -\nu$. But earlier we had shown that in such a situation $h_{a,G,f}$ is an integer. So ν is an integer, and we have the exceptional case of Theorem 2. Theorem 2 is proved.

Remark 9. We have also established that under the assumptions of Theorem 2 all statements of Theorems 8.3, 8.4 in the notation (10.1) are true as well (under the following additional requirements: when the exceptional case and (5.10) are not valid in Theorem 8.3 and when CG is non-polar in Theorem 8.4).

11. Proof of Theorems 3, 4 and 6

11.1. Proof of Theorem 3. Fix an arbitrary point $w \in (\partial_f G)_r$.

Let us check that we may apply Theorem 1 with w as a, and the function

$$\phi(\zeta) := h(\zeta) - h(w) \qquad (\zeta \in G)$$

instead of $h(\zeta)$. Indeed, from (4.2.1) we get (4.1.3) with w as a, and ϕ as h. Since for every finely connected component T of G the function $h|_{\widetilde{T}\cap\mathbf{C}}$ is finely continuous and $w \in (\partial_f G)_r \subset b(CG)$, therefore h satisfies the condition (A, w), and the same is true for ϕ.

One of the conditions (A, ∞) or (B_0, ∞) is assumed to hold for h, and it implies the same condition for ϕ.

Hence, Theorem 1 is applicable in the situation mentioned above. Since $w \in (\partial_f G)_r$, we have $G \neq \mathbf{C} \setminus \{w\}$ and the exceptional case of Theorem 1 is impossible in the situation under consideration. Thus we get (4.1.4) for ϕ instead of h, and the estimate of (4.2.2) with w as z. Because of the choice of w, it gives us (4.2.2). Theorem 3 is proved.

11.2. Proof of Theorem 4. Consider the function

$$\phi(\zeta) := h(\zeta) - h(z_0) \qquad (\zeta \in G).$$

Since ϕ is finely continuous (and finite) at z_0, one can show (on the basis of the argument used in the proof of Theorem 2) that ϕ is finely holomorphic in a fine neighbourhood of the point z_0 and

$$\phi(\zeta) = \operatorname{fine} O(|\zeta - z_0|^k) \qquad (\zeta \to z_0,\ \zeta \in G \setminus \{z_0\}) \tag{11.2.1}$$

with some integer $k \geq 1$.

Let us check that Theorem 1 is applicable to $G \setminus \{z_0\}$, z_0, $\phi|_{G\setminus\{z_0\}}$ instead of G, a, h, respectively. Indeed, from (4.2.1) we get (4.1.3) with z_0 as a, and ϕ as h.

From the assumptions of Theorem 4 including (4.2.3) we see that for the situation under consideration, the condition (B, z_0) is valid for ϕ in $G \setminus \{z_0\}$, and one of the conditions $(A, \infty), (B_0, \infty)$ for ϕ in $G\setminus\{z_0\}$ holds as well. Hence Theorem 1 is applicable in the mentioned situation, and as a result of such an application we get either the inequality (4.2.4), or the exceptional case of Theorem 1.

If the exceptional case in the application of Theorem 1 holds true, then

$$\begin{aligned}
&\mu(x) = \beta x^m \quad \text{for all } x > 0,\\
&\phi(\zeta) = c(\zeta - z_0)^m \quad \text{for all } \zeta \in G,\\
&h(\zeta) = c(\zeta - z_0)^m + h(z_0) \quad \text{for all } \zeta \in G
\end{aligned}$$

with constants $c \in \mathbf{C}$, $\beta \geq 0$, m, where m is an integer and $|c| > \beta$. From here and (11.2.1) we see that $m \geq k \geq 1$.

If $\partial_f G$ contains z_0, it contains no other point because otherwise (4.2.1) would fail for this couple of points, a contradiction.

If $m = 1$, then μ and h are linear, and $\partial_f G$ contains at most one point, because otherwise we got the same contradiction with (4.2.1) as above.

The last statement of Theorem 4 is obviouos. Theorem 4 is proved.

11.3. Proof of Theorem 6. We shall apply Lemma 2 with $D := G$ and $p := h$. Fix arbitrary $\delta > 0$, $\varepsilon > 0$, and denote by $A(\delta)$ and $B(\delta)$ the right-hand and the left-hand sides of (4.3.1), respectively. Because of Lemma 2, then there exist points $a \in (\partial_f G)_r$ and $w \in \widetilde{G}$ such that $|a - w| = \delta$ and

$$|h(a) - h(w)| > A(\delta) - \varepsilon = B(\delta) - \varepsilon.$$

One may check that in this case all requirements of Theorem 3 are fulfilled, and using it we get

$$\sup_{\zeta \in \widetilde{G},\, |\zeta - a| = \delta} |h(\zeta) - h(a)| \leq \mu(\delta).$$

Consequently, $B(\delta) - \varepsilon \leq \mu(\delta)$. Letting $\varepsilon \to 0$, we obtain $B(\delta) \leq \mu(\delta)$. Theorem 6 is proved.

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