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THE SECOND MOMENT OF THE SYMMETRIC SQUARE L-FUNCTIONS

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Abstract. In this paper we investigate the second power moment of symmetric square L-functions on the critical line, which are associated with primitive cusp forms. We establish an upper bound which is sharp with respect to the level.

1. Introduction

Estimates for L-functions on the critical line appear in many applications in number theory. The Riemann hypothesis, if true, provides essentially the best possible results. Quite often estimates on average over suitable families of L-functions are sufficient for practical purpose and the results are as good as the Riemann hypothesis can do.

In this paper we are interested in estimating the second power moment of L-functions associated with the symmetric square representations of GL_2 automorphic forms in the conductor aspect. According to the Gelbart–Jacquet lift [GJ] these correspond to automorphic forms on GL_3. Our main result is

Theorem 1.1. *Let k, N be positive integers, k even, N squarefree. Let $\mathscr{B}_k^*(N)$ be the set of primitive cusp forms of weight k with respect to the group $\Gamma_0(N)$. For each $f \in \mathscr{B}_k^*(N)$ let $L(\mathrm{sym}^2 f, s)$ be the corresponding symmetric square L-function. Let $\mathrm{Re}\, s = \frac{1}{2}$. We have*

$$\sum_{f\in\mathscr{B}_k^*(N)} |L(\mathrm{sym}^2 f, s)|^2 \ll |s|^8 N^{1+\varepsilon} \tag{1.1}$$

for any $\varepsilon > 0$, the implied constant depending only on ε and k.

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Note that the number of primitive forms satisfies (see (2.72) of [ILS])

$$|\mathscr{B}_k^*(N)| \sim \frac{k-1}{12}\varphi(N) \tag{1.2}$$

so that (1.1) implies

$$L(\mathrm{sym}^2 f, s) \ll N^{\varepsilon} \tag{1.3}$$

for almost all f (the implied constant depending on ε, k and s), while the Riemann hypothesis would imply this for every f. Ignoring all but one form in (1.1) we get

$$L(\mathrm{sym}^2 f, s) \ll N^{(1/2)+\varepsilon} \tag{1.4}$$

for every $f \in \mathscr{B}_k^*(N)$. This last estimate is nearly as good as the convexity bound which follows from the functional equation (2.23).

Inspite of three gamma factors in the functional equation for $L(\mathrm{sym}^2 f, s)$ (which fact is consistent with $\mathrm{sym}^2 f$ being a GL_3 form) in the N aspect (the conductor) $L(\mathrm{sym}^2 f, s)$ behaves like $L^2(s, \chi_N)$ for the real primitive Dirichlet character χ_N of conductor N. In that sense Theorem 1.1 can be compared with the estimate of Heath-Brown [HB] for the fourth power moment of Dirichlet L functions over the real characters on the critical line

$$\sideset{}{^\flat}\sum_{q\leq Q} |L(s,\chi_q)|^4 \ll |s|^A Q^{1+\varepsilon} \tag{1.5}$$

where (and throughout this paper) $\sum^\flat$ indicates restriction to positive odd squarefree integers. Here A is an absolute positive constant, ε is any positive number, and the implied constant depends only on ε. This analogy is justified because both $L(\mathrm{sym}^2 f, s)$ and $L^2(s, \chi_N)$ can be well approximated by the corresponding partial sums of comparable length relative to the conductor N. However the case of the symmetric square is somewhat harder. The point of distinction is that we are having the Fourier coefficients $\lambda_f(n^2)$ in place of the characters $\chi_N(n)\tau(n)$ for $n \ll N$, and the squares are sparse while the divisor function is quite regular by comparison. Heath-Brown derives (1.5) (among some other results) directly from his more general

Theorem 1.2. *For any complex numbers a_n we have*

$$\sideset{}{^\flat}\sum_{m\leq M}\left|\sideset{}{^\flat}\sum_{n\leq N} a_n\left(\frac{n}{m}\right)\right|^2 \ll (MN)^{\varepsilon}(M+N)\sideset{}{^\flat}\sum_{n\leq N}|a_n|^2 \tag{1.6}$$

with any $\varepsilon > 0$, the implied constant depending only on ε.

This is a deep and indeed powerful estimate. In the proof of our Theorem 1.1 we shall also use (1.6), but not as directly. Our method does not yield an asymptotic formula for the second power moment of $L(\mathrm{sym}^2 f, s)$ because the estimate (1.6) is not precise enough.

Since our result (1.1) just yields the convexity bound (1.4) one is encouraged to use the amplification method to get an improvement for individual $L(\mathrm{sym}^2 f, s)$. This would be possible if we could prove that

$$\sum_{f \in \mathscr{B}_k^*(N)} \lambda_f(l^2) |L(\mathrm{sym}^2 f, s)|^2 \ll l^{-\beta} N^{1+\varepsilon} \tag{1.7}$$

for every $l \leq N^\alpha$ with some small $\alpha, \beta > 0$. Investigating this question, some preliminary analysis led us to character sums of type

$$\sideset{}{^\flat}\sum_{c \leq C} \sum_{\substack{a \leq A,\, b \leq B \\ ab \neq \square}} \left(\frac{ab}{cN} \right) \chi(a) \psi(b) \tag{1.8}$$

with any $\chi, \psi \pmod{l}$. One needs this triple sum (and other slightly more general sums) for $A, B, C \ll N$ to be bounded by $O(N^{2-\eta})$ for some absolute $\eta > 0$, which seems to be plausible due to extra cancellation from summation in c, but the current technology fails.

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2. A brief account of automorphic L-functions

To fix notation and normalizations we recall standard facts about automorphic forms and their L-functions. The linear space $S_k(N)$ of cusp forms of weight k and level N is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathbf{H}} f(z) \overline{g}(z) y^{k-2} \, dx \, dy.$$

We have

$$\dim S_k(N) = \nu(N) \frac{k-1}{12} + O\big((kN)^{2/3}\big) \tag{2.1}$$

where

$$\nu(N) = [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p} \right). \tag{2.2}$$

Expand $f \in S_k(N)$ into Fourier series

$$f(z) = \sum_{n \geq 1} \psi_f(n) n^{(k-1)/2} e(nz). \tag{2.3}$$

Concerning the size of the Fourier coefficients $\psi_f(n)$ on can show that

$$\sum_{n \leq X} |\psi_f(n)|^2 \leq 3\omega_f \left(1 + \frac{80\pi X}{kN}\right) \tag{2.4}$$

where

$$\omega_f = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f \rangle. \tag{2.5}$$

Although the proof is elementary (we do not use (2.4) here) the result is quite good (only the constants in the bound (2.4) can be improved). The ω_f given by (2.5) appears to be a natural normalizing factor in the spectral trace

$$\Delta(m, n) = \sum_{f \in \mathscr{B}_k(N)} \omega_f^{-1} \overline{\psi}_f(m) \psi_f(n). \tag{2.6}$$

Here $\mathscr{B}_k(N)$ is any orthogonal basis of $S_k(N)$ and the sum (2.6) is basis independent. We have (cf. Theorem 3.6 of [I]) the following formula of Petersson

Proposition 2.1. *For any $m, n \geq 1$*

$$\Delta(m, n) = \delta(m, n) + 2\pi i^{-k} \sum_{c \equiv 0(N)} c^{-1} S(m, n; c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{2.7}$$

where $\delta(m, n)$ is the diagonal symbol, $J_{k-1}(x)$ is the Bessel function and

$$S(m, n; c) = \sum_{ad \equiv 1(c)} e\left(\frac{am + dn}{c}\right) \tag{2.8}$$

is the classical Kloosterman sum.

The Hecke operators $T_n \colon S_k(N) \to S_k(N)$ are defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad = n \\ (a, N) = 1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b \,(\mathrm{mod}\ d)} f\left(\frac{az + b}{d}\right). \tag{2.9}$$

For any $m,n\ge 1$ we have

$$T_mT_n=\sum_{\substack{d|(m,n)\\(d,N)=1}}T_{mn/d^2}. \tag{2.10}$$

Hence the Hecke operators commute. Moreover the operators T_n with $(n,N)=1$ are self-adjoint, i.e., $\langle T_nf,g\rangle=\langle f,T_ng\rangle$ for all f, g in $S_k(N)$ if $(n,N)=1$. Therefore we can choose the orthogonal basis $\mathscr{B}_k(N)$ which consists of common eigenfunctions of all the Hecke operators with $(n,N)=1$, i.e., for any $f\in\mathscr{B}_k(N)$ we require

$$T_nf=\lambda_f(n)f,\qquad \text{if } (n,N)=1. \tag{2.11}$$

We shall call any f which satisfies (2.11) a Hecke cusp form.

By (2.9), (2.10) and (2.11) one derives the formula

$$\psi_f(m)\lambda_f(n)=\sum_{d|(m,n)}\psi_f(mn/d^2) \tag{2.12}$$

for any $m,n\ge 1$ with $(n,N)=1$. In particular we obtain $\psi_f(n)=\psi_f(1)\lambda_f(n)$ if $(n,N)=1$. Moreover, it follows from (2.10) and (2.11) that

$$\lambda_f(m)\lambda_f(n)=\sum_{d|(m,n)}\lambda_f(mn/d^2),\qquad \text{if } (mn,N)=1. \tag{2.13}$$

We assume that the Hecke eigenbasis contains the subset $\mathscr{B}_k^*(N)$ of primitive forms (in the sense of Atkin–Lehner theory, see [I]). A primitive form f has many useful properties. First of all f is an eigenfunction of all the Hecke operators and the formulas (2.10), (2.12), (2.13) hold for all $m,n\ge 1$. Moreover f can be normalized so that

$$\psi_f(1)=1, \tag{2.14}$$

and the Fourier expansion becomes

$$f(z)=\sum_{n\ge 1}\lambda_f(n)n^{(k-1)/2}e(nz). \tag{2.15}$$

The associated Hecke L-function

$$L(f,s)=\sum_{n\ge 1}\lambda_f(n)n^{-s} \tag{2.16}$$

has Euler product $L(f,s)=\prod_p L_p(f,s)$ with the local factors

$$L_p(f,s)=\big(1-\lambda_f(p)p^{-s}+\chi_0(p)p^{-2s}\big)^{-1} \tag{2.17}$$

where χ_0 denotes the principal character to modulus N. Define the local factor at $p=\infty$ by

$$L_\infty(f,s)=\pi^{-s}\Gamma\Big(\frac{s}{2}+\frac{k-1}{4}\Big)\Gamma\Big(\frac{s}{2}+\frac{k+1}{4}\Big). \tag{2.18}$$

Then the complete product $\Lambda(f,s)=N^{s/2}L_\infty(f,s)L(f,s)$ is entire and it satisfies the functional equation

$$\Lambda(f,s)=\varepsilon_f\Lambda(f,1-s) \tag{2.19}$$

with the root number $\varepsilon_f=i^k\mu(N)\lambda_f(N)N^{1/2}=\pm1$.

The symmetric square L-function is defined by

$$L(\mathrm{sym}^2 f,s)=\zeta^{(N)}(2s)\sum_{n\geq1}\lambda_f(n^2)n^{-s}, \tag{2.20}$$

here and hereafter $\zeta^{(N)}(2s)$ stands for the partial zeta function with local factors at primes of N removed. This has the Euler product $L(\mathrm{sym}^2 f,s)=\prod_p L_p(\mathrm{sym}^2 f,s)$ with the local factors

$$L_p(\mathrm{sym}^2 f,s)=\big(1-\lambda_f(p^2)p^{-s}+\lambda_f(p^2)p^{-2s}-p^{-3s}\big)^{-1} \tag{2.21}$$

if $p\nmid N$ and

$$L_p(\mathrm{sym}^2 f,s)=(1-p^{-s-1})^{-1}$$

if $p|N$.

Remark. For $p\nmid N$ the $L_p(f,s)$ factors into $\big(1-\alpha_f(p)p^{-s}\big)^{-1}\big(1-\beta_f(p)p^{-s}\big)^{-1}$ with $\alpha_f(p)+\beta_f(p)=\lambda_f(p)$ and $\alpha_f(p)\beta_f(p)=1$, and then

$$L_p(\mathrm{sym}^2 f,s)=\big(1-\alpha_f^2(p)p^{-s}\big)^{-1}\big(1-\alpha_f(p)\beta_f(p)p^{-s}\big)^{-1}\big(1-\beta_f^2(p)p^{-s}\big)^{-1}.$$

Define the local factor at $p=\infty$ by

$$L_\infty(\mathrm{sym}^2 f,s)=\pi^{-3s/2}\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(\frac{s+k-1}{2}\Big)\Gamma\Big(\frac{s+k}{2}\Big). \tag{2.22}$$

Then the complete product $\Lambda(\mathrm{sym}^2 f,s)=N^sL_\infty(\mathrm{sym}^2 f,s)L(\mathrm{sym}^2 f,s)$ is entire and it satisfies the functional equation

$$\Lambda(\mathrm{sym}^2 f,s)=\Lambda(\mathrm{sym}^2 f,1-s). \tag{2.23}$$

For a primitive form one has (see Lemma 2.5 of [ILS])

$$\omega_f = \frac{k-1}{2\pi^2} N L(\mathrm{sym}^2 f, 1). \tag{2.24}$$

By the Ramanujan conjecture (proved by P. Deligne [D])

$$|\lambda_f(n)| \le \tau(N) \tag{2.25}$$

and the functional equation (2.23) we derive by convexity arguments that

$$L(\mathrm{sym}^2 f, 1) \ll (\log kN)^3 \tag{2.26}$$

where the implied constant is absolute. Hence

$$\omega_f \ll kN(\log kN)^3. \tag{2.27}$$

3. Representation of $L(\mathrm{sym}^2 f, s)$ by partial sums

The main Theorem 1.1 will be reduced to corresponding estimates for partial sums of (2.20). To this end we use the following formula which is obtained by standard contour integration of the functional equation (2.23).

Lemma 3.1. *Let A be a positive integer and $G(t) = \cos(\pi t/4A)^{-3A}$. For any s with $0 \le \mathrm{Re}\, s \le 1$ we have*

$$L(\mathrm{sym}^2 f, s) = \sum_{n\ge 1} \frac{\lambda_f(n^2)}{n^s} V_s\left(\frac{n}{N}\right) + \varepsilon(s) \sum_{n\ge 1} \frac{\lambda_f(n^2)}{n^{1-s}} V_{1-s}\left(\frac{n}{N}\right)$$

where

$$V_s(y) = \int_{(2)} G(t) \frac{L_\infty(\mathrm{sym}^2 f, s+t)}{L_\infty(\mathrm{sym}^2 f, s)} \zeta^{(N)}(2s+2t) y^{-t} \, \frac{dt}{t}$$

and $\varepsilon(s) = N^{1-2s} L_\infty(\mathrm{sym}^2 f, 1-s)/L_\infty(\mathrm{sym}^2 f, s)$.

Of course, the function $V_s(y)$ is independent of $f \in \mathscr{B}_k^*(N)$ so our notation is justified. We suppose from now on that $\mathrm{Re}\, s = \frac12$ so $|\varepsilon(s)| = 1$. If y is large we shift the contour of integration in $V_s(y)$ to $\mathrm{Re}\, t = A$, if y is small we shift the contour to $\mathrm{Re}\, t = -\frac12$, meeting poles at $t = 0$ and $t = \frac12 - s$ (eventually a double pole if $s = \frac12$). Then by Stirling's formula we derive

$$V_s(y) \ll \tau(N) \left(1 + \frac{y}{|s|^{3/2}}\right)^{-A} \log(2 + y^{-1}). \tag{3.1}$$

Moreover, taking the derivatives we derive

$$V_s^{(j)}(y) \ll \tau(N) y^{-j}\left(1+\frac{y}{|s|^{3/2}}\right)^{-A}\log(2+y^{-1}). \tag{3.2}$$

Now we apply a smooth partition of unity

$$1=\sum_{\alpha=-\infty}^{\infty} h\left(\frac{x}{2^{\alpha/2}}\right), \qquad \text{for } x>0,$$

and obtain from Lemma 3.1, (3.1) and (3.2) that for $f\in\mathscr{B}_k^*(N)$

$$L(\operatorname{sym}^2 f,s) \ll \tau(N)\sum_X |S_f(X)|\frac{\log 4NX}{\sqrt{X}}\left(1+\frac{X}{N|s|^{3/2}}\right)^{-A} \tag{3.3}$$

where $X=2^{\alpha/2}$ with $\alpha\geq -1$, and $S_f(X)$ is a sum of type

$$S_f(X)=\sum_n \psi_f(n^2) g\left(\frac{n}{X}\right) \tag{3.4}$$

where g is a smooth function supported in the interval $[1,2]$ which satisfies

$$g^{(j)}(x)\ll |s|^j, \qquad \text{for } j\geq 0 \tag{3.5}$$

and the implied constant depends on j and k only. By Theorem 5.1 for the particular sums (3.4) we have

$$\sum_{f\in\mathscr{B}_k(N)} \omega_f^{-1}|S_f(X)|^2 \ll |s|^5 (NX)^{\varepsilon}(N^{-1}X^2+X).$$

We need this only for the primitive forms. Dropping the other forms we derive by (3.3)

$$\sum_{f\in\mathscr{B}_k^*(N)} \omega_f^{-1}|L(\operatorname{sym}^2 f,s)|^2 \ll |s|^8 N^{\varepsilon}.$$

Finally applying the upper bound (2.27) for the weight ω_f we conclude Theorem 1.1. It remains to prove Theorem 5.1 for which we spend the rest of this paper.

4. Some properties of Bessel functions

In the sequel the following expression for the Bessel function will be useful (see [W, p. 206])

$$J_\kappa(x) = e^{ix}W(x) + e^{-ix}\overline{W}(x) \tag{4.1}$$

where

$$W(x) = \frac{e^{i((\pi\kappa/2)-\pi/4)}}{\Gamma(\kappa+\frac{1}{2})}\sqrt{\frac{2}{\pi x}}\int_0^\infty e^{-y}\left(y\left(1+\frac{iy}{2x}\right)\right)^{\kappa-1/2} dy.$$

When κ is a positive integer, we derive (using the Taylor expansion for $J_\kappa(x)$ if $0 < x \leq 1$, or the above integral expression for $W(x)$ if $x \geq 1$) the following bounds for the derivatives of W

$$x^j W^{(j)}(x) \ll \frac{x}{(1+x)^{3/2}} \tag{4.2}$$

for any $j \geq 0$, the implied constant depending on j and κ.

5. The mean-square of a partial sum

We consider g a smooth function supported in the interval $[1,2]$ which satisfies

$$g^{(j)}(x) \ll P^j \tag{5.1}$$

for some $P \geq 1$ and any $j \geq 0$, the implied constant depending on j only.

For $X \geq 1$ we define the partial sums

$$S_f(X) = \sum_n \psi_f(n^2) g\left(\frac{n}{X}\right)$$

and their mean-square

$$S(X) = \sum_{f \in \mathscr{B}_k(N)} \omega_f^{-1} |S_f(X)|^2.$$

Our goal is the following

Theorem 5.1. *For $X \geq 1$*

$$S(X) \ll P^3 (NPX)^\varepsilon (N^{-1}X^2 + X)$$

for any $\varepsilon > 0$, the implied constant depending on ε only.

6. Application of Petersson's formula

By Petersson's formula (2.6) we transform $S(X)$ into sums of Kloosterman sums

$$S(X) = R_0(X) + 2\pi i^k \sum_{c \equiv 0 (N)} R_c(X) \tag{6.1}$$

with

$$R_0(X) = \sum_n \left| g\left(\frac{n}{X}\right)\right|^2 \ll X \tag{6.2}$$

and

$$R_c(X) = \frac{1}{c} \sum_m \sum_n S(m^2, n^2; c) J_{k-1}\left(\frac{4\pi mn}{c}\right) \overline{g}\left(\frac{m}{X}\right) g\left(\frac{n}{X}\right).$$

Our next step will be to apply the Poisson summation formula for $R_c(X)$ to the n variable. However, this operation would produce poor results for c very large, so first we split the sum of $R_c(X)$ into sums of the type

$$R(C; X) = \sum_{\substack{C < c \le 2C \\ c \equiv 0 (N)}} R_c(X) \tag{6.3}$$

for $C \le N^{2000}$ and estimate the tail (the sum over the $c > 2N^{2000}$) by Weil's bound for individual Kloosterman sums getting $O(X^4 N^{-1000})$. From now on we deal with each remaining $R(C; X)$ separately with $c \sim C \le N^{2000}$.

Applying Poisson's formula on the n variable in $R_c(X)$, we get

$$R_c(X) = \sum_m \sum_h G(m, h; c) \overline{g}\left(\frac{m}{X}\right) \int_0^\infty J_{k-1}(4\pi m x) g\left(\frac{cx}{X}\right) e(-hx)\, dx \tag{6.4}$$

where

$$G(m, h; c) = \frac{1}{c} \sum_{a (c)} e\left(\frac{ah}{c}\right) S(m^2, a^2; c).$$

The complete sum $G(m, h; c)$ is computed in Section 8; it is shown there that $G(m, h; c) = G(D; c)$, where $D = h^2 - 4m^2$ and $G(D; c)$ is defined in (8.3). In particular we see that $G(m, h; c)$ depends on $D = h^2 - 4m^2$ rather than on h and m separately. Moreover an important feature is the factorization $D = (h - 2m)(h + 2m) = ab$, say, with $a \equiv b \pmod 4$. Making the change of variables $h = \frac{1}{2}(a + b)$, $m = \frac{1}{4}(b - a)$, we arrive at

$$R_c(X) = \sum_{D \equiv 0,1 (4)} G(D; c) \sum_{\substack{ab = D \\ a \equiv b (4)}} I_c(a, b)$$

where

$$I_c(a, b) = \overline{g}\left(\frac{b - a}{4X}\right) \int J_{k-1}(\pi(b - a)x) g\left(\frac{cx}{X}\right) e\left(-\frac{(a + b)x}{2}\right) dx.$$

7. Separation of the variables a, b

In order to separate the variables a and b we proceed by Fourier transform, and a very clean way to do it is to make use of the properties of the Bessel function which we have recalled in Section 4. From (4.1) we have a decomposition

$$
\begin{aligned}
I_c(a,b) &= \overline{g}\Big(\frac{b-a}{4X}\Big)\int e(-ax)W\big(\pi(b-a)x\big)g\Big(\frac{cx}{X}\Big)\,dx \\
&\quad + \overline{g}\Big(\frac{b-a}{4X}\Big)\int e(bx)\overline{W}\big(\pi(b-a)x\big)g\Big(\frac{cx}{X}\Big)\,dx \\
&= I_c^+(a,b)+I_c^-(a,b),
\end{aligned}
\tag{7.1}
$$

say. According to (7.1) we decompose $R(C;X) = R^+(C;X)+R^-(C;X)$. We treat $R^+(C;X)$, the other term being similar.

First of all we need to localize the range of a, b. By repeated integration by parts using (4.2) and (5.1) we obtain

Proposition 7.1. *We have,*

$$
I_c(a,b) \ll \left|g\Big(\frac{b-a}{4X}\Big)\right|\frac{X}{C}\Big(1+\frac{|a|X}{CP}\Big)^{-\alpha}
\tag{7.2}
$$

for any $\alpha \geq 0$, the implied constants depending on α only.

Using this estimate we see that, for any $\varepsilon > 0$, the contribution to $R^+(C;X)$ from the a, b such that $|a| \geq Y(NX)^{\varepsilon}$ (where $Y = CPX^{-1}$) is negligible, in fact $\ll N^{-A}$ for all $A > 0$ (the implied constant depending on ε and A). Note that the range of b is controlled by a since $|b-a| \ll X$. For the remaining portion we split the a, b sum into dyadic ranges and assume from now on that

$$
|a| \sim A, \quad |b| \sim B, \qquad \text{for some } A \leq Y(NX)^{\varepsilon}, \quad B \leq (NX)^{\varepsilon}(X+Y).
\tag{7.3}
$$

Accordingly we denote by $R^+(A,B,C;X)$ the corresponding portion of $R^+(C;X)$.

We have for $x \sim X/C$,

$$
\overline{g}\Big(\frac{z}{4X}\Big)W(\pi zx) = \int_{\mathbf{R}} h(t,x)e(tz)\,dt
$$

with (using twice partial integration and (4.2))

$$
h(t,x) = \int_{\mathbf{R}} \overline{g}\Big(\frac{z}{4X}\Big)W(\pi zx)e(-tz)\,dz \ll X\Big(1+\frac{|t|X}{P}\Big)^{-2}\frac{xX}{(1+xX)^{3/2}}
$$

so that

$$
\int_{\mathbf{R}} |h(t,x)|\,dt \ll \frac{PxX}{(1+xX)^{3/2}} \ll \frac{PX^2/C}{(1+X^2/C)^{3/2}}.
$$

Hence we have

$$
\begin{aligned}
R^+(A,B,C;X) &= \int_{X/2C}^{2X/C}\int_{-\infty}^{+\infty} h(t,x) \\
&\quad\times \sum_{\substack{c\sim C\\ c\equiv 0(N)}} g\left(\frac{cx}{X}\right)\sum_{D} G(D;c) \sum_{\substack{a\sim A,\, b\sim B\\ ab=D,\, a\equiv b(4)}}\sum e\big(-(t+x)a\big)e(tb)\,dt\,dx \\
&\ll \frac{X}{C}\frac{PX^2/C}{(1+X^2/C)^{3/2}} \sum_{\substack{c\sim C\\ c\equiv 0(N)}} \left|\sum_{D\equiv 0,1(4)} G(D;c) \sum_{\substack{a\sim A,\, b\sim B\\ ab=D,\, a\equiv b(4)}}\sum x_a y_b\right|
\end{aligned}
\tag{7.4}
$$

where x_a, y_b are complex numbers of modulus bounded by 1. Finally we detect the congruence $a\equiv b(4)$ by means of additive characters, and at the expense of changing the x_a and y_b we may remove this congruence condition. This finishes the process of separation of the variables a,b.

Now we may apply Proposition 10.1 to (7.4) which gives us

$$
\begin{aligned}
R^+(A,B,C;X) &\ll (PNX)^\varepsilon \frac{PX^3/C^2}{(1+X^3/C^2)^{3/2}} \\
&\qquad\times\left[Y(X+Y)\left(\frac{C}{N}+Y\right)\left(\frac{C}{N}+X+Y\right)\right]^{1/2} \\
&\ll (PNX)^\varepsilon P^3(N^{-1}X^2+X).
\end{aligned}
\tag{7.5}
$$

Summing over relevant A, B we get $R^+(C;X)\ll (PNX)^\varepsilon P^3(N^{-1}X^2+X)$. The same bound can be proved for $R^-(C;X)$. Finally summing these bounds over relevant $C\le N^{2000}$ by (6.1) and (6.2) we conclude the proof of Theorem 5.1. It remains to prove Proposition 10.1 for which we spend the rest of this paper.

8. Evaluation of an exponential sum

Our objective in this section is to compute the following exponential sum

$$
G(m,h;c) = \frac{1}{c}\sum_{a(c)} e\left(\frac{ah}{c}\right) S(m^2,a^2;c). \tag{8.1}
$$

Our computations will be completed in the next section. In Section 10 we will use the results to provide bounds for certain sums of bilinear forms involving $G(m,h;c)$.

Opening the Kloosterman sum and changing the variables we get

$$
G(m,h;c) = \frac{1}{c}\sum_{a\,(\mathrm{mod}\ c)} S(0,a^2+ah+m^2;c)
$$

where $S(0,w;c)$ is the Ramanujan sum. This sum has a simple formula

$$S(0,w;c)=\sum_{\substack{bd=c\\ b|w}}\mu(d)b.$$

Hence we find that

$$G(m,h;c)=\sum_{bd=c}\mu(d)\nu_b(h^2-4m^2)$$

where $\nu_b(h^2-4m^2)$ is the number of solutions to the congruence

$$a^2+ah+m^2\equiv 0 \pmod{b}$$

in $a \pmod{b}$. Note that by completing square this is also equal to half the number of solutions to the congruence

$$x^2\equiv D \pmod{4b} \tag{8.2}$$

in $x \pmod{4b}$ with $D=h^2-4m^2$. Therefore $G(m,h;c)=G(D;c)$ for $D=h^2-4m^2$, where

$$G(D;c):=\sum_{bd=c}\mu(d)\nu_b(D) \tag{8.3}$$

with $\nu_b(D)=\frac{1}{2}|\{x \pmod{4b},\ x^2\equiv D \pmod{4b}\}|$. From now on we consider the sum (8.3) for any integer $D\equiv 0,1 \pmod{4}$. Note that $G(D;c)$ is multiplicative in c so

$$G(D;c)=\prod_{p^\gamma\|c}G(D;p^\gamma)=\prod_{p^\gamma\|c}\big(\nu_{p^\gamma}(D)-\nu_{p^{\gamma-1}}(D)\big). \tag{8.4}$$

(In particular note that $G(D;1)=1$ since $D\equiv 0,1(4)$.) Let $\gamma\geq 1$. If $p\neq 2$ then $\nu_{p^\gamma}(D)$ is the number of solutions to $x^2\equiv D \pmod{p^\gamma}$ while $\nu_{2^\gamma}(D)$ is half the number of solutions to $x^2\equiv D \pmod{2^{\gamma+2}}$. Suppose $p\neq 2$ and $p^\alpha \| D$. If $\gamma\leq\alpha$, then $\nu_{p^\gamma}(D)=p^{[\gamma/2]}$. If $\gamma>\alpha$ then

$$\nu_{p^\gamma}(D)=\left(1+\left(\frac{Dp^{-\alpha}}{p}\right)\right)p^{\alpha/2}$$

when α is even, and $\nu_{p^\gamma}(D)=0$ when α is odd. Hence we conclude from (8.4) that $G(D;p^\gamma)$ is given by

$$\begin{array}{rl} 0, & \text{if } \gamma>\alpha+1,\\ \left(\dfrac{Dp^{-\alpha}}{p}\right)p^{(\gamma-1)/2}, & \text{if } \gamma=\alpha+1 \text{ odd},\\ -p^{(\gamma-2)/2}, & \text{if } \gamma=\alpha+1 \text{ even},\\ \left(1-\dfrac{1}{p}\right)p^{\gamma/2}, & \text{if } \alpha\geq\gamma \text{ even},\\ 0, & \text{if } \alpha\geq\gamma \text{ odd}. \end{array} \tag{8.5}$$

Now let $p=2$. If $\gamma>\alpha$ even then

$$\nu_{2^\gamma}(D)=\tfrac{1}{2}\bigl(1+\chi_4(D2^{-\alpha})\bigr)\bigl(1+\chi_8(D2^{-\alpha})\bigr)2^{\alpha/2},$$

if $\gamma>\alpha$ odd then $\nu_{2^\gamma}(D)=0$, if $\alpha=\gamma$ even then

$$\nu_{2^\gamma}(D)=\tfrac{1}{2}\bigl(1+\chi_4(D2^{-\alpha})\bigr)2^{\alpha/2},$$

if $\alpha=\gamma$ odd then $\nu_{2^\gamma}(D)=0$, if $\alpha=\gamma+1$ even then

$$\nu_{2^\gamma}(D)=\tfrac{1}{2}2^{\alpha/2},$$

if $\alpha=\gamma+1$ odd then $\nu_{2^\gamma}(D)=0$, and finally if $\alpha\geq\gamma+2$

$$\nu_{2^\gamma}(D)=2^{[\gamma/2]}.$$

Here χ_4, χ_8 are primitive characters of moduli 4 and 8 respectively. From the above results we conclude that $G(D;2^\gamma)$ is given by

$$\begin{aligned}
&\bigl(1+\chi_4(D2^{-\alpha})\bigr)\chi_8(D2^{-\alpha})2^{(\gamma-3)/2}, && \text{if } \gamma=\alpha+1 \text{ odd},\\
&\chi_4(D2^{-\alpha})2^{(\gamma/2)-1}, && \text{if } \gamma=\alpha \text{ even},\\
&-2^{(\gamma/2)-1}, && \text{if } \gamma=\alpha-1 \text{ even},\\
&2^{(\gamma/2)-1}, && \text{if } \gamma<\alpha-1 \text{ even},
\end{aligned}\tag{8.6}$$

and $G(D;2^\gamma)=0$ in the remaining cases. From the above computations we deduce the following

Proposition 8.1. *Let $D\equiv 0,1 \pmod 4$ and $c\geq 1$. Let $G(D;c)$ be the sum defined in (8.3). It is a multiplicative function of c and for $c=p^\gamma$ a prime power it is given by (8.5) for $p\neq 2$ and by (8.6) for $p=2$. Factoring uniquely c into qr, where q is squarefree, $4r$ is squareful and $(q,2r)=1$, we have*

$$G(D;c)=G(D;q)G(D;r)=\left(\frac{D}{q}\right)G(D;r).$$

9. Computation of $G(D;r)$

We now give a synthetic formula for $G(D;r)$ when $4r$ is squareful. For that we consider the (unique) factorization of D into $D=D_1D_2$, with D_1 squarefree odd, $4D_2$ squareful and $(D_1,D_2)=1$. Next we factor r accordingly into $r=r_1'r_2$ with $r_1'|D_1^\infty$, $(r_2,D_1)=1$ (again this decomposition is unique), so that

$$G(D;r)=G(D_1D_2;r_1'r_2)=G(D_1D_2;r_1')G(D_1D_2;r_2).$$

Using (8.5) one sees that $G(D;r) = 0$, unless $r_1' = r_1^2$ with $r_1|D_1$ (so r_1 is squarefree) in which case we have

$$G(D;r) = \mu(r_1)G(D_1D_2;r_2). \tag{9.1}$$

Finally we see from (8.5) and (8.6) that $G(D_1D_2;r_2) = 0$, unless D_2 and r_2 admit simultaneously factorization of the form

$$r_2 = 2^\gamma b_1^* b_1^2 b_2^2 b_3^2, \qquad D_2 = 2^\alpha b_1^2 (b_2^2/b_2^*) b_3^2 b_3' = 2^\alpha d_1 d_2 d_3 \tag{9.2}$$

with $2, b_1, b_2, b_3$ pairwise coprime, $\gamma \le \alpha + 1$, $b_3'|b_3^\infty$ and b_1^* being the product of distinct prime divisors of b_1 (if this simultaneous factorization exists, it is unique). Then we have

$$\begin{aligned} G(D;r_2) &= G(D_1D_2;2^\gamma)G(D_1D_2;b_1^*b_1^2)G(D_1D_2;b_2^2)G(D_1D_2;b_3^2) \\ &= G(D_1D_2;2^\gamma)\left(\frac{2^\alpha D_1 d_2 d_3}{b_1^*}\right) b_1 \mu(b_2^*)\frac{b_2}{b_2^*}\varphi(b_3) \\ &= \left(\frac{2^\alpha D_1}{b_1^*}\right)\varphi(D_1,D_2,r_2), \end{aligned} \tag{9.3}$$

say, with

$$\varphi(D_1,D_2,r_2) = G(D_1D_2;2^\gamma)\left(\frac{2^\alpha d_2 d_3}{b_1^*}\right) b_1 \mu(b_2^*)\frac{b_2}{b_2^*}\varphi(b_3). \tag{9.4}$$

If the factorization (9.2) does not hold we define $\varphi(D_1,D_2,r_2)$ to be zero. Summarizing what we have found so far we obtain

Proposition 9.1. *Let $D \equiv 0,1 \pmod 4$ and r be such that $4r$ is squareful. Let $D = D_1D_2$ and $r = r_1'r_2$ be the factorizations such that D_1 is squarefree, $4D_2$ is squareful, $(D_1, 2D_2) = 1$, $r_1'|D_1^\infty$ and $(D_1, r_2) = 1$. Then $G(D;r) = 0$, unless $r_1' = r_1^2$ with $r_1|D_1$ and r_2, D_2 admit the factorizations (9.2). In this case one has*

$$G(D;r) = \mu(r_1)\left(\frac{D_1}{b_1^*}\right)\varphi(D_1,D_2,r_2)$$

where b_1^* is defined by (9.2) and $\varphi(D_1,D_2,r_2)$ is given by (9.4).

Remark. It will be useful to know that for D_2, r_2 fixed $\varphi(D_1,D_2,r_2)$ depends only on the congruence class of D_1 mod 8, see (8.6).

10. An upper bound for a sum of quadratic forms

Given $A, B \geq 1$, $C \geq N$, and two sequences $\{x_a\}_{a\geq 1}$, $\{y_b\}_{b\geq 1}$ of complex numbers of modulus bounded by one, our goal is to obtain an upper bound for the average of the quadratic forms

$$\mathscr{B}(A,B,C) = \sum_{\substack{c\leq C\\ c\equiv 0(N)}} \Bigg| \sum_{D\equiv 0,1(4)} G(D;c) \sum_{\substack{a\leq A, b\leq B\\ ab=D}}\sum x_a y_b \Bigg|.$$

To this end we use the computations of $G(D;c)$ from the previous sections.

Proposition 10.1. *For x_a, y_b any complex numbers of modulus bounded by one,*

$$\mathscr{B}(A,B,C) \ll (ABC)^{\varepsilon}(AB)^{1/2}\Big(\frac{C}{N}+A\Big)^{1/2}\Big(\frac{C}{N}+B\Big)^{1/2}$$

for all $\varepsilon > 0$, the implied constant depending on ε.

Proof. In view of Propositions 8.1 and 9.1, we have

$$\mathscr{B}(A,B,C) = \sum\sum\sum_{\substack{q,\, r_1,\, r_2\\ qr_1^2 r_2\sim C\\ qr_1^2r_2\equiv 0(N)}} \Bigg| \sum\sum_{\substack{D_1,D_2,(D_1,D_2)=1\\ D_1D_2\equiv 0,1(4),\, r_1|D_1}} \Big(\frac{D_1}{b_1^*}\Big)\varphi(D_1,D_2,r_2)$$
$$\times \sum\sum_{\substack{a\leq A,\, b\leq B\\ ab=D_1D_2}} x_a y_b\Big(\frac{ab}{q}\Big)\Bigg|$$

where q and D_1 are squarefree and odd, $4r_2$ and $4D_2$ are squareful and have the same prime factors, and b_1^* is a certain positive number depending on the pair $\{D_2, r_2\}$. We write $a = a_1a_2$, $b = b_1b_2$ with $a_1 = (a, D_1)$, $b_1 = (b, D_1)$ (recall that D_1 is squarefree) so that $a_1b_1 = D_1$. We decompose the sum according to the congruence classes of a_1, b_1 modulo 8 (see the remark at the end of Section 9) and so our sum is bounded by 16 sums each one of the type

$$\sum_{D_2}\sum\sum_{\substack{r_1,r_2\\ (r_1,D_2)=1}} |\varphi(uv,D_2,r_2)| \sum_{a_2b_2=D_2} \sideset{}{^\flat}\sum_{q\equiv 0(N/(N,r_1r_2))} \Bigg| \sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{a,b,\ r_1|ab\\ (a,b)=(ab,2D_2)=1}} x_{aa_2}y_{bb_2}\Big(\frac{ab}{q}\Big)\Bigg|$$

where u, v are odd congruence classes modulo 8 and the last three sums are restricted by $qr_1^2r_2 \leq C$, $aa_2 \leq A$, $bb_2 \leq B$, respectively. Here the x_a, y_b may differ from the original ones by multiplication by complex numbers of modulus 0 or 1 (in particular we transfer the congruence conditions $a \equiv u(8)$, $b \equiv v(8)$ into the coefficients x_a, y_b). Next we estimate the last three sums by $\tau(r_1)$ sums of the form

$$\sideset{}{^\flat}\sum_{q\le Q}\Bigg|\sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{a,b\\(a,b)=1}} x_a y_b\left(\frac{ab}{q}\right)\Bigg|$$

with $Q = C(N, r_1r_2)/Nr_1^2r_2$, $a \le A/c$, $b \le B/d$, $cd = r_1D_2$. Then we use the following result which is an immediate consequence of Theorem 1.6.

Proposition 10.2. *For any complex numbers x_a, y_b we have*

$$\sideset{}{^\flat}\sum_{q\le Q}\Bigg|\sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{a\le A,\, b\le B\\(a,b)=1}} x_a y_b\left(\frac{ab}{q}\right)\Bigg| \ll (ABQ)^\varepsilon (Q+A)^{1/2}(Q+B)^{1/2}\|x\|\,\|y\|$$

for all $\varepsilon > 0$, the implied constant depending on ε only.

Using Proposition 10.2 we infer that $\mathscr{B}(A,B,C)$ satisfies

$$\mathscr{B}(A,B,C) \ll_\varepsilon (ABC)^\varepsilon (AB)^{1/2} \sum_{D_2\le AB}\ \sum\sum_{r_1^2r_2\le C} \frac{|\varphi(u,D_2,r_2)|}{D_2^{1/2}r_1^{1/2}}$$
$$\times\left[\frac{C(N,r_1r_2)}{Nr_1^2r_2} + \frac{C^{1/2}(N,r_1r_2)^{1/2}}{N^{1/2}r_1r_2^{1/2}}(A^{1/2}+B^{1/2}) + \frac{(AB)^{1/2}}{D_2r_1}\right]$$

where u is an odd congruence class modulo 8. First we sum over r_1, using the following three estimates

$$\sum_{r_1\le C^{1/2}} \frac{(N,r_1)}{r_1^{5/2}},\qquad \sum_{r_1\le C^{1/2}} \frac{(N,r_1)^{1/2}}{r_1^{3/2}},\qquad \sum_{r_1\le C^{1/2}} \frac{1}{r_1^{3/2}} \ll \tau(N),$$

we arrive at

$$\sum\sum_{D_2\le AB,\, r_2\le C} |\varphi(u,D_2,r_2)|\left(\frac{(N,r_2)}{D_2^{1/2}r_2} + \frac{(N,r_2)^{1/2}}{D_2^{1/2}r_2^{1/2}} + \frac{1}{D_2^{3/2}}\right).$$

Now appealing to the definition of $\varphi(u,D_2,r_2)$ (see (9.4)) we find that this is bounded by $O_\varepsilon\big((ABCN)^\varepsilon\big)$ for all $\varepsilon > 0$ (to see this we recall that the variables D_2, r_2 run over lacunary sets of integers, essentially squarefull numbers). Combining the above estimates we complete the proof of Proposition 10.1. □

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