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THE HYPERBOLIC METRIC OF A RECTANGLE

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Abstract. By using the theory of elliptic integrals we give an exact formula for the hyperbolic density of a rectangle at its centre. We compare this to the hyperbolic density of an infinite strip and obtain (in this special case) a quantitative version of the Carathéodory Kernel Theorem.

1. Introduction

The upper half-plane $\mathbf{H}$ supports the hyperbolic metric $\lambda_{\mathbf{H}}(z)\,|dz|$, where $\lambda_{\mathbf{H}}(z) = 1/\operatorname{Im}[z]$, and if f is any conformal map of $\mathbf{H}$ onto a domain D, then the hyperbolic metric on $\mathbf{H}$ transfers to the hyperbolic metric $\lambda_D(z)\,|dz|$ on D, where

$$\lambda_D\bigl(f(z)\bigr)|f'(z)| = \lambda_{\mathbf{H}}(z). \tag{1.1}$$

The *hyperbolic distance* $d_D(z_1,z_2)$ between points z_1 and z_2 in D is then the infimum of the integral of $\lambda_D(z)\,|dz|$ along γ taken over all curves γ joining z_1 to z_2 in D. The function λ_D is the *hyperbolic density* of D and one of its most important properties is its monotonicity: *if U and V are conformally equivalent to $\mathbf{H}$, and if $U \subset V$, then $\lambda_V \le \lambda_U$ on U.* This attractive property allows one to estimate the hyperbolic density of a given domain by comparing it with domains whose hyperbolic densities are known; however, its usefulness is severely limited by the scarcity of such comparison domains. With this in mind, we explore these ideas in greater depth in the case of rectangular domains. For more details on the hyperbolic metric see, for example, [1] and [8].

It is convenient to work with normalized rectangles and throughout, we shall be considering the rectangle

$$R(l) = (-l,l) \times (-\pi/2, \pi/2),$$

where $l > 0$, and where we identify $\mathbf{R}^2$ with $\mathbf{C}$. If $l = +\infty$ then $R(l)$ is the infinite strip given by points $x+iy$ where $|y| < \pi/2$, and we prefer to denote this by S. The function $z \mapsto \log z - \pi i/2$ maps $\mathbf{H}$ conformally onto S and using (1.1) we see that $\lambda_S(x) = 1$ for all real x. By using the classical theory of elliptic integrals, we are able to give the following exact formula for $\lambda_{R(l)}(0)$ (clearly one can obtain a corresponding result for any rectangle by applying a scaling map).

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Theorem 1.1. *With $R(l)$ as above, we have*

$$\lambda_{R(l)}(0) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{\cosh 2nl}.$$

We shall make three applications of this result, and we now describe these. First, it is well known that if D is a convex domain, then

$$\frac{1}{\operatorname{dist}(z, \partial D)} \le \lambda_D(z) \le \frac{2}{\operatorname{dist}(z, \partial D)}$$

(see [8]). These inequalities are best possible, and they show that if D is a square of side $2a$ and centre 0, then $1/a \le \lambda_D(0) \le 2/a$. As a by-product of the proof of Theorem 1.1 we obtain the following result.

Theorem 1.2. *Let D be a square of side $2a$ and centred at the origin. Then*

$$\lambda_D(0) = \frac{K(1/\sqrt{2}\,)}{a} = \frac{1\cdot 8541\ldots}{a},$$

where K is the elliptic integral given by

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1.2}$$

Our second application concerns the Carathéodory Kernel Theorem. This result implies that if D_n is an increasing sequence of simply connected domains whose union D is conformally equivalent to $\mathbf{H}$, then $\lambda_{D_n}(z)$ decreases strictly, and monotonically, to $\lambda_D(z)$ as $n \to \infty$ (see [3] and [6]). It seems difficult to obtain a *quantative* version of this general result; however, if we apply it to $R(l)$ and S (as $l \to +\infty$) we can establish, at least in this case, the following accurate estimate of the rate of convergence of $\lambda_{R(l)}(0)$ to 1.

Corollary 1.3. *Suppose that $l > \pi/2$. Then*

$$1 + \frac{2}{\cosh 2l} < \lambda_{R(l)}(0) < 1 + \frac{2}{\cosh 2l} + \frac{5}{e^{4l}}.$$

We remark that by symmetry (and scaling) there is no loss of generality in assuming that $l > \pi/2$ here. Note that this shows that $\lambda_{R(l)}(0) \sim 1 + 4e^{-2l}$ as $l \to +\infty$, and it also gives an explicit estimate of the error term. We shall see later (in the proof of the next result) that one can give similar estimates for $\lambda_{R(l)}(x)$ for any real x.

Our final application concerns estimates of the hyperbolic length in $R(l)$. Hayman ([5, Lemma 6, p. 170]) has used a method closely related to extremal length (but not involving the hyperbolic density) to obtain estimates of the hyperbolic distance in $R(l)$. He shows that if $0 < x < l - \pi/2$, then $x \le d_{R(l)}(0,x) \le x + \pi/2$. It is natural to expect that $d_{R(l)}(0,x) = x + o(1)$ as $l - x \to +\infty$, and we shall establish such a result (which does not seem to follow from Hayman's method). Our result (which could be improved slightly) is as follows.

Theorem 1.4. *Suppose that* $0 < x < l - \pi/2$. *Then*

$$x < d_{R(l)}(0,x) < x + \frac{4}{e^{2(l-x)}}. \tag{1.3}$$

This shows, for example, that as $l \to \infty$, $d_{R(l)}(0, l/2) = l/2 + O(e^{-l})$.

The plan of the paper is as follows. In Section 2 we discuss the hyperbolic density of a rectangle in terms of two standard elliptic integrals, and we give some of the properties of these integrals. We prove Theorems 1.1 and 1.2, and Corollary 1.3, in Section 3, and we prove Theorem 1.4 in Section 4. In Section 5 we make some further remarks. The author thanks the referee for several helpful comments.

2. A preliminary result

We begin with the elliptic integral K given in (1.2) and its companion integral

$$K'(k) = -i \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where $0 < k < 1$, and where the integrals K and K' are taken over the real intervals $[0,1]$ and $[1, 1/k]$, respectively. For brevity we shall often write K and K' for $K(k)$ and $K'(k)$. For more details the reader can consult, for example, any of [2] (which we recommend), [4], [7], [9, p. 280] and [11]. The single result in this section shows that the hyperbolic density of a rectangle is intimately connected to the two elliptic integrals K and K'.

Theorem 2.1. *Suppose that* $R = (-a,a) \times (-b,b)$, *where* a *and* b *are positive. Then there exists a unique positive* t *such that*

$$(-at, at) \times (-bt, bt) = \big(-K(k), K(k)\big) \times \big(-K'(k), K'(k)\big)$$

for some (unique) k, *and then* $\lambda_R(0) = t$.

Theorem 2.1 shows the importance of the elliptic integrals K and K' for our discussion of the hyperbolic metric of a rectangle, and because of this we briefly recall some of their properties. It is well known that $k \mapsto K(k)$ is a strictly increasing map of $(0,1)$ onto $(\pi/2, +\infty)$ that is given by the power series

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1^2}{2^2}k^2 + \frac{1^2 3^2}{2^2 4^2}k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6 + \cdots\right) \tag{2.1}$$

(see [4, p. 90]). As $K'(k) = K\big(\sqrt{1-k^2}\big)$, it follows that the map $k \mapsto K(k)/K'(k)$ is a strictly increasing map of $(0,1)$ onto $(0, +\infty)$. As usual, we define the function q on $(0,1)$ by

$$q(k) = \exp\big(-\pi K'(k)/K(k)\big); \tag{2.2}$$

then $k \mapsto q(k)$ is a homeomorphism of $(0,1)$ onto itself whose inverse is given by

$$k^2 = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^8 . \tag{2.3}$$

For these facts, see [9, pp. 281–289]. As we can express k as a function of q, we can also express K as a function of q, and the formula for this is

$$K\big(k(q)\big) = \frac{\pi}{2}\left(1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}\right) \tag{2.4}$$

(see [2, (3.14), p. 50]).

The proof of Theorem 2.1. Suppose that $0 < k < 1$ and let

$$R_0 = (-K, K) \times (0, K'), \quad R_1 = (-K, K) \times (-K', K').$$

The unique conformal map F of the upper half-plane $\mathbf{H}$ onto R_0 that maps $-1/k$, -1, 1, $1/k$ to the points $-K + iK'$, $-K$, K, $K + iK'$, respectively, is given by the Schwarz–Christoffel formula

$$F(z) = \int_0^z \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}} \tag{2.5}$$

(see [9, p. 280–282]). The map $g(z) = (z^2-1)/(z^2+1)$ maps the right half-plane Σ (given by $x > 0$) onto the complex plane $\mathbf{C}$ cut from $-\infty$ to -1, and from 1 to $+\infty$, which we denote by Ω. As F maps $(-1,1)$ onto $(-K,K)$, and $\mathbf{H}$ onto R_0, we can use the Reflection Principle to extend F to an analytic map of Ω onto R_1. Thus $F \circ g$ maps Σ conformally onto R_1, and because $F(0) = 0$, $F'(0) = 1$ and $\lambda_\Sigma(z) = 1/\operatorname{Re}[z]$, an application of (1.1) shows that $\lambda_{R_1}(0) = 1$. In particular, if $a = K(k)$ and $b = K'(k)$ for some k in $(0,1)$ then $R = R_1$ for this k and so $\lambda_R(0) = 1$.

As the function $k \mapsto K(k)/K'(k)$ maps $(0,1)$ monotonically onto $(0,+\infty)$ there is a unique value of k such that $b/a = K'(k)/K(k)$, and hence a unique value of t such that $at = K(k)$ and $bt = K'(k)$. Then (from the result above) $1 = \lambda_{R_1}(0) = t^{-1}\lambda_R(0)$ and the proof is complete.

3. The proofs of Theorems 1.1, 1.2 and Corollary 1.3

We begin with the proof of Theorem 1.1. Given any positive l, we can choose k such that $K/K' = \pi/(2l)$, and we note that with this choice of k, $q = e^{-2l}$. Now let $R = (-K,K) \times (-K',K')$. Then $g(z) = \pi i z/2K$ maps R onto $R(l)$, and from (1.1), Theorem 2.1 and (2.4), we have

$$\lambda_{R(l)}(0) = \frac{2K(k)}{\pi}\lambda_R(0) = \frac{2K(k)}{\pi} = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{\cosh 2nl}$$

as required.

The proof of Theorem 1.2. We use the same notation as above. There is exactly one value, say k_0, of k such that $K(k_0) = K'(k_0)$ and it is easy to see that $k_0 = 1/\sqrt{2}$. It follows from Theorem 2.1 that if R^* is the rectangle $(-c, c) \times (-c, c)$, where $c = K(1/\sqrt{2}) = 1 \cdot 8541\ldots$, then $\lambda_{R^*}(0) = 1$. The result for a general rectangle now follows from (1.1) by applying the map $z \mapsto az/c$.

The proof of Corollary 1.3. The lower bound of $\lambda_{R(l)}(0)$ is immediate from Theorem 1.1. The upper bound also follows easily from Theorem 1.1 if we use the inequality $2\cosh x > e^x$ for all positive x and then sum the resulting geometric series.

4. The proof of Theorem 1.4

The first inequality in (1.3) holds because if γ is the hyperbolic geodesic segment in $R(l)$ that joins 0 to x, then

$$d_{R(l)}(0, x) = \int_\gamma \lambda_{R(l)}(t)\, dt > \int_\gamma \lambda_S(t)\, dt \geq x$$

because $\lambda_S(z) \geq 1$ throughout S. Now suppose that $0 < t < l - \pi/2$, write $l' = l - t$ and let R' be the rectangle $(t - l', t + l') \times (-\pi/2, \pi/2)$. Then (from the monotonicity of the metric, and Corollary 1.3)

$$\lambda_{R(l)}(t) < \lambda_{R'}(t) = \lambda_{R(l')}(0) < 1 + \frac{2}{\cosh 2(l - t)} + \frac{5}{e^{4(l-t)}},$$

and as

$$d_{R(l)}(0, x) \leq \int_0^x \lambda_{R(l)}(t)\, dt$$

the second inequality in (1.3) follows.

5. Closing remarks

In this section we continue our discussion of the hyperbolic metric of rectangles and elliptic integrals, and we present some ideas which may be of use in other circumstances. First, we note the following variation on Theorem 2.1.

Theorem 5.1. *Let* $R = (-K, K) \times (0, K')$, *where* $0 < k < 1$. *Then* $\lambda_R(iK'/2) = 1 + k$.

Proof. The function F given by (2.5) is the conformal map of $\mathbf{H}$ onto the rectangle R with $-1/k$, -1, 1, $1/k$ mapping to the vertices of R as described earlier. The inverse of F is the Jacobian function sn: $R \to \mathbf{H}$ which can be continued analytically over $\mathbf{C}$ to give an elliptic function with periods $4K$ and $2iK'$, simple zeros at $2nK + 2imK'$ (and no other zeros), and simple poles at

$2nK+2imK'+iK'$ (and no other poles), where m and n are integers. It follows that the function $\mathrm{sn}(z)\mathrm{sn}(iK'-z)$ is elliptic and without poles and so is constant. By equating its values at K and $iK'/2$, and writing $\eta = iK'/2$, we obtain $\mathrm{sn}(\eta)^2 = -1/k$, and we deduce from this that $\mathrm{sn}(\eta) = i/\sqrt{k}$. Now from (1.1), $\lambda_R(\eta) = \lambda_{\mathbf{H}}\big(\mathrm{sn}(\eta)\big)\,|\mathrm{sn}'(\eta)| = \sqrt{k}\,|\,\mathrm{sn}'(\eta)|$. Associated to the function $\mathrm{sn}(z)$ are the Jacobi functions $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$, and these satisfy the relations $\mathrm{sn}^2+\mathrm{cn}^2=1$, $k^2\mathrm{sn}^2+\mathrm{dn}^2=1$ and $\mathrm{sn}'=\mathrm{cn}\,\mathrm{dn}$. We deduce that

$$|\mathrm{sn}'(\eta)| = \sqrt{1-\mathrm{sn}(\eta)^2}\,\sqrt{1-k^2\mathrm{sn}(\eta)^2}$$

from which we obtain $\lambda_R(\eta) = 1+k$ as required.

We end this paper with a brief discussion of the rectangle $R(l)$ when l is large. As $K(k)\to\pi/2$ when $k\to 0$, and as we know how the hyperbolic metric transforms under a scaling map $z\mapsto \mu z$, it suffices to study the the rectangle $R = (-K,K)\times(-K',K')$ as $k\to 0$. When k is small and positive R is approximately the rectangle $(-\pi/2,\pi/2)\times(-K',K')$ and so its shape (or modulus) depends on the nature of the singularity of $K'(k)$ at $k=0$. It is known that

$$\lim_{k\to 0}\left(K'(k)-\log\frac{4}{k}\right) = 0 \tag{5.1}$$

so that when k is small R is approximately $(-\pi/2,\pi/2)\times(-\log 4/k, \log 4/k)$. There are many proofs of (5.1) in the literature, and an elementary proof, valid for $0<k<1$ (which is sufficient for our purpose), is given in [11, p. 522]. Other proofs, and other inequalities, can be found in the more recent [10, p. 45], and proofs for complex k occur in [4, pp. 91 and 178], [11, pp. 299 and 521–522], and [7, pp. 25–27 and 73–75]. Here, we provide a short and elementary proof (which is perhaps new) of the following result.

Theorem 5.2. *As* $k\to 0$,

$$K'(k) = \left(1+\frac{k^2}{4}\right)\log\frac{4}{k} + O(k^2).$$

Proof. Throughout we assume that k and hence q, lie in $(0,1)$. We begin by proving that $k^2+k^4 > 16q$ when $0<k<\sqrt{3/8}$. First, from (2.3) we have

$$k^2 > \frac{16q}{(1+q)^8} > 16q(1-8q) \tag{5.2}$$

(because $(1+x)^8(1-8x)$ is strictly decreasing for $x\geq 0$) and it follows from this that

$$k^2+k^4 > 16q + 128q^2(1-32q) + 2^{14}q^4.$$

Now the inequality in (5.2) shows that if $q = 1/32$ then $k^2 > 3/8$; thus if $0 < k < \sqrt{3/8}$ then $0 < q < 1/32$ and so $k^2 + k^4 > 16q$ as claimed. Now (2.2) implies that

$$K' = \frac{K}{\pi}\log\frac{1}{q} = \frac{2K}{\pi}\log\frac{4}{k} - \frac{K}{\pi}\log\left(\frac{16q}{k^2}\right),$$

and so using (2.3) again and the inequality $k^2 + k^4 > 16q$, we see that

$$\frac{2K}{\pi}\log\frac{4}{k} \geq K' \geq \frac{2K}{\pi}\log\frac{4}{k} - \frac{K}{\pi}\log(1+k^2).$$

Now $\log(1+x) < x$ when $x > 0$, and using this and (2.1), Theorem 5.1 follows easily.

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