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COLLARS IN PSL(2,H)

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Abstract. Let $M = H^4/\Gamma$ denote a 4-dimensional oriented hyperbolic manifold of finite volume. By identifying $\mathrm{Iso}^+(H^4)$ with $\mathrm{PSL}(2, \mathbf{H})$, we construct embedded tubular neighborhoods around short simple closed geodesics in M whose collar width depends on the length of g, only. We show that two non-intersecting short geodesics have disjoint collars. Moreover, the constructed collars do not intersect the canonical cusps associated to parabolic elements in Γ. Finally, we discuss some applications and provide bounds for the injectivity radius and the number of short simple closed geodesics in M.

0. Introduction

Consider the group of orientation preserving Möbius transformations of $\widehat{E}^3 = E^3 \cup \{\infty\}$ which acts by direct isometries on hyperbolic 4-space realized in the upper half space model. As such, the group $\mathrm{Iso}^+(H^4)$ can be identified with the group $\mathrm{PSL}(2, \mathbf{H})$ of Clifford matrices with quaternion coefficients (cf. Section 1). In this setting, we study the geometry of discrete groups in $\mathrm{Iso}^+(H^4)$ and quotient manifolds of finite volume.

The main result of this paper is a collar theorem for hyperbolic 4-manifolds M providing around each simple closed geodesic g in M of length $l \le l_0 = (\sqrt{3}/4\pi)\log^2 2 \simeq 0.06622$ an embedded tubular neighborhood whose collar width depends on the length l of g, only (cf. Section 2.1). One important tool in the proof (cf. Section 2.2) is the inequality for discrete non-elementary two generator groups of Clifford matrices due to P. Waterman [Wat] which generalizes the well-known trace inequality for $\mathrm{PSL}(2, \mathbf{C})$. Moreover, we make use of properties of the generalized cross ratio for an ordered quadruple of vectors in a Clifford algebra as developed by C. Cao and P. Waterman [CW, Section 6].

In the remaining part of Section 2, we prove some results about the size and the position of the collars in the thin part of M. In Section 2.3, we show that distinct simple closed geodesics have disjoint collars. In Section 2.4, we investigate parabolic group elements and cusps in a non-compact manifold M of finite volume (cf. also [K2]). We show that the canonical cusps and the collars do not intersect in M. Moreover, for a sequence of loxodromic elements converging to a parabolic

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one, the collars tend to the canonical cusp. In this way, we obtain a fairly good picture about the thin part of a hyperbolic 4-manifold M.

Finally, in Section 3, we discuss some applications concerning the geometry and topology of hyperbolic 4-manifolds. For example, the injectivity radius $i(M)$ of a compact hyperbolic 4-manifold M satisfies the inequality $i(M) \geq \mathrm{const} \cdot \mathrm{vol}_4(M)^{-2}$ which improves a result of A. Reznikov [Re].

Our results extend the well-known collar theorems for Riemannian surfaces of genus > 1 and hyperbolic 3-manifolds (cf. [Bu, p. 94] for relevant references) to discrete subgroups of $\mathrm{Iso}^+(H^4)$ and their quotient manifolds. In [CW, Section 9], Cao and Waterman derived a collar theorem for hyperbolic manifolds of arbitrary dimensions $n \geq 2$. They approach isometries of hyperbolic n-space by means of Clifford matrices as well and utilize similar inequalities for discrete two generator groups as mentioned above. The methods differ where they make use of certain extremal values associated to the rotational part of loxodromic elements. Therefore, by specializing to the thin part of hyperbolic manifolds of dimension $n = 4$, it is not a surprise that our results are stronger when compared to theirs.

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1. Hyperbolic isometries and Clifford matrices

1.1. Some hyperbolic geometry. Let H^n denote the hyperbolic n-space realized in the Poincaré conformal model of the upper half space $E^n_+ = \{x = (x_1, \dots, x_n) \in E^n \mid x_n > 0\}$ equipped with the metric

$$\cosh \mathrm{dist}(x, y) = 1 + \frac{|x-y|^2}{2x_n y_n}.$$

The compactification $\overline{H^n} = H^n \cup \partial H^n$ consists of H^n together with the set $\partial H^n = \widehat{E}^{n-1} := E^{n-1} \cup \{\infty\}$ of its points at infinity.

The group $I(H^n)$ of isometries of H^n is isomorphic to the subgroup $M(E^n_+) \subset M(\widehat{E}^n)$ of Möbius transformations of $\widehat{E}^n$ that leave E^n_+ invariant. By means of Poincaré extension, there are the isomorphisms

$$I(H^n) \cong M(E^n_+) \cong M(\widehat{E}^{n-1}).$$

According to the fixed point behavior a Möbius transformation is either elliptic, parabolic, or loxodromic. For example, if $\varphi \in M(E^n_+)$ has precisely one, respectively two, fixed points in $\widehat{E}^{n-1}$ and none in E^n_+, then φ is parabolic, respectively loxodromic.

Γ is elementary if Γ has a finite orbit Γp for some point $p \in \overline{H^n}$. Moreover, Γ is said to be of parabolic, respectively hyperbolic, type if Γ has one, respectively two, different fixed points in ∂H^n and no further finite orbits in $\overline{H^n}$. There are the following characterizations (cf. [Ra]). Γ is discrete, elementary and of parabolic type if and only if Γ is conjugate to an infinite discrete subgroup of the isometry group $I(E^{n-1})$ of E^{n-1}. Let $S(E^{n-1})_*$ denote the group of all Möbius transformations in $M(E^n_+)$ leaving invariant the set $\{0,\infty\}$. Then, Γ is discrete, elementary and of hyperbolic type if and only if Γ is conjugate in $M(E^n_+)$ to an infinite discrete subgroup of $S(E^{n-1})_*$.

Finally, if $\varphi, \psi \in M(E^n_+)$ are such that ψ is loxodromic with one fixed point in common with φ, then the subgroup $\langle \varphi, \psi \rangle$ generated by φ, ψ is not discrete (cf. [Ra, Theorem 5, 5.4]). Therefore, a discrete elementary group Γ containing a loxodromic (parabolic) element, consists of loxodromic (parabolic) elements, only, and they all have the same fixed points.

1.2. Discrete groups of Clifford matrices. Let $\Gamma \subset I^+(H^n)$ denote a discrete group of orientation preserving isometries of H^n. We are interested in the geometrical behavior of Γ such as the uniform isolation of id in Γ. To this end, following K.Th. Vahlen [V] and L.V. Ahlfors [A1], [A2], we make use of the very elegant and important description of Möbius transformations as Clifford matrices. In the following, we provide a summary about the Clifford calculus in n variables.

The Clifford algebra C_n is the associative algebra over $\mathbf{R}$ generated by $n-1$ elements $i_1, \ldots, i_{n-1}$ subject to the relations $i_k i_l = -i_l i_k$ $(k \neq l)$ and $i_k^2 = -1$. Each element $a \in C_n$ can be uniquely represented in the form

$$a = \sum_I a_I I, \quad a_I \in \mathbf{R},$$

where I runs through all products $i_{k_1} \cdots i_{k_r}$ with $0 < k_1 < \cdots < k_r < n$. Here, the empty product is included and identified with $i_0 = i_{k_0} := 1$. The number $r \geq 0$ is called the degree of I.

Examples are $C_1 = \mathbf{R}$, $C_2 = \mathbf{C}$ and $C_3 = \mathbf{H}$.

C_n is a real vector space of dimension 2^{n-1} which can be turned into a normed space by imposing the Euclidean norm

$$|a|^2 := \sum_I a_I^2 \quad \text{for} \quad a = \sum_I a_I I.$$

Moreover, C_n admits a direct sum decomposition

$$C_n = \bigoplus_{r=0}^{n-1} C_n(r),$$

where $C_n(r) \subset C_n$ is spanned by the products I of degree r. Therefore, we can write

$$a = a(0) + a(1) + \cdots + a(n-1) \quad \text{with} \quad a(r) \in C_n(r).$$

There are three involutions on C_n. The mapping $a \mapsto a^*$ is defined by sending each $I = i_{k_1} \cdots i_{k_r}$ to $I^* := i_{k_r} \cdots i_{k_1}$, while $a \mapsto a'$ is given by replacing each factor i_k by $-i_k$. The conjugation $a \mapsto \overline{a}$ is the composition $\overline{a} := a'^*$. We obtain (cf. [Wad, p. 126])

$$\overline{a} = a(0) - a(1) - a(2) + a(3) + \cdots + (-1)^{(n-1)n/2} a(n-1). \tag{1.1}$$

Clifford numbers of the form $x = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$ are called vectors. They satisfy $x = x^*$. Vectors form an n-dimensional linear subspace of C_n which is usually identified with E^n. Non-zero vectors are invertible with inverse

$$x^{-1} = \frac{1}{|x|^2}(x_0 - x_1 i_1 - \cdots - x_{n-1} i_{n-1}).$$

Hence, products of non-zero vectors are invertible and give rise to a multiplicative subgroup, the Clifford group G_n. For $a, b \in G_n$, one has $|ab| = |a| \cdot |b|$. Embedding $E^n \subset E^{n+1}$ in the natural way, one obtains $G_n < G_{n+1}$.

A Clifford matrix is an element of the set

$$\mathrm{SL}(2, C_n) := \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in G_n \cup \{0\};\ ab^*, cd^*, c^*a, d^*b \in E^n; \right.$$
$$\left. ad^* - bc^* = 1 \right\}.$$

By a result of Vahlen and H. Maass (cf. [A2, p. 221]), $\mathrm{SL}(2, C_n)$ is a group under matrix multiplication with

$$T^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix} \quad \text{for} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n). \tag{1.2}$$

There are different ways to introduce the notion of trace for a matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n).$$

The quantity $\mathrm{tr}(T) = a + d^*$ is often called the trace of T. One checks that the scalar part of $\mathrm{tr}(T)$ is a conjugacy invariant (cf. [Wat, p. 99]) which we denote by

$$\mathrm{Tr}(T) := \mathrm{tr}(T)(0) = (a + d^*)(0). \tag{1.3}$$

Consider the subgroup

$$\mathrm{PSL}(2, C_n) := \mathrm{SL}(2, C_n)/\{\lambda E \mid \lambda \in \mathbf{R} - \{0\}\},$$

which satisfies $\mathrm{PSL}(2, C_n) < \mathrm{PSL}(2, C_{n+1})$. The group $\mathrm{PSL}(2, C_n)$ acts bijectively on $\widehat{E}^n$ by

$$T(x) = (ax + b)(cx + d)^{-1}, \quad T(0) = bd^{-1}, \quad T(\infty) = ac^{-1},$$

and this action can be extended to $\widehat{E}^{n+1} = \{x_0 + x_1 i_1 + \cdots + x_n i_n \mid x_k \in \mathbf{R}\} \cup \{\infty\}$, which we denote by the same symbol. Furthermore, $\mathrm{PSL}(2, C_n)$ is isomorphic to the group $M^+(\widehat{E}^n)$ of orientation preserving Möbius transformations of $\widehat{E}^n$ where matrix multiplication corresponds to composition of mappings. Each $T \in \mathrm{PSL}(2, C_n)$ preserves the upper half space (cf. Section 1.1) since

$$[T(x)]_n = \frac{x_n}{|cx + d|^2}; \quad \frac{|T'(x)|}{[T(x)]_n} = \frac{1}{x_n}. \tag{1.4}$$

Therefore, we have

$$I^+(H^{n+1}) \cong \mathrm{PSL}(2, C_n) \quad \text{for} \quad n \geq 1. \tag{1.5}$$

Among the generators of $\mathrm{PSL}(2, C_n)$ one has the

$$\text{dilation} \quad \begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix}, \quad x \mapsto \rho^2 x, \ \rho \neq 0; \tag{1.6}$$

$$\text{direct orthogonal} \quad \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, \quad x \mapsto a x a^*, \ |a| = 1; \tag{1.7}$$

$$\text{inversion} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x \mapsto -x^{-1}; \tag{1.8}$$

$$\text{translation} \quad \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad x \mapsto x + \mu, \mu \in E^n. \tag{1.9}$$

By [CW, Section 6], a notion of generalized cross ratio for an ordered tuple $\{x_1, x_2, x_3, x_4\}$ of four points in E^n, no three of which coincide, is defined by

$$[x_1, x_2, x_3, x_4] := (x_1 - x_3)(x_1 - x_2)^{-1}(x_2 - x_4)(x_3 - x_4)^{-1} \tag{1.10}$$

and extended to allow x_1, x_2 or x_4 to be infinite. This cross ratio satisfies several properties such as

$$\begin{gathered} [x_1, x_2, x_3, x_4] = [x_4, x_2, x_3, x_1]^{-1}; \\ [\infty, x, 0, y] = (y - x)y^{-1}. \end{gathered} \tag{1.11}$$

Finally, for a Möbius transformation

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, C_n),$$

there is the following pseudo-invariance

$$[T(x_1), T(x_2), T(x_3), T(x_4)] = (cx_3 + d)^{*\,-1}[x_1, x_2, x_3, x_4](cx_3 + d)^*. \tag{1.12}$$

We are interested in the geometry of a finitely generated discrete group of Möbius transformations, or more precisely, of its two generator subgroups. For $\mathrm{PSL}(2, \mathbf{C})$, Jørgensen's trace inequality (cf. [J], [Be, Section 5.4]) provides a very satisfactory picture. Using similar techniques, P. Waterman [Wat] generalized the result in various ways.

Theorem 1.1 ([Wat, Theorem 9]). *Let*

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix}$$

generate a discrete and non-elementary subgroup in $\mathrm{PSL}(2, C_n)$. *Let* $\tilde{\lambda} = 2\lambda(0) - \lambda$. *Then,*

$$|\bar{\tilde{\lambda}} - \lambda^{-1}|^2 \cdot (1 + |bc|) \geq 1 \quad \text{if } |\lambda| \neq 1.$$

Theorem 1.2 ([Wat, Theorem 8]). *Let*

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \in E^n,$$

generate a discrete and non-elementary subgroup in $\mathrm{PSL}(2, C_n)$. *Then,*

$$|c| \cdot |\mu| \geq 1.$$

Proof. We present a proof different in parts from [Wat] by adapting ideas of [Sh, p. 42]. Define recursively the sequence

$$S_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} := S, \quad S_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} := S_{n-1} T S_{n-1}^{-1} \quad \text{for} \quad n \geq 1.$$

By (1.2), one computes

$$\begin{aligned} a_n &= -a_{n-1}\mu c_{n-1}^* + a_{n-1}d_{n-1}^* - b_{n-1}c_{n-1}^*, \\ b_n &= a_{n-1}\mu a_{n-1}^* - a_{n-1}b_{n-1}^* + b_{n-1}a_{n-1}^*, \\ c_n &= -c_{n-1}\mu c_{n-1}^* + c_{n-1}d_{n-1}^* - d_{n-1}c_{n-1}^*, \\ d_n &= c_{n-1}\mu a_{n-1}^* + (a_{n-1}d_{n-1}^* - b_{n-1}c_{n-1}^*)^*. \end{aligned} \tag{1.13}$$

Since $a_k d_k^* - b_k c_k^* = 1$, $k \geq 0$, and since $x = x^*$ for the vectors $x = a_{n-1}b_{n-1}^*$, $c_{n-1}d_{n-1}^*$, we deduce

$$\begin{aligned} a_n &= 1 - a_{n-1}\mu c_{n-1}^*, \\ b_n &= a_{n-1}\mu a_{n-1}^*, \\ c_n &= -c_{n-1}\mu c_{n-1}^*, \\ d_n &= 1 + c_{n-1}\mu a_{n-1}^*. \end{aligned}$$

By induction, one obtains

$$|c_n \mu| = |c\mu|^{2^n},$$

and, by means of the triangle inequality,

$$|a_n - 1| = |d_n - 1| \leq |c\mu|^{2^{n-1}} \sum_{k=0}^{\infty} |c\mu|^k + |a_0 - 1| \cdot |c\mu|^{2^{n-1}+\cdots+1}.$$

Suppose that $|c\mu| < 1$. Then,

$$|a_n - 1| = |d_n - 1| \leq \frac{|c\mu|^{2^{n-1}}}{1 - |c\mu|} + |a - 1| \cdot |c\mu|^{2^n - 1}.$$

Therefore, $c_n \to 0$, $a_n, d_n \to 1$ and $b_n \to \mu$, that is, the sequence S_n converges to T. But $\langle S, T\rangle$ is discrete, so that $S_n = T$—and especially $c_n = 0$—for all n sufficiently large. Hence, $c = 0$, and the group $\langle S, T\rangle$ is elementary of parabolic type which is a contradiction to the assumption. □

1.3. Loxodromic elements in $\mathrm{PSL}(2,\mathbf{H})$**.** Consider hyperbolic space H^4 as subset of $\{x = x_0 + x_1 i_1 + x_2 i_2 + y i_3 \in C_3 \mid y > 0\}$. By (1.5), we have $I^+(H^4) \cong \mathrm{PSL}(2,\mathbf{H})$. The coefficient algebra $\mathbf{H}$ of quaternions is a real vector space with basis

$$1,\ i := i_1,\ j := i_2,\ k := ij \quad \text{satisfying} \quad i^2 = j^2 = k^2 = -1.$$

Accordingly, each element $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbf{H}$ can be written as $a = Sa + Va$ with scalar part $Sa := a_0$ and, by abuse of language, vector part $Va := a_1 i + a_2 j + a_3 k$. Since

$$\begin{aligned} \overline{a} &= Sa - Va, \\ |a|^2 &= a\overline{a} = (Sa)^2 - (Va)^2, \end{aligned}$$

a unit quaternion $a \in \mathbf{H}$, i.e., $|a|^2 = a_0^2 + (a_1^2 + a_2^2 + a_3^2) = 1$, may be expressed as

$$a = \cos\alpha + q\sin\alpha \quad \text{for some } \alpha \in [0, 2\pi), \tag{1.14}$$

where q is a pure unit quaternion, i.e., q satisfies $Sq = 0$ and therefore $q = -\overline{q}$. Actually, $q = Va/|Va|$. Since $q^2 = -1$, de Moivre's theorem yields (cf. [Co])

$$a^r = \cos r\alpha + q \sin r\alpha, \quad \text{for all } r \in \mathbf{R}. \tag{1.15}$$

The above motivates the notation $\exp(q\alpha) := \cos\alpha + q\sin\alpha$. For a *fixed* q, this function satisfies all well-known properties such as

$$\exp(q\alpha)\cdot\exp(q\beta) = \exp\bigl(q(\alpha+\beta)\bigr) = \exp(q\beta)\cdot\exp(q\alpha).$$

Notice that by (1.1) this picture fails for Clifford numbers $a \in C_n$ for $n > 3$.

Now, consider a loxodromic element $\varphi \in I^+(H^4)$ with fixed points $u, v \in \partial H^4$. The geodesic joining u and v is the unique line left invariant by φ, the so-called axis a_φ of φ, restricted to which φ acts as a translation with the translational length $\tau_1 = 2\log\rho$, say. Globally, φ acts as a dilation φ_1 with multiplier $e^{\tau_1} = \rho^2 \neq 1$ followed by a special orthogonal transformation φ_2, rotating in some 2-plane by an angle $\tau_2 = 2\omega$, say, and having one rotational axis (cf. (1.6), (1.7)).

In order to represent φ by a matrix in $\mathrm{PSL}(2,\mathbf{H})$, we simplify first. Conjugate $\varphi \in M^+(\widehat{E}^3)$ to obtain the Möbius transformation $\widetilde{\psi}(x) = \rho^2 Ax$ in $M^+(\widehat{E}^3)$ rotating in the plane spanned by 1, i with fixed j-axis, i.e.,

$$A = \begin{pmatrix} \cos 2\omega & -\sin 2\omega & 0 \\ \sin 2\omega & \cos 2\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3).$$

Then, the loxodromic transformation $\widetilde{\psi}(x) = \rho^2 Ax$ can be represented by

$$T = T_{\tilde{\psi}} = \begin{pmatrix} \rho\exp(i\omega) & 0 \\ 0 & \rho^{-1}\exp(-i\omega) \end{pmatrix} \in \mathrm{PSL}(2,\mathbf{H}). \tag{1.16}$$

By (1.3), $\mathrm{Tr}(T) = 2\cosh(\tau_1/2)\cdot\cos(\tau_2/2)$. In analogy to the complex case (cf. [M, p. 1040]), we call the conjugacy invariant pair $\tau = (\tau_1, \tau_2)$ the (quaternionic) length vector of the loxodromic transformation $\varphi \in I^+(H^4)$ (and of $T \in \mathrm{PSL}(2,\mathbf{H})$).

Later, we shall need the following property of loxodromic elements (cf. Proposition 2.6). Consider an arbitrary loxodromic element $T \in \mathrm{PSL}(2,\mathbf{H})$ with quaternionic length vector $\tau = (\tau_1, \tau_2)$.

Lemma 1.3. *Let $T \in \mathrm{PSL}(2,\mathbf{H})$ be a loxodromic element with axis a_T and with quaternionic length vector (τ_1, τ_2). Let $p \in H^4$ such that $p \notin a_T$ and assume that the perpendicular from p to a_T meets a_T at $\hat{p}$. Denote by $\alpha = \alpha(p)$ the angle at $\hat{p}$ in the triangle $\bigl(p, \hat{p}, T_2(p)\bigr)$. Let $d = \mathrm{dist}\bigl(p, T(p)\bigr)$ and $\delta = \mathrm{dist}(p, a_T)$. Then,*

$$\cosh d = \cosh\tau_1 + \sinh^2\delta\cdot\bigl(\cosh\tau_1 - \cos\alpha\bigr). \tag{1.17}$$

Proof. Assume without loss of generality that

$$T = \begin{pmatrix} e^{\tau_1/2}\exp(i\tau_2/2) & 0 \\ 0 & e^{-\tau_1/2}\exp(-i\tau_2/2) \end{pmatrix},$$

that is, a_T equals the positive i_3-axis, and T_2 is a rotation in the $(1,i)$-plane of $\{x = x_0 + x_1 i + x_2 j + y i_3 \in C_3\}$.

Observe that $\delta = \operatorname{dist}(p,\hat{p}) = \operatorname{dist}(|p|i_3, p) = \operatorname{dist}\big(|p|i_3, T_2(p)\big)$ satisfying $\cosh\delta = |p|/p_3$. Now, project the hyperbolic triangle $\big(p, |p|i_3, T_2(p)\big)$ orthogonally down to $\{y=0\} \simeq E^3$ and use Euclidean trigonometry to verify that

$$\cos\alpha = \frac{(p_0^2+p_1^2)\cos\tau_2 + p_2^2}{p_0^2+p_1^2+p_2^2} \geq \cos\tau_2. \tag{1.18}$$

Write $p = z + p_2 j + p_3 i_3$ for some $z = r\exp(i\psi)$ with $r^2 = p_0^2 + p_1^2$. Then, we obtain

$$\begin{aligned}
\cosh d &= 1 + \frac{|p - T(p)|^2}{2p_3^2 e^{\tau_1}} = 1 + \frac{|p - e^{\tau_1}\exp(i\tau_2/2)p\exp(i\tau_2/2)|^2}{2p_3^2 e^{\tau_1}} \\
&= 1 + \frac{|\big(r\exp(i\psi), p_2, p_3\big) - e^{\tau_1}\big(r\exp(i(\psi+\tau_2)), p_2, p_3\big)|^2}{2p_3^2 e^{\tau_1}} \\
&= 1 + \frac{|r\exp(i\psi) - re^{\tau_1}\exp\big(i(\psi+\tau_2)\big)|^2 + (p_2^2+p_3^2)(e^{\tau_1}-1)^2}{2p_3^2 e^{\tau_1}} \\
&= \cosh\tau_1 + 2\frac{p_2^2}{p_3^2}\sinh^2(\tau_1/2) + \frac{p_0^2+p_1^2}{p_3^2}\big(\cosh\tau_1 - \cos\tau_2\big) \\
&= \cosh\tau_1 + 2\sinh^2(\tau_1/2)\frac{p_0^2+p_1^2+p_2^2}{p_3^2} + \frac{p_0^2+p_1^2-(p_0^2+p_1^2)\cos\tau_2}{p_3^2} \\
&= \cosh\tau_1 + \sinh^2\delta\cdot\big(\cosh\tau_1 - \cos\alpha\big),
\end{aligned}$$

since $\sinh^2\delta = (p_0^2+p_1^2+p_2^2)/p_3^2$. This proves (1.17). □

2. The collar theorem

2.1. Statement of the result. Let M denote an oriented hyperbolic 4-manifold of finite volume. Then M can be written as a quotient H^4/Γ by a discrete torsion-free subgroup $\Gamma < \mathrm{PSL}(2,\mathbf{H})$ which is non-elementary.

Let g be a simple (i.e., with no self-intersection) closed geodesic in M of length $l(g)$. Consider a collar or tubular neighborhood

$$T_g(r) = \{p \in M \mid \operatorname{dist}(p,g) < r\}$$

around g embedded in M of collar width or radius $r = r(g) \geq 0$. We present a non-trivial lower bound for r depending on $l(g)$, only, which generalizes results of Brooks–Matelski [BM] and Meyerhoff [M].

Theorem 2.1. *Let $l_0 = (\sqrt{3}\,/4\pi)\log^2 2 \simeq 0.06622$. Then, each simple closed geodesic g in M of length $l(g) \le l_0$ has a collar of radius r satisfying*

$$\cosh(2r) = \frac{1-3k}{k}, \quad \textit{where} \quad k = \cosh\sqrt{\frac{4\pi l(g)}{\sqrt{3}}} - 1.$$

2.2. The proof. Each lift $\tilde{g}$ of g in H^4 is the axis of a loxodromic transformation in Γ given by some matrix $T \in \mathrm{PSL}(2, \mathbf{H})$ having length vector $\tau = (\tau_1, \tau_2)$. Here, $\tau_1 = l(g)$ and τ_2 encodes the twisting of T. We call τ also the length vector of g.

Two different lifts $\tilde{g}_1, \tilde{g}_2$ of g give rise to Γ-conjugate loxodromic elements T_1, T_2 with disjoint axes a_{T_1}, a_{T_2} but equal length vector τ. Denote by p the common perpendicular line of a_{T_1}, a_{T_2}. We have to estimate the length $2r$ of the segment of p between a_{T_1} and a_{T_2} in terms of the translational length τ_1. This will be done in two steps.

Since τ is a conjugacy invariant, we conjugate Γ such that we may assume that (cf. Section 1.3)

$$\begin{aligned} T_1 &= \begin{pmatrix} e^{\tau_1/2}\exp(j\tau_2/2) & 0 \\ 0 & e^{-\tau_1/2}\exp(-j\tau_2/2) \end{pmatrix}, \\ T_2 &= U T_1 U^{-1} \quad \text{with} \quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

Since $\langle T_1, T_2\rangle$ is non-elementary, $\langle T_1, U\rangle$ is non-elementary as well (cf. Section 1.1). Now, Theorem 1.1 applied to $\langle T_1, U\rangle$ yields

$$(2.1) \quad |e^{\tau_1/2}\exp(j\tau_2/2) - e^{-\tau_1/2}\exp(-j\tau_2/2)|^2 \cdot \big(1 + |bc|\big) = 2k \cdot \big(1 + |bc|\big) \ge 1,$$

where $k = \cosh\tau_1 - \cos\tau_2$.

Next, we have to relate $|bc|$ to $2r$ (cf. [Be, p. 112] for the case of dimension 3). For this, we take a Möbius transformation

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{H})$$

which maps $0, \infty, U(0), U(\infty)$, the fixed points of T_1, T_2, to $-w, w, -1, 1$ with $|w| > 1$, say. Hence, the common perpendicular p is mapped to the positive i_3-axis, and

$$2r = \mathrm{dist}(a_{T_1}, a_{T_2}) = \mathrm{dist}(a_{UT_1}, a_{UT_2}) = \log|w|.$$

Furthermore, by Section 1.2 and (1.12), we have

$$[-1, 1, -w, w] = {\delta^*}^{-1}[bd^{-1}, ac^{-1}, 0, \infty]\delta^*.$$

A short computation, using $ad^* - bc^* = 1$ and (1.11), yields

$$\tfrac{1}{4}(1-w)^2 w^{-1} = {\delta^*}^{-1} bc^* \delta^*. \tag{2.2}$$

Now, write $w = \rho \exp(I\omega)$ in E^3 for some $\omega \in [0, 2\pi)$ and a unit pure element $I \in \mathbf{H}(1)$. Then,

$$2r = \log \rho.$$

Putting $z := (2r + I\omega)/2$, we obtain

$$w = e^{2r} \exp(I\omega) =: \exp(2r + I\omega) = \exp(2z).$$

Next, define

$$\sinh z := \tfrac{1}{2}\{\exp(z) - \exp(-z)\}.$$

Then, by (1.15),

$$\sinh^2 z = \tfrac{1}{4}\{w + w^{-1} - 2\} = \tfrac{1}{4}(1-w)^2 w^{-1}.$$

Moreover, one computes

$$|\sinh z|^2 = \tfrac{1}{2}\big(\cosh(2r) - \cos\omega\big) \le \tfrac{1}{2}\big(\cosh(2r) + 1\big). \tag{2.3}$$

Thus, (2.1)–(2.3) yield

$$\tfrac{1}{2}\big(\cosh(2r) + 1\big) \ge |\sinh z|^2 = |{\delta^*}^{-1} bc^* \delta^*| = |bc| \ge \frac{1}{2k} - 1.$$

Hence, we proved the following (cf. [BM, Theorem 1, p. 166] and [M, Theorem, p. 1042] in the case of dimension 3 and [CW, Lemma 9.1, p. 133] in the case of arbitrary dimension).

Lemma 2.2. *Let g be a simple closed geodesic in M with quaternionic length vector τ such that $k = k(\tau) = \cosh\tau_1 - \cos\tau_2 < \frac{1}{4}$. Then, there is a collar $T_g(r)$ around g in M of radius r satisfying*

$$\cosh(2r) = \frac{1-3k}{k}.$$

The second step consists in estimating $\cosh(2r)$ in terms of $l(g) = \tau_1$ alone. This can be achieved by following the lines of proof in [M, pp. 1044–1046]. That is, observe first that Lemma 2.2 holds also for

$$k(T^n) = k(n\tau) = \cosh(n\tau_1) - \cos(n\tau_2) < \tfrac{1}{4}, \quad \text{for all } n \in \mathbf{N}.$$

Then, use the following lemma which is due to D. Zagier.

Lemma 2.3. ([M, Lemma, p. 1045]). *For* $0 < \tau_1 < \pi\sqrt{3}$ *and* $\tau_2 \in [0, 2\pi)$, *there exists a number* $n \in \mathbf{N}$ *such that*

$$\cosh(n\tau_1) - \cos(n\tau_2) \leq \cosh\sqrt{\frac{4\pi\tau_1}{\sqrt{3}}} - 1.$$

By choosing $\tau_1 = (\sqrt{3}\,/4\pi)\log^2 2$, Lemma 2.3 implies that

$$k(n\tau) \leq \cosh(\log 2) - 1 = \tfrac{1}{4}.$$

2.3. The relative size of a collar in M. In this paragraph we investigate some properties of the collars just constructed around sufficiently short closed geodesics in M such as the growth behavior of a collar in terms of the collar width and the mutual position of the collars around disjoint loops.

Let g be a simple closed geodesic in M of length $l = l(g)$. If $l \leq l_0 = (\sqrt{3}\,/4\pi)\log^2 2$, then Theorem 2.1 yields a collar $T_g(r)$ around g of radius r given by

$$\cosh(2r) = \frac{1-3k}{k}, \quad \text{where } k = \cosh\sqrt{\frac{4\pi l}{\sqrt{3}}} - 1.$$

Remark. It is easy to see that $r = r(l)$ is strictly decreasing.

We show that the volume of $T_g(r)$ as a function of l is strictly decreasing as well.

Lemma 2.4. *Let* $\mathrm{Cyl}(r,l) \subset H^n$ *denote a hyperbolic* n*-cylinder of radius* r *with axis of length* l. *Then, the volume* $\mathrm{vol}_n\big(\mathrm{Cyl}(r,l)\big)$ *of* $\mathrm{Cyl}(l,r)$ *is given by*

$$\mathrm{vol}_n\big(\mathrm{Cyl}(r,l)\big) = \frac{2\pi}{n-1}\cdot l\cdot \sinh^{n-1} r.$$

Proof. Consider the line $h \subset H^n$ determined by the axis of $\mathrm{Cyl}(l,r)$, and let S be the unit sphere in its orthogonal complement. A point $x \in H^n$ can be represented in the coordinates (t,y,s) where $t = \mathrm{dist}(x,h), y$ is the projection from x to h, and $s \in S$ is the tangent vector of the segment yx at y. With respect to these spherical-equidistant (or Fermi-) coordinates, the volume element of H^n turns into (cf. [G, (20), p. 91])

$$d\mathrm{vol}_n = \cosh t \sinh^{n-2} t\, dt\, dy\, ds.$$

Therefore, we can write

$$\begin{aligned}\mathrm{vol}_n\big(\mathrm{Cyl}(r,l)\big) &= \int_0^r \int_0^l \int_0^{2\pi} \cosh t \sinh^{n-2} t\, dt\, dy\, ds\\ &= 2\pi\cdot l\cdot \int_0^{\sinh r} u^{n-2}\, du\\ &= \frac{2\pi}{n-1}\cdot l\cdot \sinh^{n-1} r. \;\square\end{aligned}$$

Proposition 2.5. *Let* g *denote a simple closed geodesic in* M *of length* $l \leq l_0$. *Then, the volume* $\mathrm{vol}_4(T_g(r))$ *of the collar* $T_g(r)$ *of radius* $r = r(l)$ *is a strictly decreasing function of* l.

Proof. We have to investigate the growth of $\mathrm{vol}_4\big(T_g(r)\big) = \mathrm{vol}_4\big(\mathrm{Cyl}(r,l)\big)$. By Lemma 2.4, we can write

$$\mathrm{vol}_4\big(\mathrm{Cyl}(r,l)\big) = 2\pi/3 \cdot l \cdot \sinh^3 r = 2/3 \cdot \sinh r \cdot \mathrm{vol}_3\big(\mathrm{Cyl}(r,l)\big). \tag{2.4}$$

Now, $\sinh r$ is strictly monotonely decreasing in terms of l by the above remark. Moreover, the same is true for $\mathrm{vol}_3\big(\mathrm{Cyl}(r,l)\big) = \pi \cdot l \cdot \sinh^2 r(l)$. Namely, since $l \cdot k'(l) > k(l)$ by [M, p. 1047], one sees that

$$\frac{d}{dl}\big(l \cdot \sinh^2 r(l)\big) = \frac{d}{dl}\left(l \cdot \frac{1-4k(l)}{2k(l)}\right) < 0, \quad \text{where } k(l) = \cosh\sqrt{\frac{4\pi l}{\sqrt{3}}} - 1.$$

Hence, by (2.4), the assertion is proved. □

Proposition 2.6. *Let* g, g' *be two simple closed geodesics in* M *of lengths* $l, l' \leq l_1 := (\sqrt{3}/4\pi)\log^2 \frac{17}{9}$ $(< l_0)$ *which do not intersect. Then, the collars* T_g, $T_{g'}$ *of radius* $r(l)$, $r(l')$ *are disjoint.*

Proof. Let $\tilde{g}$, $\tilde{g}'$ be lifts to H^4 of g, g' in $M = H^4/\Gamma$, and denote by $T, T' \in \Gamma$ loxodromic elements with axes $\tilde{g}$, $\tilde{g}'$ and quaternionic length vectors τ, τ'. Put $\delta = \mathrm{dist}(\tilde{g}, \tilde{g}')$. We have to show that $\delta \geq r + r'$, where $r = r(l)$, $r' = r(l')$.

For this, conjugate T, T' in $\mathrm{PSL}(2, \mathbf{H})$ in order to obtain the elements

$$X = \begin{pmatrix} e^{\tau_1/2}\exp(j\tau_2/2) & 0 \\ 0 & e^{-\tau_1/2}\exp(-j\tau_2/2) \end{pmatrix}, \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of quaternionic length vectors equal to τ, τ'. Consider the element YXY^{-1} conjugate to X whose axis $a_{YXY^{-1}} = Y(a_X)$ is disjoint from a_X and a_Y. Let $p \in a_X$ denote the point such that $\delta = \mathrm{dist}(a_X, a_Y) = \mathrm{dist}(p, a_Y)$; p is the foot point on a_X of the common perpendicular of a_X, a_Y. By construction, $d := \mathrm{dist}\big(p, Y(p)\big) \geq 2r$. Now, using the notation $k' := k(Y) = \cosh\tau_1' - \cos\tau_2'$, Lemma 1.3 and (1.18) imply that

$$\begin{aligned} \cosh(2r) \leq \cosh d &= \cosh\tau_1' + \sinh^2\delta(\cosh\tau_1' - \cos\alpha) \\ &\leq k' + 1 + \sinh^2\delta \cdot k' = \cosh^2\delta \cdot k' + 1. \end{aligned}$$

By Theorem 2.1, we deduce that

$$\begin{aligned} \cosh(2\delta) &= 2\cosh^2\delta - 1 \geq 2 \cdot \frac{\cosh(2r) - 1}{k'} - 1 = 2 \cdot \frac{1-4k}{kk'} - 1 \\ &= \frac{1-4k}{kk'} + \frac{1-4k-kk'}{kk'}. \end{aligned}$$

Suppose that $k' \geq k$ (otherwise, exchange the role of X and Y). Then, we obtain

$$\cosh(2\delta) \geq \frac{\sqrt{1-4k}}{k} \cdot \frac{\sqrt{1-4k'}}{k'} + \frac{1-4k-kk'}{kk'}.$$

By assumption, $l \leq l_1 = (\sqrt{3}/4\pi)\log^2 17/9 < (\sqrt{3}/4\pi)\log^2(11+\sqrt{40})/9$. Therefore, by Theorem 2.1,

$$k = \cosh\sqrt{\frac{4\pi l}{\sqrt{3}}} - 1 < \frac{2}{9}.$$

Hence,

$$\cosh(2r) = \frac{1-3k}{k} < \frac{\sqrt{1-4k}}{k},$$

and similarly for $\cosh(2r')$. In order to conclude that $\cosh(2\delta) \geq \cosh(2r+2r') = \cosh(2r)\cdot\cosh(2r') + \sinh(2r)\cdot\sinh(2r')$, we will show that

$$\frac{1-4k-kk'}{kk'} \geq \frac{\sqrt{1-4k-k^2}}{k} \cdot \frac{\sqrt{1-4k'-k'^2}}{k'} \geq \sinh(2r)\cdot\sinh(2r').$$

First, it is very easy to see that

$$(1-4k-k^2)\cdot(1-4k'-k'^2) \leq (1-4k-kk')\cdot(1-4k'-kk').$$

Therefore,

$$\begin{aligned}\frac{1-4k-kk'}{kk'} &\geq \frac{\sqrt{1-4k-kk'}}{k} \cdot \frac{\sqrt{1-4k'-kk'}}{k'} \\ &\geq \frac{\sqrt{1-4k-k^2}}{k} \cdot \frac{\sqrt{1-4k'-k'^2}}{k'}.\end{aligned}$$

Secondly, since $k < \frac{2}{9}$, it can be checked that

$$\sinh(2r) = \frac{\sqrt{1-6k+8k^2}}{k} < \frac{\sqrt{1-4k-k^2}}{k},$$

and similarly for $\sinh(2r')$. This finishes the proof. □

2.4. Cusps and collars. In this paragraph, assume that $M = H^4/\Gamma$ is a non-compact oriented manifold of finite volume. That is, $\Gamma < \mathrm{PSL}(2,\mathbf{H})$ is a discrete non-elementary group without torsion containing parabolic elements which give rise to cusps in M.

In general, a cusp $C \subset H^n/\Gamma$ can be written as $C = C_q = V_q/\Gamma_q$ for some point $q \in \partial H^n$, where $\Gamma_q < \Gamma$ is of parabolic type with fixed point q (cf. Section 1.1), and where $V_q \subset H^n$ is some precisely invariant horoball based at q. Actually, one can associate to Γ_q a particular horoball B_q based at q such that B_q/Γ_q embeds in H^n/Γ. Assume for simplicity that $q = \infty$. By a theorem of Bieberbach, the subgroup $\Lambda = \Lambda(\infty) \subset \Gamma_\infty$ consisting of all translations (cf. (1.9)) is a lattice of finite index and rank $n-1$. Let $\mu \in \Lambda \cong E^{n-1}$ denote a shortest nontrivial vector with associated translation t_μ (cf. (1.9)). Then,

$$B(\mu) = B_\infty(\mu) = \left\{x \in H^n \mid \sinh \tfrac{1}{2}\mathrm{dist}\big(x, t_\mu(x)\big) < \tfrac{1}{2}\right\} = \{x \in H^n \mid x_n > |\mu|\}$$

is called the canonical horoball of Γ_∞. Generalizing [Be, Theorem 5.4.4]), S. Hersonsky [H] showed that $B_\infty(\mu)/\Gamma_\infty$ embeds in M and that the canonical horoballs associated to inequivalent parabolic elements in Γ are disjoint. The following two lemmata provide a different proof; this one is inspired by [Sh].

Lemma 2.7. *Let $\Gamma \subset \mathrm{PSL}(2, C_{n-1})$ be a discrete non-elementary subgroup containing the translation $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$. Define the horoball*

$$B(\mu) := \{x \in H^n \mid x_n > |\mu|\}.$$

Then, for all $S \in \Gamma - \Gamma_\infty$,

$$S\big(B(\mu)\big) \cap B(\mu) = \emptyset.$$

Proof. Suppose that there is an $S \in \Gamma - \Gamma_\infty$ such that $S\big(B(\mu)\big) \cap B(\mu) \neq \emptyset$. Hence, there are points $x, y \in B(\mu)$ such that $S(x) = y$. Write $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $S \notin \Gamma_\infty$, one has $c \neq 0$ and $S^{-1}(\infty) = -c^{-1}d \in E^{n-1}$ with $\left[c^{-1}d\right]_n = 0$. By (1.4), we deduce

$$|\mu| < y_n = \big[S(x)\big]_n = \frac{x_n}{|cx + d|^2} \leq \frac{1}{|c|^2 \cdot x_n}, \tag{2.8}$$

since $|cx + d|^2 = |c|^2 \cdot |x + c^{-1}d|^2 \geq |c|^2 \cdot x_n^2$. Therefore, $|c|^2 \cdot |\mu|^2 < 1$, which contradicts Theorem 1.2. ◻

Lemma 2.8. *Let $\Gamma \subset \mathrm{PSL}(2, C_{n-1})$ be a discrete non-elementary subgroup. For $i = 1, 2$, let $T_i \in \Gamma$ be a parabolic element with fixed point q_i such that $S_i^{-1} T_i S_i = \begin{pmatrix} 1 & \mu_i \\ 0 & 1 \end{pmatrix}$ for some $S_i \in \mathrm{PSL}(2, C_{n-1})$, and put $K_i := S_i\big(B(\mu_i)\big) = \{S_i(x) \mid x_n > |\mu_i|\}$. Then,*

$$U(K_1) \cap K_2 \neq \emptyset \text{ for some } U \in \Gamma \quad \Longrightarrow \quad U(q_1) = q_2.$$

Proof. Consider the matrix $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} := S_2^{-1} U S_1 \in \mathrm{PSL}(2, C_{n-1})$. By assumption, there are $y_i \in B(\mu_i)$, $i = 1, 2$, such that $V(y_1) = y_2$. Since $S_i(\infty) = q_i$, $i = 1, 2$, we have to show that $V(\infty) = \infty$, or equivalently, $c = 0$.

Let $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := VS_1^{-1}T_1S_1V^{-1}$. Then, $W = S_2^{-1}UT_1U^{-1}S_2$ implying that $W \in S_2^{-1}\Gamma S_2$, and by (1.13),

$$\gamma = -c\mu_1 c^* + cd^* - dc^* = -c\mu_1 c^*.$$

Since $V(y_1) = y_2$, we obtain in analogy to (2.8) that $|c|^2|\mu_1\mu_2| < 1$, whence

$$|\gamma| \cdot |\mu_2| = |c|^2|\mu_1\mu_2| < 1. \tag{2.9}$$

Consider now the elements

$$W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S_2^{-1}T_2S_2 = \begin{pmatrix} 1 & \mu_2 \\ 0 & 1 \end{pmatrix}$$

in the discrete group $S_2^{-1}\Gamma S_2$. They generate a discrete subgroup which, by (2.9) and Theorem 1.2, is elementary. This implies $\gamma = c = 0$. □

Consider an oriented hyperbolic 4-manifold M with cusps. We show now that the canonical cusps, i.e., those covered by canonical horoballs, and the collars around short closed geodesics, i.e., simple closed geodesics g of length $l(g) \leq l_0 = (\sqrt{3}/4\pi)\log^2 2 \simeq 0.06622$, are disjoint.

Theorem 2.9. *Let M denote a non-compact oriented hyperbolic 4-manifold of finite volume. Then, the canonical cusps and the collars around* short *closed geodesics in M do not intersect.*

Proof. Let $M = H^4/\Gamma$, and assume without loss of generality that Γ contains a parabolic element fixing ∞. Denote by

$$X = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$$

the translation by a vector $\mu \neq 0$ of minimal length in $\Lambda(\Gamma_\infty)$. Hence, $B(\mu)/\Gamma_\infty$ is a cusp in M.

Let g be a short closed geodesic in M. Denote by $Y \in \Gamma$ a loxodromic element with quaternionic length vector $\tau = (\tau_1, \tau_2)$ (cf. Section 1.3) whose axis projects to g. Consider an embedded collar $T_g(r)$ around g of radius r satisfying (cf. Lemma 2.2)

$$\cosh(2r) = \frac{1-3k}{k}, \quad \text{where } k = \cosh\tau_1 - \cos\tau_2 < \frac{1}{4}.$$

We have to show that $T_g(r)$ is disjoint from $B(\mu)/\Gamma_\infty$.

Since $\langle X,Y\rangle\subset\Gamma$ is discrete, by Section 1.1, the fixed points of Y lie in $\{y=0\}\simeq E^3$. Denote by $z\in E^3$ the center of the circle which contains the axis of Y. Conjugate Γ suitably by a translation which maps z to 0 and the fixed points of Y to $\pm\rho\exp(J\omega)$. Conjugating further by a suitable rotary dilation, we may assume that this new Y has fixed points ±1. That is, we may work with

$$X=\begin{pmatrix}1 & \mu/\rho\\ 0 & 1\end{pmatrix},\quad Y=\begin{pmatrix}a & b\\ b & a\end{pmatrix}\quad \text{for some } a,b\in\mathbf{H},$$

generating a discrete and non-elementary subgroup of Γ, and we have to show that

$$|\mu|/\rho\geq e^r.$$

By Theorem 1.2, we know that

$$|b|\cdot|\mu|/\rho\geq 1.$$

Therefore, it is enough to prove that

$$|b|^2\cdot e^{2r}\leq 1. \tag{2.10}$$

First, observe that

$$e^{2r}=\cosh(2r)+\sinh(2r)\leq 2\cosh(2r)=\frac{2}{k}(1-3k)<\frac{2}{k}. \tag{2.11}$$

Next, we estimate $|b|^2$ proceeding as in the proof of Theorem 2.1. Let

$$V=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{PSL}(2,\mathbf{H})$$

be a Möbius transformation mapping 0, ∞, $Y(0)$,$Y(\infty)$ to $-w$, w, -1, 1 (with $w\notin\mathbf{R}$). By Section 1.2 and (1.12), we deduce that

$$[-1,1,-w,w]=\delta^{*-1}[ba^{-1},ab^{-1},0,\infty]\delta^*,$$

and therefore

$$\tfrac{1}{4}(1-w)^2w^{-1}=\delta^{*-1}bb^*\delta^*.$$

We can write $w=e^{\tau_1}\exp(I\tau_2)=:\exp(\tau_1+I\tau_2)=\exp(2z)$ for some unit pure element $I\in\mathbf{H}(1)$. Since $\sinh^2 z=\frac{1}{4}(1-w)^2w^{-1}$, we obtain

$$|b|^2=|\sinh z|^2=\tfrac{1}{2}(\cosh\tau_1-\cos\tau_2)=\tfrac{1}{2}k. \tag{2.12}$$

Now, (2.12) combined with (2.11) yields (2.10). □

Proposition 2.10. *Consider a sequence of discrete torsion-free groups* $\Gamma_n <$ $\mathrm{PSL}(2,\mathbf{H})$ *containing loxodromic elements* $S_n \in \Gamma_n$ *with axes* a_{S_n} *and length vectors* τ_n. *For* $\tau_n \to 0$, *let* $S_n \to \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ *for some* $\mu \in E^3 - \{0\}$. *Then, the collars* T_n *around* a_{S_n} *of radius* $r(\tau_n)$ *tend to the horoball* $B(\mu) = \{x \in H^4 \mid x_n > |\mu|\}$ *in* H^4.

Proof. We adopt ideas of [BM, pp. 167–168]. The translation $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ has fixed point ∞. As usually, let $\mu = |\mu| \exp(I\omega)$.

We may conjugate Γ_n by

$$U := \begin{pmatrix} 0 & |\mu|^{1/2}\exp(I\omega/2) \\ |\mu|^{-1/2}\exp(-I\omega/2) & 0 \end{pmatrix} \in \mathrm{PSL}(2,\mathbf{H}),$$

so that

$$U^{-1}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =: P,$$

which fixes 0, and assume that $S_n \in \Gamma_n$ are loxodromic elements with limit P for $\tau_n \to 0$. That is, $S_n(\infty) \neq \infty$ for $n \geq n_0$, and the axis a_{S_n} of S_n, $n \geq n_0$, is a semicircle in E^4_+ perpendicular to $\{y = 0\}$. This semicircle can be shifted by a translation t_{μ_n} to a semicircle centered at 0 and by a direct rotation with matrix A_n such that the fixed points are $\pm\rho_n^{-1} \in \mathbf{R}$. This, together with the fixed point condition, yields the representation

$$S_n = \begin{pmatrix} 1 & -\mu_n \\ 0 & 1 \end{pmatrix} A_n^{-1} \begin{pmatrix} a_n & \rho_n^{-1} b_n \\ \rho_n b_n & a_n \end{pmatrix} A_n \begin{pmatrix} 1 & \mu_n \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

The axis a_{S_n} is at hyperbolic distance $|\log \rho_n|$ from the Euclidean hyperplane $\{y = 1\}$. For simplicity, assume that $\rho_n > 1$.

Since $S_n \to P$, the endpoints of a_n tend to $0 \in E^4$, and we also have $A_n t_{\mu_n} \to E$. Moreover, by looking at the non-diagonal entries of S_n in (2.13), the same reasoning as in the proof of Theorem 2.9 (cf. (2.12)) yields

$$|\rho_n b_n| = \rho_n |\sinh(\tau_n/2)| \to 1,$$
$$|\rho_n^{-1} b_n| = \rho_n^{-1} |\sinh(\tau_n/2)| \to 0.$$

In the difference, this gives the asymptotic behavior

$$\sinh(\log \rho_n) \sim \frac{1}{2|\sinh(\tau_n/2)|} \quad \text{for } n \geq n_0. \tag{2.14}$$

Now, associate to each S_n, $n \geq n_0$, its tubular neighborhood T_n of radius $r_n = r(\tau_n)$ given by (cf. Lemma 2.2)

$$\sinh(r_n) = \sqrt{\frac{1-2k(\tau_n)}{2k(\tau_n)}} = \sqrt{\frac{1}{4|\sinh(\tau_n/2)|^2} - 1}. \tag{2.15}$$

The hyperbolic distance from the top of T_n to the Euclidean hyperplane $\{y = 1\}$ equals $\log \rho_n - r_n$. Hence, the corresponding Euclidean distance behaves according to

$$\exp(\log \rho_n - r_n) - 1 \sim \frac{\sinh(\log \rho_n)}{\sinh r_n} - 1,$$

and tends to 0 by (2.4) and (2.15). □

3. Some applications

Let M be a compact oriented hyperbolic 4-manifold with Euler-characteristic $\chi(M)$. Closed geodesics in M come along with embedded tubular neighborhoods whose volumes increase the shorter the geodesics are. This property allows to derive some global assertions about the geometry and topology of M.

Consider the injectivity radius $i(M)$ of M which is related to the length of a shortest (and necessarily simple) closed geodesic g_0 in M by $i(M) = \frac{1}{2}l(g_0)$. Since M is 4-dimensional, by [Re, Theorem], we have the inequality

$$i(M) \geq \mathrm{const} \cdot \mathrm{vol}_4(M)^{-5}. \tag{3.1}$$

The following result improves the exponent of $\mathrm{vol}_4(M)$ in (3.1).

Proposition 3.1. *Let M denote a compact oriented hyperbolic 4-manifold. Then, the injectivity radius and the volume of M are related by*

$$i(M) \geq \mathrm{const} \cdot \mathrm{vol}_4(M)^{-2}. \tag{3.2}$$

Proof. Assume that there is a short simple closed geodesic g of length l in M. By Theorem 2.1, there is a tubular neighborhood $T_g(r) \subset M$ around g of radius r satisfying

$$\sinh^2 r = \frac{1}{2k} - 2, \quad \text{where } k = \cosh\sqrt{\frac{4\pi l}{\sqrt{3}}} - 1.$$

By Lemma 2.4, we get

$$\mathrm{vol}_4(M) \geq \mathrm{vol}_4\big(T_g(r)\big) = \frac{2\pi}{3} \cdot l \cdot \sinh^3 r.$$

Since $\sinh^3 r \sim \text{const} \cdot l^{-3/2}$, for small l, we obtain

$$l \geq \text{const} \cdot \mathrm{vol}_4(M)^{-2}$$

as desired. □

Remark. By [S, Theorem], the first positive element $\lambda_1(M)$ in the discrete spectrum associated to the eigenvalue problem $\triangle f + \lambda f = 0$ shows the same behavior (3.2) with respect to the volume $\mathrm{vol}_4(M)$ like the injectivity radius $i(M)$, namely, $\lambda_1(M) \geq \text{const} \cdot \mathrm{vol}_4(M)^{-2}$. More generally, there is a close relationship between the length (or geometric) spectrum and the eigenvalue spectrum of a compact hyperbolic n-manifold M (cf. [BB] and [Ri]). Well understood are their asymptotic distributions. For example, if $\Pi_M(t)$ denotes the number of all free homotopy classes of length $\leq t$ on M, then $\Pi_M(t) < \infty$, and

$$\Pi_M(t) \sim \frac{e^{(n-1)t}}{(n-1)t} \quad \text{for } t \to \infty.$$

Moreover, two compact hyperbolic n-manifolds have the same eigenvalue spectrum if and only if they have the same length spectrum.

Next, we present a bound for the number of different short closed geodesics in M. Two closed geodesics g, g' in M are called inequivalent if they do not belong to the same free homotopy class of closed curves in M (cf. [Bu, Section 1.6]).

Proposition 3.2. *Let M denote a compact oriented hyperbolic 4-manifold with Euler-characteristic $\chi(M)$. Let $g_1, \ldots, g_m$ be pairwise disjoint inequivalent simple closed geodesics in M of length < 0.04. Then, $m < 100\chi(M)$.*

Proof. As above, each of the simple closed geodesics g_i, $i = 1, \ldots, m$, of length $l(g_i) < l_1 := 0.04 < l_0 = (\sqrt{3}/4\pi)\log^2 2$ has a collar $T_i = T_i(r_i)$ embedded in M of radius $r_i = r\big(l(g_i)\big)$. By Proposition 2.6, the collars $T_1, \ldots, T_m$ are pairwise disjoint. Therefore, $\mathrm{vol}_4(M) \geq \sum_{i=1}^m \mathrm{vol}_4(T_i)$. By Lemma 2.4 and by Proposition 2.5, we obtain

$$\sum_{i=1}^{m} \mathrm{vol}_4(T_i) \geq m \cdot \mathrm{vol}_4\left(T\big(r(l_1)\big)\right) = \frac{2\pi}{3} \cdot m \cdot l_1 \cdot \sinh^3\big(r(l_1)\big).$$

On the other hand, the theorem of Gauss–Bonnet–Chern (cf. [Ch]) gives

$$\mathrm{vol}_4(M) = \tfrac{4}{3}\pi^2\chi(M),$$

which implies the bound $m < 100\chi(M)$. □

Remark. The proof shows that the data in the assertion of Proposition 3.2 can be varied. In general, the number m of pairwise disjoint inequivalent simple closed geodesics in M decreases the smaller the upper bound on their lengths is.

Example. There is one explicit geometrical construction of a compact oriented hyperbolic 4-manifold M^* which is due to M. Davis [D]. His construction is based on identifying through transvections opposite dodecahedral facets in the regular polytope $P = \{5,3,3\} \subset H^4$, the so-called 120-cell, with dihedral angles equal to $\frac{2}{5}\pi$. The polytope P can be decomposed into 14,400 congruent orthoschemes each of volume $13\pi^2/5{,}400$. This implies that the manifold M^* has Euler-characteristic 26 (cf. [D] and also [K1, p. 92]). The trigonometry of the dissecting orthoschemes can be well controlled. In particular, the edge length L of P can be computed by

$$\cosh L = \frac{3\tau}{2-\tau}, \quad \text{where } \tau = 2\cos\frac{\pi}{5},$$

which yields $L \simeq 3.23384$. Therefore, shortest closed geodesics in M^* are of lengths ≤ 3.23384. On the other side, proceeding as in the proof of Proposition 3.2, an arbitrary compact oriented hyperbolic 4-manifold M with $\chi(M) = 26$ has closed geodesics of lengths $> 9.82 \cdot 10^{-8}$.

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