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JOHN DISKS AND THE PRE-SCHWARZIAN DERIVATIVE

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Abstract. We present necessary and sufficient conditions on the pre-Schwarzian f''/f' for $G = f(\mathbf{D})$ to be a John disk. Our results extend theorems proved by Chuaqui, Osgood and Pommerenke. In the last part of this paper we obtain some new results connecting a functional of f''/f' introduced by Gehring and Pommerenke with John disks.

1. Introduction

This paper is concerned with the connection between the pre-Schwarzian or logarithmic derivative f''/f' and certain geometrical properties of $G = f(\mathbf{D})$ when f is a conformal mapping of the unit disc $\mathbf{D}$ in $\mathbf{C}$. In particular, we study the situation when G is a (bounded) John disk. We are able to give a sufficient condition on the norm

$$\|f''/f'\|_1 = \sup_{z\in\mathbf{D}}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$$

for G to be a John disk. This is the main result of the present paper. Similar results on the Schwarzian and the pre-Schwarzian derivatives for G to be a quasidisk are proved earlier by Ahlfors and Weill, [AW], and by Becker and Pommerenke [BP]. Our proof is based on ideas in a paper of Chuaqui, Osgood and Pommerenke, [COP], which again rely on a theorem of Pommerenke. In Section 2 we prove a quantitatively refined version of Pommerenke's theorem, and this improvement is used later in our paper. In Section 3 we give a new sufficient condition on the pre-Schwarzian for $G = f(\mathbf{D})$ to be a John disk, and in Section 4 we also obtain a necessary condition on f''/f' when G is a John disk. Theorem 4.3 is the main result mentioned above. In Section 5 we prove some new results concerning a function σ_ζ introduced in [COP].

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2. Bounded John disks and conformal mappings

In this section we first give a new proof of one version of Pommerenke's theorem mentioned in the introduction. The growth condition on f' is expressed by constants that are related to the constant $c \in [1,\infty)$, G being a c-John disk in the terminology of Näkki and Väisälä, [NV]. Our proof will hopefully also shed more light on the geometric ideas in the argument, and at the same time we fill in a gap in the (original) proof both in [P1] and [P2] in using Corollary 1.6, [P2]. But the main reason for giving this proof is that a closer inspection of the argument leads to a sharper version of Pommerenke's result.

We first need some definitions and basic results. The following definition is based on the classical definition by John, [J].

Definition 2.1. A bounded simply connected plane domain G is called a c-*John disk* for $c \geq 1$ with *John center* $w_0 \in G$ if for each $w_1 \in G$ there exists a rectifiable arc γ, called a *John curve*, in G with end points w_1 and w_0 such that

$$\sigma_l(w) \leq cd(w,\partial G) \tag{1}$$

for all w on γ, where $\sigma_l(w)$ denotes the euclidean length of $\gamma[w_1,w]$, the subarc of γ between w_1 and w, and $d(w,\partial G)$ denotes the distance between w and the boundary ∂G of G.

Remark. Unbounded John disks, and more generally unbounded John domains, are introduced in [NV], but for the rest of this paper we will understand by a John disk a bounded John disk.

Using the terminology of [NV] the following set is called a *length carrot* with *core* γ and *vertex* w_1:

$$\operatorname{car}_l(\gamma,c) = \bigcup\bigl\{B(w,\sigma_l(w)/c);\ w \in \gamma \setminus \{w_0,w_1\}\bigr\} \tag{2}$$

where: $B(x,r) = \{y; |x-y| < r\}$.

We also need the following result from [GHM] or [NV]:

Theorem 2.2. *A bounded simply connected domain G is a c-John disk with John center $w_0 \in G$ if and only if, up to constants, $\operatorname{car}_l(\gamma,c) \subset G$ for every hyperbolic segment γ in G terminating in w_0.*

Remark. It is well known that any point $w_0 \in G$ can be chosen as John center by modifying the constant c if necessary.

The following result is due to Pommerenke, ([P1]; see also [P2, p. 97]).

Theorem 2.3. *Let f map $\mathbf{D} = B(0,1)$ conformally onto G. Then the following conditions are equivalent*:

(i) *G is a John disk.*

(ii) *There exist constants* $M > 0$ *and* $\delta \in (0,1)$ *such that for each* $\zeta \in \mathbf{T}$, *the unit circle, and for* $0 \le r \le \varrho < 1$

$$|f'(\varrho\zeta)| \le M|f'(r\zeta)|\left(\frac{1-\varrho}{1-r}\right)^{\delta-1}.$$

Proof. (i) $\Rightarrow$ (ii): Knowing that G is a John disk, we choose $w_0 = f(0)$ as the John center and the hyperbolic segments as the John curves; G can be assumed to be a c-John disk with respect to this choice, where $c \in [1,\infty)$. Hence for $w = f(r\zeta)$ and $w_1 = f(\varrho\zeta)$, we have

$$\sigma_l(w) \le c\, d(w,\partial G) \qquad \text{for all } \varrho \in [r,1).$$

Equivalently:

$$\int_r^1 |f'(t\zeta)|\, dt \le c\, d(w,\partial G).$$

By the well-known distortion inequality, [P2, p. 9], we obtain from this inequality

$$\int_r^1 |f'(t\zeta)|\, dt \le c\,|f'(r\zeta)|(1-r^2) \le 2c|f'(r\zeta)|(1-r). \tag{3}$$

We next define

$$\psi(r) = (1-r)^{-1/2c}\int_r^1 |f'(t\zeta)|\, dt,$$

and we obtain

$$\begin{aligned}\psi'(r) &= \frac{1}{2c}(1-r)^{-(1/2c)-1}\int_r^1 |f'(t\zeta)|\, dt - (1-r)^{-1/2c}|f'(r\zeta)| \\ &= (1-r)^{-1/2c}\left[\frac{1}{2c}(1-r)^{-1}\int_r^1 |f'(t\zeta)|\, dt - |f'(r\zeta)|\right] \le 0,\end{aligned}$$

where the inequality follows from (3). Therefore ψ is non-increasing on $(0,1)$ and hence

$$(1-r)^{-1/2c}\int_r^1 |f'(t\zeta)|\, dt \ge (1-\varrho)^{-1/2c}\int_\varrho^1 |f'(t\zeta)|\, dt, \tag{4}$$

for $0 \le r \le \varrho < 1$.

Now we need the following:

Lemma 2.4. *If* $\varrho \le t \le \frac{1}{2}(1+\varrho)$, *then* $|f'(\varrho\zeta)| \le 16|f'(t\zeta)|$.

Proof. From [O] we have that for every conformal mapping $f\colon \mathbf{D} \to \mathbf{C}$, the following inequality is valid

$$|f''(z)/f'(z)| \le 4(1-|z|)^{-1}.$$

Hence

$$\left|\log\left|\frac{f'(t\zeta)}{f'(r\zeta)}\right|\right| \le \int_r^t \left|\frac{f''(s\zeta)}{f'(s\zeta)}\right| ds \le \int_r^t \frac{4ds}{1-s} \le \log 16$$

when $r < t \le \frac{1}{2}(1+r)$, and the inequality of the lemma follows. □

From this lemma we obtain

$$\int_\varrho^1 |f'(t\zeta)|\,dt \ge \int_\varrho^{(1+\varrho)/2} |f'(t\zeta)|\,dt \ge \frac{1}{16}|f'(\varrho\zeta)| \int_\varrho^{(1+\varrho)/2} dt = \frac{1}{32}|f'(\varrho\zeta)|(1-\varrho).$$

Combining this inequality with (4) we obtain

$$\begin{aligned}(1-\varrho)^{1-(1/2c)}|f'(\varrho\zeta)| &\le 32(1-\varrho)^{-1/2c}\int_\varrho^1 |f'(t\zeta)|\,dt\\ &= 32\psi(\varrho) \le 32\psi(r) = 32(1-r)^{-1/2c}\int_r^1 |f'(t\zeta)|\,dt\\ &\le 64c(1-r)^{1-(1/2c)}|f'(r\zeta)|,\end{aligned}$$

where the last inequality is a consequence of (3). In other words,

$$\left|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\right| \le 64c\left(\frac{1-\varrho}{1-r}\right)^{(1/2c)-1}$$

whenever $0 \le r \le \varrho < 1$. This is (ii) of our theorem, with $M = 64c$ and $\delta = 1/2c$.

(ii) $\Rightarrow$ (i): We assume that (ii) holds and want to calculate:

$$\begin{aligned}\sigma_l(w) &= \int_r^\varrho |f'(t\zeta)|\,dt \le M|f'(r\zeta)|\int_r^1 \left(\frac{1-t}{1-r}\right)^{\delta-1} dt\\ &= M|f'(r\zeta)|(1-r)^{1-\delta}\int_r^1 (1-t)^{\delta-1}\,dt\\ &= M|f'(r\zeta)|(1-r)^{1-\delta}\frac{1}{\delta}(1-r)^\delta\\ &\le \frac{M}{\delta}|f'(r\zeta)|(1-r^2) \le \frac{4M}{\delta}\,d(w,\partial G).\end{aligned}$$

The last inequality is a consequence of the well-known distortion inequality. Hence G is a $4M/\delta$-John disk with John center in $w_0 = f(0)$ and with the hyperbolic lines terminating in w_0 as the John curves. □

Remarks. (a) In [P1] and [P2] two more conditions are proved to be equivalent to (i) and (ii) of Theorem 2.3. We omit these conditions since they are of less interest in the following.

(b) If we take a closer look at the constants of our theorem, we observe that if $4M/\delta < 1$, our proof will lead to the impossible conclusion that

$$\sigma_l(w) < d(w, \partial G),$$

since inequality should hold for every w_1 on the hyperbolic line connecting $f(0)$ with the boundary. Hence we must have $4M \geq \delta$.

From the fact that

$$(1-\varrho)^s < (1-\varrho)^{s_1}$$

when $s_1 < s < 0$ for $\varrho \in (0,1)$, we also make the following observation. If we assume that

$$\left|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\right| \leq M\left(\frac{1-\varrho}{1-r}\right)^{\delta-1}; \qquad 0 \leq r \leq \varrho < 1$$

for all $\zeta \in \mathbf{T}$, with $M < \frac{1}{8}$ (and of course $4M \geq \delta$), then we obtain that G is a c-John disk with John center $f(0)$, with hyperbolic lines as John curves, and with

$$c = 4M/\delta.$$

Starting with this fact, from the second implication of Theorem 2.3 we obtain the inequality

$$\left|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\right| \leq \frac{256M}{\delta}\left(\frac{1-\varrho}{1-r}\right)^{(\delta/8M)-1}; \qquad 0 \leq r \leq \varrho < 1$$

for all $\zeta \in \mathbf{T}$. For $r = 0$, we observe that

$$|f'(\varrho\zeta)| = O\big((1-\varrho)^{(\delta/8M)-1}\big) \qquad \text{as } \varrho \to 1-, \tag{5}$$

while our starting assumption was

$$|f'(\varrho\zeta)| = O\big((1-\varrho)^{\delta-1}\big) \qquad \text{as } \varrho \to 1-. \tag{6}$$

Since in this case $\delta/8M > \delta$, (5) is a stronger condition than (6). Hence we may as well assume that $M \geq \frac{1}{8}$ in Theorem 2.3(ii).

We therefore have

Theorem 2.5. *Let $f\colon \mathbf{D} \to G$ be a conformal bijection. Then the following are true:*

(i) *If G is a c-John disk with $c \geq 1$ with John center $f(0)$ and with hyperbolic lines as the John curves, then for all $\zeta \in \mathbf{T}$ and $0 \leq r \leq \varrho < 1$, we have*

$$\left|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\right| \leq 64c\left(\frac{1-\varrho}{1-r}\right)^{(1/2c)-1}.$$

(ii) *If there exist constants $M > 0$ and $\delta \in (0,1)$ with $4M \geq \delta$ and $M \geq \frac{1}{8}$ such that for all $\zeta \in \mathbf{T}$ and all $0 \leq r \leq \varrho < 1$,*

$$\left|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\right| \leq M\left(\frac{1-\varrho}{1-r}\right)^{\delta-1},$$

then G is a $4M/\delta$-John disk with John center $f(0)$ and with hyperbolic lines as the John curves.

Proof. Follows from the proof of Theorem 2.3 and the remarks above. □

3. The Nehari class

We recall the definition of the Schwarzian derivative of a locally injective mermorphic function $f\colon \mathbf{D} \to \overline{\mathbf{C}}$ ($\overline{\mathbf{C}}$ the Riemann sphere):

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

at each point where f is analytic, and by $S_f(z) = S_{1/f}(z)$ at the poles of f. We also introduce the norm on the family of Schwarzian derivatives:

$$\|S_f\|_2 = \sup_{z\in\mathbf{D}} (1-|z|^2)^2 |S_f(z)|.$$

We have the following classical result proved by Nehari, [N] in 1949:

Theorem 3.1. *If f is locally injective and meromorphic on $\mathbf{D}$ and $\|S_f\|_2 \leq 2$, then f is injective.*

Remarks. Hille, [H], proved that the constant 2 is sharp. Chuaqui, Osgood and Pommerenke, [COP], introduced the notation *the Nehari class N* for the functions satisfying the assumption of Theorem 3.1.

Gehring and Pommerenke studied the class N in a paper in 1984, [GP]. They proved that $f(\mathbf{D})$ is a Jordan domain on the Riemann sphere except for the case when $f(\mathbf{D})$ is the image of a parallel strip

$$\{z\,;\,|\operatorname{Im} z| < \tfrac{1}{4}\pi\}$$

under a Möbius transformation. Such a domain is clearly not a John disk. In particular, the case when $f \in N$ and $f''(0) = 0$ has been studied in [GP]. Theorem 2 of this paper claims that either $f(\mathbf{D})$ is a Jordan domain on $\overline{\mathbf{C}}$ or the image of a parallell strip under a similarity map. However, the function

$$f(z) = \left[\log\left\{e^{-i\pi/4}\frac{z+i}{1-z}\right\} - i\left(\frac{\pi}{4}+2\right)\right]^{-1}$$

is in the class $N, f''(0) = 0,\ f(1) = f(-i) = 0$ and $f(\mathbf{D})$ is bounded. Hence $f(\mathbf{D})$ is the domain bounded by two circles with a common tangent at origin, and consequently not of the type discribed above.

Definition 3.2. A K-*quasidisk* is the image in $\overline{\mathbf{C}}$ of a disk or a half plane under a K-quasiconformal mapping $f\colon\ \overline{\mathbf{C}} \to \overline{\mathbf{C}}$. The boundary of a K-quasidisk is a K-*quasicircle*.

(For this definition and the definition of a quasiconformal mapping, see [L].)

It is a well-known fact that every bounded quasidisk is a John disk; [NV, pp. 40–42]. The opposite does not hold.

We also have the following classical result proved by Ahlfors and Weill in 1962, [AW]:

Theorem 3.3. *If* $f\colon \mathbf{D} \to \overline{\mathbf{C}}$ *is locally injective and meromorphic and* $\|S_f\|_2 < 2$, *then* $f(\mathbf{D})$ *is a quasidisk.*

In view of this result and the results in [GP], one may be tempted to believe that if $f \in N$ and $f(\mathbf{D})$ is a Jordan domain, then $f(\mathbf{D})$ is a quasidisk. However, according to [COP] there are $f \in N$ such that $f(\mathbf{D})$ is a Jordan domain on $\overline{\mathbf{C}}$ but $f(\mathbf{D})$ is not a John disk, and hence $f(\mathbf{D})$ is not a quasidisk. But in [COP] the following surprising result is proved:

Theorem 3.4. *If* $f \in N$ *and* $f(\mathbf{D})$ *is a John disk, then* $f(\mathbf{D})$ *is a quasidisk.*

We also need the following

Definition 3.5. The class N_0 is given by

$$N_0 = \{f \in N; f(0) = 0,\ f'(0) = 1,\ f''(0) = 0\}.$$

In [COP] also the following is proved:

Theorem 3.6. *Let* $f \in N_0$. *Then the following are equivalent:*

(i) $f(\mathbf{D})$ *is a John disk.*

(ii) $\displaystyle\limsup_{|z|\to 1}(1-|z|^2)\operatorname{Re}\left\{z\frac{f''(z)}{f'(z)}\right\} < 2.$

(iii) $\displaystyle\limsup_{|z|\to 1}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| < 2.$

We are able to prove a result of the same type in a more general setting.

Theorem 3.7. *If* $f\colon \mathbf{D} \to \mathbf{C}$ *is conformal and*

$$\limsup_{|z|\to 1}(1-|z|^2)\operatorname{Re}\Big\{z\frac{f''(z)}{f'(z)}\Big\} < 2,$$

then $f(\mathbf{D})$ *is a John disk.*

Proof. By the assumption there exists a $\beta \in (0,2)$ and $r_0 \in (0,1)$ such that when $r_0 \le \tau < 1$, then

$$\operatorname{Re}\Big\{\zeta\frac{f''(\tau\zeta)}{f'(\tau\zeta)}\Big\} \le \frac{\beta}{1-\tau^2}$$

for all $\zeta \in \mathbf{T}$. Choosing $\varepsilon \in (0,\, 2-\beta)$, we also have an $r_1 \in (0,1)$, $r_1 \ge r_0$, such that

$$\operatorname{Re}\Big\{\zeta\frac{f''(\tau\zeta)}{f'(\tau\zeta)}\Big\} < \frac{2\tau-\varepsilon}{1-\tau^2}$$

when $\tau \in [r_1, 1)$ for all $\zeta \in \mathbf{T}$.

From this we obtain, when $0 \le r_1 \le r \le \varrho < 1$,

$$\begin{aligned}\log\bigg(\frac{(1-\varrho^2)|f'(\varrho\zeta)|}{(1-r^2)|f'(r\zeta)|}\bigg) &= \int_r^\varrho \bigg(\operatorname{Re}\Big\{\zeta\frac{f''(\tau\zeta)}{f'(\tau\zeta)}\Big\} - \frac{2\tau}{1-\tau^2}\bigg)\,d\tau\\ &< -\varepsilon\int_r^\varrho \frac{d\tau}{1-\tau^2} = -\frac{\varepsilon}{2}\log\bigg(\frac{1+\varrho}{1+r}\cdot\frac{1-r}{1-\varrho}\bigg)\end{aligned}$$

or equivalently

$$\bigg|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\bigg| < \bigg(\frac{1+r}{1+\varrho}\bigg)^{1+(\varepsilon/2)}\bigg(\frac{1-\varrho}{1-r}\bigg)^{(\varepsilon/2)-1}; \qquad 0 \le r_1 \le r \le \varrho < 1.$$

Consequently, for all $\zeta \in \mathbf{T}$ and $0 \le r_1 \le r \le \varrho < 1$, we have

$$\bigg|\frac{f'(\varrho\zeta)}{f'(r\zeta)}\bigg| \le \bigg(\frac{1-\varrho}{1-r}\bigg)^{(\varepsilon/2)-1}. \tag{7}$$

To apply Theorem 2.3 in order to conclude that $G = f(\mathbf{D})$ is a John disk, we must remove the restriction $r \ge r_1$ above. To do this we first observe that from the proof of Theorem 2.3 it follows that the inequality (7) implies that $\sigma_l(w) \le cd(w,\partial G)$ for $w_1 = f(\varrho\zeta)$ and $w = f(r\zeta)$ when $0 \le r_1 \le r \le \varrho < 1$ and γ denotes the hyperbolic segment, where $c = c(\varepsilon) > 1$. This implies immediately that

$$\operatorname{diam}(\gamma[w_1,w]) \le cd(w,\partial G) \tag{8}$$

for such choice of w and w_1. If we next consider $f(\Delta_1)$ where $\Delta_1 = \{z; |z| \leq r_1\}$, we have

$$\operatorname{diam}\big(f(\Delta_1)\big) = \lambda_0 < \infty \tag{9}$$

and

$$\operatorname{dist}\big(f(\Delta_1), \partial G\big) = \delta_0 > 0. \tag{10}$$

If $\gamma(w, w_1)$ denotes the geodesic segment from $f(0)$ to $w_1 = f(\varrho\zeta)$, and if $w = f(r\zeta)$ now assuming that $0 \leq r < r_1 < \varrho < 1$, we obtain

$$\operatorname{diam}\big(\gamma(w, w_1)\big) \leq \operatorname{diam}\big(\gamma(w, w_0)\big) + \operatorname{diam}\big(\gamma(w_0, w_1)\big)$$

where $w_0 = f(r_1\zeta)$. Hence:

$$\operatorname{diam}\big(\gamma(w, w_1)\big) \leq \lambda_0 + cd(w_0, \partial G) \leq \lambda_0 + c(\delta_0 + \lambda_0) \tag{11}$$

from (8), (9) and (10).

If we now introduce

$$c_1 = (1 + c) \cdot \lambda_0/\delta_0,$$

we obtain from (11) that

$$\operatorname{diam}\big(\gamma(w, w_0)\big) \leq (c + c_1) \cdot \delta_0 \leq c_2 d(w, \partial G)$$

if $c_2 = c + c_1$.

The remaining case when $0 \leq r < \varrho < r_1 < 1$ is treated similarly. Hence we obtain

$$\operatorname{diam}\big(\gamma(w, w_0)\big) \leq c_0 d(w, \partial G)$$

for some $c_0 > 1$ and all $0 \leq r \leq \varrho < 1$. The constant c_0 is independent of $\zeta \in \mathbf{T}$, so we have proved with the notations of [NV] that $G \in \operatorname{car}_d$, and by Lemma 2.10, p. 9, [NV], it follows that $G = f(\mathbf{D})$ is a John disk. □

Remark. The proof is quite similar to the proof of Theorem 3.6 in [COP], except that our proof also considers the case when $r < r_1$.

4. The pre-Schwarzian derivative

Let $f\colon \mathbf{D} \to \mathbf{C}$ be analytic and locally injective. Then we introduce the notation

$$L_f(z) = f''(z)/f'(z),$$

the *pre-Schwarzian derivative* of f. Also, we introduce the norm

$$\|L_f\|_1 = \sup_{z\in\mathbf{D}} (1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|.$$

This notation was introduced by Astala and Gehring in 1986, [AG]. But the same quantities were actually studied earlier by Becker in 1971, [B], and by Becker and Pommerenke in 1984, [BP]. It is interesting to notice that there are several analogies between the Schwarzian and the pre-Schwarzian derivatives (see also [AG] and [AH]). Becker proved in [B], the following analogue of Nehari's theorem:

Theorem 4.1. *If $f\colon \mathbf{D} \to \mathbf{C}$ is analytic and locally injective and $\|L_f\|_1 \le 1$, then f is injective.*

In 1984 the following analogue of the Ahlfors/Weill theorem (Theorem 3.3), was proved by Becker and Pommerenke, [BP]:

Theorem 4.2. *If $f\colon \mathbf{D} \to \mathbf{C}$ is analytic and locally injective and $\|L_f\|_1 < 1$, then $G = f(\mathbf{D})$ is a quasidisk.*

As a direct consequence of Theorem 3.7 of the present paper we obtain

Theorem 4.3. *If $f\colon \mathbf{D} \to \mathbf{C}$ is conformal and $\|L_f\|_1 < 2$, then $G = f(\mathbf{D})$ is a John disk. The constant 2 is the best possible.*

Proof. We have by our assumption

$$\limsup_{|z|\to 1} (1-|z|^2)\,\mathrm{Re}\left\{ z\frac{f''(z)}{f'(z)} \right\} \le \sup_{z\in\mathbf{D}} (1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| < 2,$$

and the result follows from Theorem 3.7.

If $F_0(z) = \frac{1}{2}\log[(1+z)/(1-z)]$, then $\|L_{F_0}\|_1 = 2$ and $F_0(\mathbf{D})$ is an infinite strip and hence not a John disk. □

Remark. Returning to the Schwarzian derivative for a moment, it is a natural question to ask at this point whether there exists a constant $K > 0$ such that when f is conformal and $\|S_f\|_2 < K$, then $G = f(\mathbf{D})$ is a John disk. The answer to this question is no. This can be seen in the following way. If f is a Møbius transformation mapping $\mathbf{D}$ onto a half plane, then $\|S_f\|_2 = 0$ but $f(\mathbf{D})$ is not a John disk in our language. But if we add the assumption $f''(0) = 0$, it follows from [GP] and [AW] that if $\|S_f\|_2 < 2$, then $f(\mathbf{D})$ is a bounded quasidisk and hence a John disk. The constant 2 is the best possible, since for the function F_0 introduced above, we have $F_0''(0) = 0$ and $\|S_{F_0}\|_2 = 2$ but $F_0(\mathbf{D})$ is not a John disk. (For the case of unbounded John disks we refer to [NV, Section 9].)

Theorem 3.7 gives a sufficient condition for $f(\mathbf{D})$ to be a John disk. Can this condition also be necessary? (From Theorem 3.6 it follows that this is the case whenever $f \in N_0$.) We have not been able to settle this question completely, but we can give one necessary condition for $f(\mathbf{D})$ to be a John disk.

Proposition 4.4. *If $f\colon \mathbf{D} \to \mathbf{C}$ is conformal and $f(\mathbf{D})$ is a c-John disk for $c \ge 1$ with respect to the hyperbolic segments terminating at $f(0)$, then for each $\zeta \in \mathbf{T}$ we have*

$$\liminf_{r\to 1}(1-r^2)\,\mathrm{Re}\left\{\zeta\frac{f''(r\zeta)}{f'(r\zeta)}\right\} \le 2-\frac{1}{c}. \tag{12}$$

Proof. Assume for contradiction that $f(\mathbf{D})$ is a c-John disk in the sense of our assumption and at the same time that there exists a $\zeta_0 \in \mathbf{T}$ such that (12) does not hold. Hence, there is an $r_0 \in (0,1)$ and an $\varepsilon > 0$ such that for $\tau \in [r_0, 1)$ we have

$$(1-\tau^2)\,\mathrm{Re}\left\{\zeta_0\frac{f''(\tau\zeta_0)}{f'(\tau\zeta_0)}\right\} > 2-\frac{1}{c}+2\varepsilon.$$

From this follows for $0 \le r_0 \le r \le \varrho < 1$:

$$\begin{aligned}\log\left(\frac{(1-\varrho^2)|f'(\varrho\zeta_0)|}{(1-r^2)|f'(r\zeta_0)|}\right) &= \int_r^\varrho\left(\mathrm{Re}\left\{\zeta_0\frac{f''(\tau\zeta_0)}{f'(\tau\zeta_0)}\right\}-\frac{2\tau}{1-\tau^2}\right)d\tau\\ &> \int_r^\varrho \frac{2-1/c+2\varepsilon-2\tau}{1-\tau^2}\,d\tau\\ &> \left(\frac{1}{2c}-\varepsilon\right)\log\left(\frac{1+r}{1+\varrho}\cdot\frac{1-\varrho}{1-r}\right).\end{aligned}$$

Therefore

$$\left|\frac{f'(\varrho\zeta_0)}{f'(r\zeta_0)}\right| > \frac{1}{2\sqrt{2}}\left(\frac{1-\varrho}{1-r}\right)^{(1/2c)-1-\varepsilon} \tag{13}$$

for $0 \le r_0 \le r \le \varrho < 1$. But from Theorem 2.5 we also have

$$\left|\frac{f'(\varrho\zeta_0)}{f'(r\zeta_0)}\right| \le 64c\left(\frac{1-\varrho}{1-r}\right)^{(1/2c)-1} \tag{14}$$

for $0 \le r \le \varrho < 1$ when $f(\mathbf{D})$ is a c-John disk in our sense. Letting $\varrho \to 1-$ and fixing r, we observe that (13) is in contradiction with (14). □

5. The function σ_ζ

In [COP] the following function is introduced:

$$\sigma_\zeta(r) = \mathrm{Re}\{\zeta^2 S_f(r\zeta)\} - \tfrac{1}{2}[\mathrm{Im}\{\zeta L_f(r\zeta)\}]^2$$

for $\zeta \in \mathbf{T}$ and $r \in [0,1)$. This function is defined for f analytic and locally injective in the unit disk. It will follow from this section that certain properties of the function $\sigma_\zeta(r)$ are closely related to the question whether $\Omega = f(\mathbf{D})$ is a John disk or not.

Let $f\colon \mathbf{D} \to \Omega$ be a conformal equivalence and suppose that f has an angular limit at $\zeta \in \mathbf{T}$. Then $f(\zeta)$ is said to be *well-accessible* if there is a Jordan arc γ in $\mathbf{D}$ ending at ζ and a constant $M > 0$ such that

$$\operatorname{diam}\big(f\big(\gamma(z)\big)\big) \le M d\big(f(z), \partial\Omega\big),$$

where $\gamma(z)$ denotes the part of γ from z to ζ and diam denotes the diameter. The following result is proved in [COP, Theorem 9, p. 104]):

Proposition 5.1. *If $f\colon \mathbf{D} \to \Omega$ is a conformal equivalence and*

$$\liminf_{r\to 1}(1-r^2)^2\sigma_\zeta(r) = 2,$$

then $f(\zeta)$ is not well-accessible.

Remark. The assumption in [COP] is that f is analytic and *locally* univalent. But the argument is leaning on previous results in the same paper where the assumption that f is conformal is essential. In [COP] also the assumption stated is $\liminf_{r\to 1}(1-r^2)^2\sigma_\zeta(r) \ge 2$, but we will show later in this section that the case $\liminf_{r\to 1}(1-r^2)^2\sigma_\zeta(r) > 2$ is not compatible with the assumption that f is analytic in $\mathbf{D}$.

If f is a conformal mapping onto a John disk, then all the boundary points are well-accessible with a constant M independent of ζ. Conversely, if all boundary points are uniformly well-accessible, then Ω is a John disk, [COP, p. 81].

In [GP] the authors introduce another function $p = p_\zeta\colon \mathbf{R} \to \mathbf{R}$ which is closely related to the function σ_ζ defined above. For completeness we will give a short explanation of how p_ζ is used in the proof of the theorem in [GP] mentioned in the remark after Theorem 3.1 in our present paper.

The assumption is that f is locally univalent and meromorphic in $\mathbf{D}$, $\|S_f\|_2 \le 2$ and $f''(0) = 0$. It follows immediately from Nehari's theorem (Theorem 3.1), that f is univalent. For each fixed $\zeta \in \mathbf{T}$ we introduce the function

$$h_\zeta(t) = \zeta\frac{e^t - 1}{e^t + 1}, \qquad t \in T = \left\{w; -\frac{\pi}{2} < \mathrm{Im}\, w < \frac{\pi}{2}\right\}.$$

Next we study the function

$$g_\zeta = f \circ h_\zeta .$$

For $t \in \mathbf{R}$, we introduce $r = |h_\zeta(t)| = (e^t - 1)/(e^t + 1)$. After some calculations we obtain

$$\operatorname{Im}\{L_{g_\zeta}(t)\} = \tfrac{1}{2}(1 - r^2)\operatorname{Im}\{\zeta L_f(r\zeta)\} \tag{15}$$

for each $z = t\zeta$ that is not a pole of f, and

$$\operatorname{Re}\{S_{g_\zeta}(t)\} = -\tfrac{1}{2} + \tfrac{1}{4}(1 - r^2)^2 \operatorname{Re}\{\zeta^2 S_f(r\zeta)\}. \tag{16}$$

Next we introduce the function $v_\zeta \colon \mathbf{R} \to \mathbf{R}$ by

$$v_\zeta(t) = \begin{cases} |g_\zeta'(t)|^{-1/2} & \text{for } g_\zeta(t) \neq \infty, \\ 0 & \text{for } g_\zeta(t) = \infty. \end{cases}$$

Clearly v_ζ is a non-negative, continuous function with possible zeros only at the poles of g_ζ. If we now introduce the function

$$p_\zeta(t) = -\tfrac{1}{2}\operatorname{Re}\{S_{g_\zeta}(t)\} + \bigl(\tfrac{1}{2}\operatorname{Im}\{L_{g_\zeta}(t)\}\bigr)^2, \tag{17}$$

we obtain after some calculations

$$v_\zeta'' = p_\zeta \cdot v_\zeta \tag{18}$$

except where g_ζ has a pole. From the assumption $\|S_f\|_2 \le 2$ it follows that

$$(1 - r^2)^2 \operatorname{Re}\{\zeta^2 S_f(r\zeta)\} \le 2.$$

Combining this with (16), we obtain that $\operatorname{Re}\{S_{g_\zeta}(t)\} \le 0$, and from (17) we then obtain that $p_\zeta(t) \ge 0$. From (18) and the continuity of v_ζ this leads to the conclusion that v_ζ is a non-negative and convex function on $\mathbf{R}$. Using the condition $f''(0) = 0$, we obtain that $v_\zeta'(0) = 0$, which then implies that v_ζ has its minimum at $t = 0$ where $v_\zeta(0) > 0$ since the condition $f''(0) = 0$ implies that g_ζ has no pole at 0. These observations now lead to the conclusion that $v_\zeta(t) > 0$ for all $t \in \mathbf{R}$, i.e. g_ζ has no poles. Since this is true for all $\zeta \in \mathbf{T}$, we conclude f must be analytic and hence conformal.

Observe that the condition $\|S_f\|_2 \le 2$ so far has only been used to establish the fact that $p_\zeta(t) \ge 0$ wherever it is defined. Hence we will obtain the same information about v_ζ by simply assuming $p_\zeta(t) \ge 0$ for a fixed $\zeta \in \mathbf{T}$. The more restrictive assumption $p_\zeta(t) \ge 0$ for all $t \in \mathbf{R}$ and all $\zeta \in \mathbf{T}$ likewise implies that f is analytic in $\mathbf{D}$. It is therefore a natural question to ask what we can conclude about $\Omega = f(\mathbf{D})$ under the condition $p_\zeta(t) \ge 0$ for all $t \in \mathbf{R}$ and all $\zeta \in \mathbf{T}$. It turns out that we obtain some of the same information about Ω as in the case when we assume that $f \in N$. But a natural condition should relate to f and not to g_ζ. To this end, we need the following:

Lemma 5.2. *If f is analytic and locally univalent in* $\mathbf{D}$ *and* p_ζ *and* σ_ζ *are defined as above, we have*

$$p_\zeta(t) = \tfrac{1}{8}\big[2 - \sigma_\zeta(r)(1-r^2)^2\big]$$

where $\zeta \in \mathbf{T}$ *and* $t = \log(1-r)/(1-r)$.

Proof. From (15), (16) and (17) we obtain

$$\begin{aligned} p_\zeta(t) &= -\tfrac{1}{2}\operatorname{Re}\{S_{g_\zeta}(t)\} + \big(\tfrac{1}{2}\operatorname{Im}\{L_{g_\zeta}(t)\}\big)^2 \\ &= \tfrac{1}{4} - \tfrac{1}{8}(1-r^2)^2\operatorname{Re}\{\zeta^2 S_f(r\zeta)\} + \tfrac{1}{16}(1-r^2)^2\big(\operatorname{Im}\{\zeta L_f(r\zeta)\}\big)^2 \\ &= \tfrac{1}{8}\big[2 - (1-r^2)^2\big(\operatorname{Re}\{\zeta^2 S_f(r\zeta)\} - \tfrac{1}{2}\big(\operatorname{Im}\{\zeta L_f(r\zeta)\}\big)^2\big)\big] \\ &= \tfrac{1}{8}\big[2 - \sigma_\zeta(r)(1-r^2)^2\big]. \ \square \end{aligned}$$

We are now able to prove the following analogue of Theorem 2, [GP]:

Proposition 5.3. *If* $f\colon \mathbf{D} \to \Omega$ *is a conformal eqivalence satisfying* $f''(0) = 0$ *and*

$$\sup_{z\in\mathbf{D}}(1-r^2)^2\sigma_\zeta(r) \le 2 \qquad (z = r\zeta) \tag{19}$$

then either

(i) Ω *is unbounded, or*

(ii) f *has a continuous extension to* $\overline{\mathbf{D}}$, *and there exist positive constants* M_1 *and* M_2 *such that*

$$|f(z) - f(z')| \le M_1\left(\log\frac{3}{|z-z'|}\right)^{-1} \qquad (z, z' \in \overline{\mathbf{D}})$$

and

$$|f(r\zeta) - f(\zeta)| \le M_2\big[\operatorname{dist}\big(f(r\zeta), \partial\Omega\big)\big]^{1/2} \qquad (r \in [0,1),\ \zeta \in \mathbf{T}).$$

Proof. According to Lemma 5.2 the assumption (19) is equivalent to $p_\zeta(t) \ge 0$ for all $t \in \mathbf{R}$ and all $\zeta \in \mathbf{T}$. Returning to the proof of Theorem 2, [GP], we observe that under the assumptions $p_\zeta(t) \ge 0$ and $f''(0) = 0$, we can conclude that v_ζ is convex and non-negative on $\mathbf{R}$ with its minimum at $t = 0$. However, at this point our argument deviates from the course of the proof of [GP], since we cannot conclude in our case that $\operatorname{Im}\{L_{g_\zeta}(t)\} = 0$ on an interval $[0, t_0]$ if $p_\zeta(t) = 0$ on this interval. (To do this we would need $\operatorname{Re}\{\zeta S_f(r\zeta)\}(1-r^2)^2 \le 2$.) We now consider two different cases:

(a) $p_{\zeta_0}(t) \equiv 0$ on $[0, \infty)$ for at least one $\zeta_0 \in \mathbf{T}$.

(b) $p_\zeta(t_1) > 0$ for some $t_1 = t_1(\zeta)$ for all $\zeta \in \mathbf{T}$.

Case (a): It follows from equation (18) that $v''_\zeta \equiv 0$ on $[0,\infty)$, and therefore $v_\zeta(t) \equiv v_\zeta(0) = \alpha > 0$ since $v'_\zeta(0) = 0$. Hence $|g'_\zeta(t)| \equiv \alpha^{-2} = A > 0$ by the definition of v_ζ. Therefore

$$l(g_\zeta[0,t]) = \int_0^t |g'_\zeta(s)|\, ds = A \cdot t$$

for all $t \in [0,\infty)$. In particular $l\bigl(f[0,\zeta)\bigr) = l\bigl(g_\zeta[0,\infty)\bigr) = \infty$. Also, the connection between g_ζ and f implies that

$$|f'(r\zeta)| = 2(1-r^2)^{-1}|g'_\zeta(t)| = 2(1-r^2)^{-1} \cdot A.$$

The well-known distortion inequalities now give

$$\tfrac{1}{2}A = \tfrac{1}{4}(1-r^2)|f'(r\zeta)| \le d\bigl(f(r\zeta), \partial\Omega\bigr) \le (1-r^2)|f'(r\zeta)| = 2A.$$

Assume now for contradiction that $\gamma = f\bigl([0,\zeta)\bigr)$ is a bounded set in $\mathbf{C}$. Then the set

$$E = \bigl\{z \in \mathbf{C}; d(z,\gamma) \le \tfrac{1}{4}A\bigr\}$$

is a compact subset of Ω containing γ as a subset. Since $f^{-1}\colon \Omega \to \mathbf{D}$ is continuous, $f^{-1}(E)$ is a compact subset of $\mathbf{D}$. But $[0,\zeta) \subset f^{-1}(E)$, and $[0,\zeta)$ is not contained in a compact subset of $\mathbf{D}$. Hence γ is unbounded and therefore Ω is unbounded.

Case (b): We first claim that our assumption leads to the existence of a smallest $t_0 = t_0(\zeta) < \infty$ for each $\zeta \in \mathbf{T}$ such that $v'_\zeta(t) > 0$ for all $t > t_0(\zeta)$. We first observe that the assumption that $p_\zeta(t_1) > 0$ and the fact that v_ζ is non-decreasing leads to the fact that

$$v''_\zeta(t_1) \ge p_\zeta(t_1) \cdot v_\zeta(0) = B > 0.$$

Hence there exists a $\delta > 0$ such that $t_1 - \delta < t < t_1 + \delta$ implies that $v''_\zeta(t) > \frac{1}{2}B > 0$. By the mean value theorem from calculus we then obtain $v'_\zeta(t_1) = v'_\zeta(t_1 - \delta) + v''_\zeta(\tau) \cdot \delta$ for some $\tau \in (t_1 - \delta, t_1)$ and hence by the convexity of v_ζ and the fact that $v'_\zeta(0) = 0$, we have

$$v'_\zeta(t_1) \ge \tfrac{1}{2}B \cdot \delta > 0.$$

By the convexity of v_ζ we then obtain

$$v'_\zeta(t) > 0 \qquad \text{for all } t \ge t_1.$$

For fixed ζ, we will introduce the notation

$$t_0(\zeta) = \inf\{t;\, v'_\zeta(t) > 0\}.$$

Next we claim that $\tau_0 = \sup\bigl\{t_0(\zeta); \zeta \in \mathbf{T}\bigr\} < \infty$.

Assume for contradiction that there is a sequence $\{\zeta_n\}$ in $\mathbf{T}$ such that $\lim_{n\to\infty} t_0(\zeta_n) = \infty$. Without loss of generality we may assume that $\{\zeta_n\}$ converges to a $\zeta_0 \in \mathbf{T}$. We claim that this together with the fact that $t_0(\zeta_0) < \infty$ will lead to a contradiction. In fact, $v'_\zeta(t)$ is continuous as a function of ζ and t by the definition. Hence since $v'_{\zeta_0}(t_2) = C > 0$ for some $t_2 \in (t_0(\zeta_0), \infty)$, there exists $\delta > 0$ and $\varepsilon > 0$ such that when

$$t \in (t_2 - \delta, t_2 + \delta) \wedge \zeta \in \{e^{is}\zeta_0; |s| < \varepsilon\},$$

then $v'_\zeta(t) > \frac{1}{2}C$, and consequently $t_0(\zeta) < t_2 - \delta$ for all such ζ. But choosing n large enough, we obtain $t_0(\zeta_n) > t_2 - \delta$ and $\zeta_n \in \{e^{is}\zeta_0;\, |s| < \varepsilon\}$, a contradiction.

Let $\alpha = \min\{v'_\zeta(\tau_0 + 1);\, \zeta \in \mathbf{T}\}$. Clearly $\alpha > 0$. Hence $v'_\zeta(t) \geq \alpha$ for $t \geq \tau_0 + 1$ for all ζ, and we can continue the argument as in the proof of the corresponding case of Theorem 2 in [GP].

Remark. We have not been able to prove that f has a homeomorphic extension to $\overline{\mathbf{D}}$ as in the corresponding cases in the [GP] approach, neither to prove that Ω in the unbounded case is the image of a strip $T = \bigl\{w; -\frac{1}{2}\pi < \operatorname{Im} w < \frac{1}{2}\pi\bigr\}$ under a Möbius transformation. □

We will now take a closer look at the connection between $\sigma_\zeta(t)$ and John disks. We will first return to our remark following Proposition 5.1.

We can prove the following:

Proposition 5.4. *If $f\colon \mathbf{D} \to \mathbf{C}$ is analytic and locally univalent, then*

$$\liminf_{r\to 1} (1 - r^2)^2 \sigma_\zeta(r) \leq 2$$

for each $\zeta \in \mathbf{T}$.

To prove this result we need the classical Sturm's comparison theorem:

Lemma 5.5. *If $y = y_j(x)$ is a nontrivial solution of the differential equation*

$$(f_j y')' + g_j y = 0, \qquad j = 1, 2,$$

and furthermore, $f_1 \geq f_2 > 0$ and $g_1 \leq g_2$, then there is at least one zero of y_2 between each pair of consecutive zeros of y_1, or $y_2 \equiv C y_1$ in the interval between these two zeros.

(Proof of Lemma [K, p. 125].)

Proof of Proposition 5.4. We assume for contradiction that for some ζ

$$\liminf_{r\to 1}(1-r^2)^2\sigma_\zeta(r) > 2.$$

From Lemma 5.2, this assumption is equivalent to

$$\limsup_{t\to\infty} p_\zeta(t) < 0.$$

Hence, there exists an $\varepsilon > 0$ and a $t_0 \in \mathbf{R}$ such that for $t \geq t_0$ we have $p_\zeta(t) < -\varepsilon$. The differential equation

$$y'' + \varepsilon y = 0$$

has the (non-trivial) general solution

$$y = C_1 \cos\bigl(\sqrt{\varepsilon}\cdot t\bigr) + C_2 \sin\bigl(\sqrt{\varepsilon}\cdot t\bigr) \tag{20}$$

where $(C_1, C_2) \neq (0,0)$.

Using Lemma 5.5 with $f_1 = f_2 \equiv 1$, $g_1 = \varepsilon$ and $g_2 = -p_\zeta$, we conclude that a non-trivial solution of

$$v_\zeta'' - p_\zeta \cdot v_\zeta = 0$$

has at least one zero between two of the zeros of (20). This leads to the situation that f has a pole at $z = t\zeta$ if $v_\zeta(t) = 0$, while the case when $v_\zeta(t) = 0$ for $t > t_0$ implies that $g_\zeta' \equiv \infty$. Hence we can conclude that under the assumption that f is analytic in $\mathbf{D}$,

$$\liminf_{r\to 1}(1-r^2)^2\sigma_\zeta(r) \leq 2. \quad \square$$

In the remaining part of this paper we shall need the following lemma.

Lemma 5.6. *If v, w, P and Q are real continuous functions on $[0,\infty)$ satisfying the following conditions*:

(i) $v'' + Pv \geq 0$ *and* $v > 0$,
(ii) $w'' + Qw \leq 0$ *and* $w > 0$,
(iii) $Q \geq P$,
(iv) $v(0) = w(0)$ *and* $v'(0) \geq w'(0)$,

then

$$\frac{v(t)}{v(t_0)} \geq \frac{w(t)}{w(t_0)} \qquad \text{for all } t \geq t_0 \geq 0.$$

Proof. We introduce the function $\omega(t) = v(t)/w(t)$. Our goal is to prove that this function is non-decreasing. From our assumption it follows that

$$\omega(0) = 1 \qquad \text{and} \qquad \omega'(0) = [v'(0) - w'(0)]/w(0) \geq 0.$$

Furthermore we obtain

$$(\omega' w^2)' = (v'\cdot w - v\cdot w')' = v''\cdot w - v\cdot w'' = w\cdot v\left[\frac{v''}{v} - \frac{w''}{w}\right] \geq w\cdot v[-P+Q] \geq 0.$$

Hence $\omega' w^2$ is non-decreasing, and in particular

$$\omega'(t)\cdot w(t)^2 \geq \omega'(0)\cdot w(0)^2,$$

and since $\omega'(0) \geq 0$, this implies that $\omega'(t) \geq 0$. $\square$

(The idea of this proof is due to N. Steinmetz, [S].)

Theorem 5.7. *If* $f\colon \mathbf{D} \to \mathbf{C}$ *is conformal and* $f''(0) = 0$ *and furthermore*

$$\sup_{r\zeta \in \mathbf{D}} (1 - r^2)^2 \sigma_\zeta(r) < 2, \tag{21}$$

then $\Omega = f(\mathbf{D})$ *is a John disk.*

Proof. We can without loss of generality assume that $|f'(0)| = 2$ by multiplication with a constant if necessary. Introducing v_ζ as before for each $\zeta \in \mathbf{T}$ we obtain that

$$v_\zeta'' - p_\zeta v_\zeta = 0, \qquad v_\zeta(0) = |g_\zeta'(0)|^{-1/2} = \big(2/|f'(0)|\big)^{1/2} = 1$$

and $v_\zeta'(0) = 0$ since $f''(0) = 0$. The condition (21) is by Lemma 5.2 equivalent to the condition that there exists an $\alpha > 0$ such that

$$p_\zeta(t) \geq \alpha > 0 \quad \text{for all } t \in [0, \infty) \text{ and for all } \zeta \in \mathbf{T}.$$

In order to apply Lemma 5.6, we consider the initial value problem

$$w'' - \alpha w = 0, \qquad w(0) = 1, \qquad w'(0) = 0,$$

which has the solution

$$w(t) = \tfrac{1}{2}\big[e^{t\sqrt{\alpha}} + e^{-t\sqrt{\alpha}}\big].$$

Clearly $w > 0$, and with $P = -p_\zeta$ and $Q = -\alpha$, all the conditions of Lemma 5.6 are satisfied. We also obtain

$$w(t)/w(t_0) = \frac{e^{t\sqrt{\alpha}} + e^{-t\sqrt{\alpha}}}{e^{t_0\sqrt{\alpha}} + e^{-t_0\sqrt{\alpha}}} \geq \frac{1}{2} e^{(t-t_0)\sqrt{\alpha}} \quad \text{for } t \geq t_0 \geq 0.$$

By the lemma we conclude that

$$v_\zeta(t)/v_\zeta(t_0) \geq \tfrac{1}{2} e^{(t-t_0)\sqrt{\alpha}} \qquad \text{for } t \geq t_0 \geq 0. \tag{22}$$

As before we have

$$e^{t-t_0} = \frac{1+r}{1-r} \cdot \frac{1-r_0}{1+r_0},$$

where

$$r = \frac{e^t - 1}{e^t + 1}, \qquad r_0 = \frac{e^{t_0} - 1}{e^t + 1},$$

and from (22) we then obtain

$$v_\zeta(t)/v_\zeta(t_0) \geq \frac{1}{2}\left[\frac{1+r}{1-r} \cdot \frac{1-r_0}{1+r_0}\right]^{\sqrt{\alpha}} > \frac{1}{2}\left(\frac{1-r_0}{1-r}\right)^{\sqrt{\alpha}}. \tag{23}$$

Again, using the connection between f, g_ζ and v_ζ and (23) we obtain

$$\begin{aligned}\left|\frac{f'(r\zeta)}{f'(r_0\zeta)}\right| &= \left|\frac{g'_\zeta(t)}{g'_\zeta(t_0)}\right|\left(\frac{1-r_0^2}{1-r^2}\right)\\ &= \left|\frac{v_\zeta(t_0)}{v_\zeta(t)}\right|^2\left(\frac{1-r_0^2}{1-r^2}\right)\\ &< 2\left(\frac{1-r}{1-r_0}\right)^{2\sqrt{\alpha}}\cdot\left(\frac{1-r^2}{1-r_0^2}\right)^{-1}\\ &\le 2\left(\frac{1-r}{1-r_0}\right)^{2\sqrt{\alpha}-1} \qquad \text{for } 0\le r_0\le r<1.\end{aligned}$$

This condition holds for all $\zeta\in\mathbf{T}$ with the same constant α. Hence the conclusion follows from Theorem 2.3. □

Closing remark. It seems as the condition $f''(0)=0$ is essential for the proof of Theorem 5.7. It is an open question whether or not this condition can be omitted. Another question is whether condition (21) can be weakened to

$$\limsup_{r\to 1}(1-r^2)^2\sigma_\zeta(r)\le\alpha<2, \tag{24}$$

where α does not depend on ζ. It seems likely that condition (24) should be sufficient to conclude that $\Omega=f(\mathbf{D})$ is a John disk, but our argument does not seem to work immediately in this case. Nevertheless, we have a feeling that the function $\sigma_\zeta(r)$ rather than the Schwarzian or the pre-Schwarzian is the natural function to study in connection with John disks.

Another natural question to ask is if there is a condition on the pre-Schwarzian derivative which is both necessary and sufficient for $f(\mathbf{D})$ to be a John disk.

Is it possible to obtain similar results as in [COP] and in the present paper concerning John disks that are not necessarily bounded?

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