Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 26, 2001, 225–231

ON FOLDING SOLUTIONS OF THE BELTRAMI EQUATION

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Abstract. A sufficient condition for the existence of a local folding solution to an alternating Beltrami equation is presented, and the uniformization of local folding solutions is studied.

1. Introduction

1.1. Consider the Beltrami equation

$$w_{\bar z} = \mu(z) w_z \tag{B}$$

in the unit disk $D = \{z \in \mathbf{C} : |z| < 1\}$, where $\mu\colon D \to \overline{\mathbf{C}}$ is measurable, μ is locally bounded in $D_1 = \{z \in D : \operatorname{Im} z > 0\}$, i.e. $\|\mu|K\|_\infty < 1$ for every compact set K in D_1, and $1/\mu$ is locally bounded in $D_2 = \{z \in D : \operatorname{Im} z < 0\}$.

Equation (B) can be rewritten in the symmetric form

$$A(z) w_z + B(z) w_{\bar z} = 0 \tag{B$'$}$$

where $A(z)$ and $B(z)$ are complex-valued measurable functions such that $\mu = -A/B$ in D_1 and $1/\mu = -B/A$ in D_2. Let $E = D \cap \mathbf{R}$.

1.2. Let μ be as in 1.1. A function $f\colon D \to \mathbf{C}$ is said to be a *solution* of (B), if f is continuous, $f \mid (D \setminus \mathbf{R})$ is in $\mathrm{W}^{1,2}_{\mathrm{loc}}$, and its weak derivatives satisfy (B$'$) a.e. in D. A solution f in D is a *folding solution* if it is injective in D_1, in D_2 and in E. If, in addition, $f(D_1) = f(D_2)$ f is a *proper folding solution.*

In Section 2 below, we provide a sufficient condition for the existence of a local folding solution, i.e. a local solution which is a folding, and in Section 3, we check to what extend local folding solutions can be uniformized to a global folding solution.

2000 Mathematics Subject Classification: Primary 30C62, 35J70,30B40.

2. Existence of local folding solutions

2.1. Theorem. *Let D, D_1, D_2, E and μ be as in 1.1. If there exist $r \in (0,1)$, a non-negative integer m, a real-valued real analytic function $\theta(z)$, $|z| < r$ with $|\theta(z)| < \frac{1}{2}\pi$ and a complex-valued real analytic function $M(x,y)$ in $|z| < r$, $z = x + iy$, with*

$$\operatorname{Re} M(0,0) > 0 \tag{2.2}$$

and such that for $|z| < r$,

$$\mu(z) = e^{2i\theta(z)}[1 - y^{2m+1}M(x,y)], \tag{2.3}$$

then (B) *has a folding solution in some neighborhood of* 0.

Proof. In [SY2, Theorem 1.1] we considered a Beltrami equation (B) with

$$\mu(z) = e^{2i\theta(z)}\big[1 - \varrho(y)M(z)\big] \tag{M}$$

where θ and ϱ are real-valued functions, and M is a complex-valued function which satisfy certain conditions. We then applied a process, which we called a deformation of the complex dilatation, and reduced (B) to a Beltrami equation in the (ξ,η)-plane with complex dilatation

$$\tilde{\mu}(\xi,\eta) = 1 - \varrho(\eta)\widetilde{M}(\xi,\eta)$$

where ϱ is the same as in (M), and $\widetilde{M}$ satisfies a similar condition as M, see [SY2, pp. 475–477]. The same method can be applied here. Thus we will assume that $\theta(z) \equiv 0$. We will also assume that $m = 0$ and that M is a polynomial. The general case is proved in a similar way.

In view of (2.2), M has the form

$$M(x,y) = 2a + 2ib + \sum_{n+k=1}^{N} q_{nk} z^n \bar{z}^k \tag{2.4}$$

where $N \ge 1$, a and b are real and $a > 0$.

As in [L, pp. 23–24] we set in (B)

$$w = f(z) = z + c_{01}\bar{z} + \sum_{n+k=2}^{\infty} c_{nk} z^n \bar{z}^k \tag{2.5}$$

with $c_{n0} = 0$ for all $n > 1$. With application of (2.3), one obtains all c_{nk} (uniquely). Again, exactly as in [L], one can show that the series in (2.5) converges uniformly in $U_0 = \{z : |z| < r_0\}$ for some $r_0 \in (0,r)$.

It remains to show that f is a folding in some neighborhood U of 0 which is contained in U_0. Let $f = u + iv$, then by (2.5)

$$u = 2x - b(x^2 + y^2) + \sum_{n+k=3}^{\infty} \alpha_{nk} x^n y^k \tag{2.6}$$

and

$$v = a(x^2 + y^2) + \sum_{n+k=3}^{\infty} \beta_{nk} x^n y^k \tag{2.7}$$

for some α_{nk} and β_{nk} which depend on the c_{nk}'s.

Next, define $\varphi\colon U_0 \to \mathbf{C}$ by letting

$$\varphi(x,y) = \big(u(x,y), y\big) \tag{2.8}$$

with u as in (2.6). Then φ has a non-zero Jacobian at 0, and hence it is a diffeomorphism in some neighborhood $U_1 \subset U_0$ of 0. Consequently, there exists a mapping $F\colon \varphi(U_1) \to \mathbf{C}$ such that $f = F \circ \varphi$. Since $\varphi \mid U_1$ is injective, it sufficies to show that F is a folding. Now, $(\varphi \mid U_1)^{-1}$ has the form

$$\varphi^{-1}(u,y) = \big(x(u,y), y\big), \tag{2.9}$$

hence,

$$F(u,y) = \big(u, \tilde{v}(u,y)\big), \tag{2.10}$$

where, by (2.7),

$$\tilde{v}(u,y) = v\big(x(u,y),y\big) = a\big[x^2(u,y) + y^2\big] + \sum_{n+k=3}^{\infty} \beta_{nk} x^n(u,y) y^k, \tag{2.11}$$

or

$$\tilde{v}(u,y) = A_0(u) + A_1(u)y + A_2(u)y^2 + \sum_{k=3}^{\infty} A_k(u) y^k \tag{2.12}$$

for some functions $A_k(u)$.

We now show that $A_1(u) \equiv 0$ and $A_2(u) \neq 0$ in some neighborhood of 0. This will imply that F is a folding in some neighborhood of 0. Indeed, in view of (2.3), the Jacobian of f vanishes at all points $(x,0)$. Hence for $y = 0$,

$$A_1(u) = \frac{\partial \tilde{v}(u,y)}{\partial y}\Big|_{y=0} = J_F|_{y=0} = J_{f\circ\varphi^{-1}}|_{y=0} = 0.$$

By (2.11) and (2.12)

$$2A_2(0) = \frac{\partial^2 \tilde{v}(0,0)}{\partial y^2} = 2a > 0,$$

and by continuity, $A_2(u) > 0$ in $|u| < \delta$ for some $\delta > 0$.

Finally, since $A_2(u) > 0$ and $A_1(u) \equiv 0$ for $|u| < \delta$ we can get from (2.12)

$$F(u,y) = \big(u, \tilde{v}(u,y)\big) = \big(u, A_0(u) + w^2(u,y)\big),$$

where

$$w(u,y) = A_2^{1/2}(u)\Big(y + \sum_{k=2}^{\infty} b_k(u) y^k\Big)$$

for some functions $b_k(u)$. This shows that F is a local folding in some neighborhood of 0, and hence so is f.

3. Uniformization

3.1. Let D, D_1, D_2, E and μ be as in 1.1. The main question addressed in this section is whether the existence of local folding solutions at every point of E implies the existence of a global folding solution in D.

In the proof of the following theorem, we will use the fact that if $f\colon U \to \mathbf{C}$ is a folding solution such that $U \cap E$ is connected, then $U \cap E$ has a simply connected neighborhood V, $V \subset U$ such that $V \cap E$ is connected, and $f \mid V$ is a proper folding solution, see [SY1, Lemma 2.2].

3.2. Theorem. *Let D, D_1, D_2, E and μ be as in* 1.1, *and suppose that every point x in E has a neighborhood U_x, $U_x \subset D$, where* (B) *has a folding solution. Then E has a neighborhood V, $V \subset \bigcup_{x\in E} U_x$, such that*

(i) (B) *has a proper folding solution in V.*

(ii) (B) *has a folding solution g in $D_1 \cup V$ and a folding solution h in $D_2 \cup V$.*

Proof. We first prove (ii). Let $x \in E$. Then x has a simply connected neighborhood V_x, $V_x \subset U_x$ such that $V_x \cap E$ is connected, and such that (B) has a proper folding solution $f_x\colon V_x \to \mathbf{C}$. Let $V = \bigcup_{x\in E} V_x$. Since μ is locally bounded in D_1, (B) has a homeomorphic solution g_1 in D_1, cf. [B]. We may assume that $g_1(D_1) = D_1$. We will show that g_1 has a homeomorphic extension, denoted again by g_1, on $D_1 \cup E$. Indeed, given $x \in E$, f_x maps $V_x \setminus D_2$ homeomorphically onto $f_x(V_x)$, and thus, every subarc of $f_x(V_x \cap E)$ is a free boundary arc of $f_x(V_x \cap D_1)$. Let $\hat{f}_x = f \mid V_x \cap D_1$. Then, $g_1 \circ \hat{f}_x^{-1}$ is conformal, and hence has a homeomorphic extension on $f_x(V_x \setminus D_2)$. Therefore, g_1 has a homeomorphic extension on $D_1 \cup (E \cap V_x)$. Consequently, g_1 has a homeomorphic extension on $D_1 \cup E$. We may assume that $g_1(E) = E$.

We now define g in $D_1 \cup V$ as follows. If $z \in D_1 \cup E$, we set $g(z) = g_1(z)$. Given x in E, we define g in $V_x \cap D_2$ by letting $g(z) = g_1 \circ \hat{f}_x^{-1} \circ f_x(z)$. The mapping g is well defined in $V \cap D_2$, since by the uniqueness theorem for proper folding solutions [SY1, Theorem 3.2], once two points have the same image by a proper folding solution, they will have the same image under any other folding solution. Clearly, g is a folding solution in $D_1 \cup V$.

The mapping h for (ii) is constructed in the same way. As for (i), note that $g \mid V$, with g and V as above, is a proper folding solution of (B) in V, as needed.

In spite of Theorem 3.2 and contrary to the uniformization theorem, we have the following two theorems.

3.3. Theorem. *Let* D, D_1, D_2, E *be as in* 1.1. *Given a point* $z_0 \in D \setminus E$, *there exists* μ *as in* 1.1 *such that*

(i) (B) *has a local folding solution at every point of* E.
(ii) (B) *has a global solution in* D.
(iii) *Every global solution of* (B) *in* D *branches at* z_0, *and in particular,* (B) *has no (global) folding solution in* D.

Proof. Let $T(z) = i(z - \frac{1}{4})(1 - \frac{1}{4}z)$ and $\psi(z) = z^2$. Let E' be the connected component of $\psi^{-1}\bigl(T^{-1}(E)\bigr)$, which contains the point $\frac{1}{2}$. Let φ be a diffeomorphism of D onto itself such that $\varphi(z_0) = 0$ and $\varphi(E) = E'$, and let $F(x,y) = (x,|y|)$. Set $f = F \circ T \circ \psi \circ \varphi$ and $\mu = \mu_f$. Then μ satisfies the conditions of 1.1, f is a solution of (B) in D, and every point x of E has a neighborhood U_x where $f \mid U_x$ is a local proper folding solution of (B). Note that f branches at z_0 with local index $i(z_0, f) = 2$.

We now show that every other global solution branches at z_0. Suppose that g is a solution in D. We may assume that $\operatorname{Im} z_0 < 0$. The case $\operatorname{Im} z_0 > 0$ is similar. Let $G_1 = D_1$, let G_2 be the subdomain of D which lies between E and $\varphi^{-1}(L)$ where L is the intersection of D with the imaginary axes, and let $G_3 = D_2 \setminus \overline{G_2}$. Next for $i = 1,2,3$, let $f_i = f \mid G_i$ and $h_i = g \circ f_i^{-1}$. Then h_1 and h_3 are defined in D_1, and h_2 is defined in $D_1 \setminus F \circ T \circ \psi(L)$. Since g and all f_i are $\mathrm{W}^{1,2}_{\mathrm{loc}}$ solutions of (B) off E it follows that each h_i is analytic. Also,

$$g = h_i \circ f_i. \tag{3.4}$$

The mappings f_1 and f_2 extend homeomorphically on $G_1 \cup E$ and $G_2 \cup E$, respectively, and they coincide on E. Therefore, h_1 and h_2 have the same boundary values on E, and hence $h_1 = h_2$ in $D_1 \setminus L'$, where $L' = T \circ \psi(L)$. However, h_1 is analytic in D_1, therefore h_2 can be extended analytically to D_1, and $h_1 = h_2$ in D_1.

Finally, f_2 and f_3 coincide on $\varphi^{-1}(L)$, therefore h_2 and h_3 have the same boundary values on L' and therefore $h_2 = h_3$ in $D_1 \setminus L'$, and hence in D_1, when continued analytically across L'.

We obtain that $h_1 = h_2 = h_3 := h$ is analytic in D_1, and has a continuous extension on $D_1 \cup E$, and in view of (3.4), satisfy $g = h \circ f$. Therefore g branches at z_0 and consequently there is no global folding solution in D.

In view of the last theorem, one may ask: Suppose that (B) has a global solution in D which folds along E, and which is locally injective off E. Does it follow that (B) has a global folding solution in D? The following example answers this question negatively.

3.5. Theorem. *Given D, D_1, D_2, E as in* 1.1, *there exists μ satisfying the conditions of* 1.1 *such that*

(i) (B) *has a global solution in D, which folds along E and is locally injective off E.*

(ii) (B) *has no global folding solution in D.*

Proof. For $k = 0, 1, \dots, 8$, let $z_k = e^{ik\pi/8}$, and let I_k denote the subarc of ∂D_1 having end points z_{k-1} and z_k. Let Q' denote the square having vertices at the points $\frac{1}{4}i$, $\frac{3}{4}i$, $\frac{3}{4}i - \frac{1}{2}$ and $\frac{1}{4}i - \frac{1}{2}$.

Let f be a mapping of $\overline{D}$ onto $\overline{D_1}$ which has the following properties:

(a) $f(z) = \bar{z}$ for $\operatorname{Im} z \le 0$.

(b) $f(z_0) = 1$, $f(z_1) = i$, $f(z_2) = \frac{3}{4}i$, $f(z_3) = \frac{3}{4}i - \frac{1}{2}$, $f(z_4) = \frac{1}{4}i - \frac{1}{2}$, $f(z_5) = \frac{1}{4}i$, $f(z_6) = \frac{1}{8}i$, $f(z_7) = i$ and $f(z_8) = -1$.

(c) f is a local diffeomorphism in D_1, injective on each arc I_k, and maps I_1 and I_8 into $|z| = 1$ and each other arc I_k into a line segment.

(d) $f(D_1) = D_1 \setminus \big(\big[\frac{1}{8}i, \frac{1}{4}i\big] \cup \big[\frac{1}{2}i, i\big] \big)$, where $f \mid D_1$ covers every point of Q' twice, and every point in $D_1 \setminus \big(\big[\frac{1}{8}i, i\big] \cup Q' \big)$ once.

Then $\mu = \mu_f$ satisfies the conditions of 1.1, and f is a solution of (B) which folds along E and is a local homeomorphism in $D \setminus E$. Let Q denote the connected component of $f^{-1}(Q')$ whose boundary meets I_4, and let $D_0 = D \setminus \big(Q \cup \big[-i, -\frac{1}{8}i\big]\big)$. Then $f_0 = f \mid D_0$ is a proper folding solution in D_0 which maps D_0 onto $E \cup D_1 \setminus \big[\frac{1}{8}i, i\big]$.

Suppose now that (B) has a global folding solution in D. Then, by the uniqueness theorem for proper folding solutions [SY1, Theorem 3.2], $g \mid D_0 = h \circ f_0$, for some mapping h which is conformal in $\operatorname{int} f(D_0) = D_1 \setminus \big[\frac{1}{8}i, i\big] := D_1'$. However, g is homeomorphic in D_2, and $g = h \circ f$, hence h has a homeomorphic extension on D_1, denoted again by h.

Now $f^{-1}(\frac{1}{2}i)$ consists of three points $b_1 \in D_1$, $b_2 \in D_2$ and $b_3 \in \partial D_1$, or more precisely $b_3 \in I_7$, and thus a small neighborhood V of b_3 relative D_1 is mapped by f into Q'. It follows that $g^{-1}(w)$ has two pre-image points in D_1, whenever $w \in g(V) = h\big(f(V)\big)$, contradicting the assumption that g is a folding.

3.6. Remark. In [SY3] we showed that the existence of local folding solution at every point of E does not imply the existence of a (global) folding solution in D. There, the proof was based on the fact that $|a_2| \le 2$ for every schlicht function $f(z) = z + a_2 z^2 + \cdots$ in D.

Acknowledgment. The research of the first author was partially supported by the Fund for the Promotion of Research at the Technion. The authors acknowledge the hospitality of the Mathematics Department at the University of Helsinki.

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Received 16 August 1999