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A RIESZ REPRESENTATION FORMULA FOR SUPER-BIHARMONIC FUNCTIONS

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Abstract. Let u be a real-valued function defined on the unit disk $\mathbf{D}$. We call u super-biharmonic provided that u is locally integrable and the bi-laplacian $\Delta^2 u$ is a positive distribution on $\mathbf{D}$. In this paper, we shall establish a representation formula for super-biharmonic functions. This formula can be regarded as an analogue of the Poisson–Jensen representation formula for subharmonic functions.

0. Introduction

Notation. We denote by $\mathbf{C}$ the complex plane, by $\mathbf{D}$ the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$, and by $\mathbf{T}$ the unit circle $\{z \in \mathbf{C} : |z| = 1\}$. The Laplace operator in the complex plane is denoted by

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad z = x + iy.$$

We write $dA(z) = \pi^{-1}\,dx\,dy$ for the normalized Lebesgue area measure on the unit disk. Similarly for $z = e^{i\theta}$, we write $d\sigma(z) = (2\pi)^{-1}\,d\theta$ for the normalized arc length measure on the unit circle. We write u_r for the dilation of u by r, $0 < r < 1$: $u_r(z) = u(rz)$. In this way, we may think of u_r as a function on the unit circle $\mathbf{T}$. A locally integrable function u on the unit disk will be considered as a distribution via the duality relation

$$\langle \phi, u \rangle = \int_{\mathbf{D}} u(z)\phi(z)\,dA(z),$$

where $\phi \in C_c^\infty(\mathbf{D})$, the space of C^∞-functions with compact support in $\mathbf{D}$.

Representation of subharmonic functions. For C^∞-smooth u on the closed unit disk $\overline{\mathbf{D}}$, the Poisson–Jensen formula, which is an immediate consequence of Green's formula, represents u as

$$u(z) = \int_{\mathbf{D}} G(z,\zeta)\Delta u(\zeta)\,dA(\zeta) + \int_{\mathbf{T}} P(z,\zeta)u(\zeta)\,d\sigma(\zeta), \qquad z \in \mathbf{D}. \tag{0-1}$$

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Here $G(z,\zeta)$ stands for the Green function for the Laplace operator in the unit disk:

$$G(z,\zeta)=\log\left|\frac{z-\zeta}{1-\bar\zeta z}\right|^2,\qquad (z,\zeta)\in\mathbf{D}\times\mathbf{D},$$

and $P(z,\zeta)$ denotes the Poisson kernel for $\mathbf{D}$:

$$P(z,\zeta)=\frac{1-|z|^2}{|z-\zeta|^2},\qquad (z,\zeta)\in\mathbf{D}\times\mathbf{T}.$$

A function $u\colon\mathbf{D}\to[-\infty,+\infty[$ is said to be *subharmonic* if it is upper semicontinuous (but not identically $-\infty$) and satisfies the sub-mean value inequality

$$u(z)\le\int_{\mathbf{T}}u(z+r\zeta)\,d\sigma(\zeta),\qquad z\in\mathbf{D},\ 0\le r<1-|z|.$$

The class of subharmonic functions coincides with the class of locally summable functions whose Laplacian is a locally finite positive Borel measure. In this identification, the locally summable function need not be upper semicontinuous, but it can be altered on an area-null set so as to have this property. The Poisson–Jensen representation formula (0-1) is valid in the context of subharmonic functions u under a growth assumption. To be specific, we assume that u is subharmonic on $\mathbf{D}$, and that

$$\sup_{0\le r<1}\int_{\mathbf{T}}u^+(rz)\,d\sigma(z)<+\infty,$$

where the superscript $+$ stands for the positive part of the function u. The positive and the negative part of a real-valued function u are defined by $u^+=\max(u,0)$ and $u^-=\max(-u,0)$. It follows under these assumptions that the Poisson–Jensen representation formula (0-1) generalizes to:

$$\text{(0-2)}\qquad u(z)=\int_{\mathbf{D}}G(z,\zeta)\,d\mu(\zeta)+\int_{\mathbf{T}}P(z,\zeta)\,d\nu(\zeta),\qquad z\in\mathbf{D},$$

where μ is a positive Borel measure on $\mathbf{D}$ with

$$\int_{\mathbf{D}}(1-|z|^2)\,d\mu(z)<+\infty,$$

and ν is a finite real-valued Borel measure on $\mathbf{T}$. The measure μ corresponds to the Laplacian Δu, and ν corresponds to the boundary values of u. Although we have not located the original manuscript where (0-2) first appeared, we nevertheless feel it should be ascribed to F. Riesz. It is possible to view (0-2) as a generalization of the classical inner-outer factorization theorem of functions in the Nevanlinna class [8]. If the function u is harmonic, the representation formula (0-2) involves only a Poisson integral. As we introduce the analytic function g with real part u, we obtain from (0-2) the Riesz–Herglotz representation for g. In [3], Hayman and Korenblum studied the Riesz–Herglotz representation in the much more general case of a premeasure ν.

Super-biharmonic functions. Let u be a locally summable complex-valued function on the unit disk. We say that u is *biharmonic* provided that $\Delta^2 u = 0$. If u is real-valued and $\Delta^2 u$ is a locally finite positive Borel measure on $\mathbf{D}$, we say that u is *super-biharmonic*. By elliptic regularity, the locally summable function u can be altered on an area-null set so as to be real-analytic, in case u is biharmonic. In case u is super-biharmonic, it is locally of regularity class $C^{1,\alpha}$ for each α, $0 < \alpha < 1$. We recall the standard notation $C^{1,\alpha}$ for the class of continuous functions whose first order partial derivatives are of Hölder class α.

We shall find a representation formula, analogous to (0-1), for a class of super-biharmonic functions. Let us first review the smooth case. The biharmonic Green function for the operator Δ^2 in the unit disk (with Dirichlet boundary conditions) is the function

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-\bar\zeta z}\right|^2 + \big(1-|z|^2\big)\big(1-|\zeta|^2\big), \qquad (z,\zeta)\in \mathbf{D}\times\mathbf{D}.$$

For fixed $\zeta \in \mathbf{D}$, it solves the following boundary value problem:

$$\begin{cases} \Delta_z^2 \Gamma(z,\zeta) = \delta_\zeta(z), & z\in\mathbf{D},\\ \Gamma(z,\zeta) = 0, & z\in\mathbf{T},\\ \partial_{n(z)}\Gamma(z,\zeta) = 0, & z\in\mathbf{T}, \end{cases}$$

where the notation $\partial_{n(z)}$ denotes the inward normal derivative with respect to the variable $z\in\mathbf{T}$. Suppose that u is a C^∞-smooth function on $\overline{\mathbf{D}}$. Applying Green's formula twice, we obtain the representation

$$\begin{aligned} u(z) = \int_{\mathbf{D}} \Gamma(\zeta,z)\Delta^2 u(\zeta)\,dA(\zeta) &- \frac{1}{2}\int_{\mathbf{T}} \partial_{n(\zeta)}\big(\Delta_\zeta \Gamma(\zeta,z)\big)u(\zeta)\,d\sigma(\zeta) \\ &+ \frac{1}{2}\int_{\mathbf{T}} \big(\Delta_\zeta\Gamma(\zeta,z)\big)\partial_{n(\zeta)}u(\zeta)\,d\sigma(\zeta), \qquad z\in\mathbf{D}. \end{aligned} \tag{0-3}$$

A computation shows that

$$\Delta_\zeta \Gamma(\zeta,z) = G(\zeta,z) + H(\zeta,z), \qquad (\zeta,z)\in\mathbf{D}\times\mathbf{D},$$

where

$$H(\zeta,z) = \big(1-|z|^2\big)\frac{1-|\zeta z|^2}{|1-\bar z\zeta|^2}, \qquad (\zeta,z)\in\overline{\mathbf{D}}\times\mathbf{D}. \tag{0-4}$$

We shall refer to $H(\zeta,z)$ as the *harmonic compensator*; it is harmonic in its first argument and is biharmonic in its second argument. Observe that $H(\zeta,z)$ is not symmetric in its arguments. Another computation shows that the function

$$F(\zeta,z) = -\tfrac{1}{2}\partial_{n(\zeta)}\Delta_\zeta\Gamma(\zeta,z), \qquad (\zeta,z)\in\mathbf{T}\times\mathbf{D},$$

has the form

$$F(\zeta,z)=\frac{1}{2}\left\{\frac{\left(1-|z|^2\right)^2}{|1-\bar{z}\zeta|^2}+\frac{\left(1-|z|^2\right)^3}{|1-\bar{z}\zeta|^4}\right\},\qquad (\zeta,z)\in\mathbf{T}\times\mathbf{D}. \tag{0-5}$$

Being biharmonic in its second argument, the function $F(\zeta, z)$ will be referred to as the *biharmonic Poisson kernel.* Note that in terms of the above kernels, (0-3) assumes the form

$$\begin{aligned}u(z)=&\int_{\mathbf{D}}\Gamma(\zeta,z)\Delta^2u(\zeta)\,dA(\zeta)+\int_{\mathbf{T}}F(\zeta,z)u(\zeta)\,d\sigma(\zeta)\\&+\frac{1}{2}\int_{\mathbf{T}}H(\zeta,z)\partial_{n(\zeta)}u(\zeta)\,d\sigma(\zeta),\qquad z\in\mathbf{D}.\end{aligned}$$

For a possibly non-smooth function u, it is natural to ask when we have the representation formula

$$u(z)=\int_{\mathbf{D}}\Gamma(\zeta,z)\,d\mu(\zeta)+\int_{\mathbf{T}}F(\zeta,z)\,d\nu(\zeta)+\int_{\mathbf{T}}H(\zeta,z)\,d\lambda(\zeta),\qquad z\in\mathbf{D},$$

where ν and λ are two real-valued finite Borel measures on $\mathbf{T}$, and μ is a positive Borel measure on $\mathbf{D}$ with

$$\int_{\mathbf{D}}\left(1-|z|^2\right)^2d\mu(z)<+\infty.$$

Clearly, u has to be super-biharmonic. Moreover, it can be seen that it meets the growth conditions

$$\sup_{0\le r<1}\int_{\mathbf{T}}u^+(rz)\,d\sigma(z)<+\infty, \tag{A}$$

which assures that $u_r\,d\sigma$ has at least one weak-star cluster point $d\nu$ as $r\to 1$, and

$$\sup_{0\le r<1}\frac{1}{1-r}\int_{\mathbf{T}}\left(u-F[\nu]\right)^-(rz)\,d\sigma(z)<+\infty, \tag{B}$$

where

$$F[\nu](z)=\int_{\mathbf{T}}F(\zeta,z)\,d\nu(\zeta),\qquad z\in\mathbf{D}. \tag{0-6}$$

It is a consequence of the second assumption (B) that the measure $d\nu$ obtained from this limit process is unique: $u_r\,d\sigma\to d\nu$ weak-star, as $r\to 1$. It now makes sense to ask whether the conditions (A) and (B) characterize the super-biharmonic functions u on $\mathbf{D}$ having the above representation.

Main Theorem. *For a locally summable function u on $\mathbf{D}$, the following two conditions are equivalent*:

(a) *u has the representation*

$$u(z) = \int_{\mathbf{D}} \Gamma(\zeta, z)\, d\mu(\zeta) + \int_{\mathbf{T}} F(\zeta, z)\, d\nu(\zeta) + \int_{\mathbf{T}} H(\zeta, z)\, d\lambda(\zeta), \qquad z \in \mathbf{D},$$

where μ is a positive Borel measure on $\mathbf{D}$ with

$$\int_{\mathbf{D}} \left(1 - |z|^2\right)^2 d\mu(z) < +\infty,$$

and ν, λ are finite real-valued Borel measures on $\mathbf{T}$.

(b) *u is super-biharmonic on $\mathbf{D}$, and meets the above growth conditions* (A) *and* (B).

1. The biharmonic extension

In this section, we study the kernel function $F(\zeta, z)$, defined by (0-5), in more detail. We denote by $C(\mathbf{T})$ the space of continuous functions on the unit circle $\mathbf{T}$. It turns out that every $f \in C(\mathbf{T})$ has a biharmonic extension Bf to the closed unit disk; moreover, Bf is continuous on $\overline{\mathbf{D}}$, and $(Bf)(rz) \to f(z)$ uniformly, as $r \to 1$. We use this fact to prove that the measure ν on $\mathbf{T}$ which is obtained as a weak-star limit of $u_r\, d\sigma$ has some kind of uniqueness property. We first note that

$$F(\zeta, z) > 0, \qquad (\zeta, z) \in \mathbf{T} \times \mathbf{D}, \tag{1-1}$$

and that

$$\int_{\mathbf{T}} F(\zeta, z)\, d\sigma(\zeta) = 1, \qquad z \in \mathbf{D}. \tag{1-2}$$

The equality (1-2) follows from the following simple calculation:

$$\begin{aligned}\int_{\mathbf{T}} F(\zeta, z)\, d\sigma(\zeta) &= \frac{1}{2}\int_{\mathbf{T}} \frac{\left(1 - |z|^2\right)^2}{|1 - \bar{z}\zeta|^2}\, d\sigma(\zeta) + \frac{1}{2}\int_{\mathbf{T}} \frac{\left(1 - |z|^2\right)^3}{|1 - \bar{z}\zeta|^4}\, d\sigma(\zeta) \\ &= \tfrac{1}{2}\left((1 - |z|^2) + (1 + |z|^2)\right) = 1.\end{aligned}$$

For $\zeta \in \mathbf{T}$, we let $I(\zeta)$ be the arc on the unit circle with center ζ and length $\delta > 0$. For $z \in \mathbf{D} \setminus \{0\}$, we write $z^* = z/|z|$ which belongs to the unit circle. It follows from (0-5) that $F(\zeta, z) \to 0$ uniformly, as $|z| \to 1$ and $z^* \in \mathbf{T} \setminus I(\zeta)$. Note that the Poisson kernel for the unit disk satisfies the relations (1-1), (1-2) and this last property as well. Let $f \in L^1(\mathbf{T})$, we then define the *F-integral* of f by

$$F[f](z) = \int_{\mathbf{T}} F(\zeta, z) f(\zeta)\, d\sigma(\zeta), \qquad z \in \mathbf{D}.$$

Since $F(\zeta, z)$, for fixed $\zeta \in \mathbf{T}$, is biharmonic, it follows that $F[f]$ is biharmonic in $\mathbf{D}$. The following proposition shows that the F-integrals of continuous functions behave well near the boundary of the unit disk.

Proposition 1.1. *Let $f \in C(\mathbf{T})$ and $z \in \mathbf{T}$. Define the biharmonic extension of f to the unit disk by*

$$(Bf)(z) = \begin{cases} f(z) & \text{if } z \in \mathbf{T}, \\ F[f](z) & \text{if } z \in \mathbf{D}. \end{cases}$$

Then Bf is continuous on $\overline{\mathbf{D}}$.

Proof. For a subset E of the complex plane, we use the notation

$$\|f\|_E = \sup\{|f(z)| : z \in E\}.$$

Assume that $f \in C(\mathbf{T})$ and that $z \in \mathbf{D}$. It follows from (1-1) and (1-2) that

$$\big|F[f](z)\big| = \left| \int_{\mathbf{T}} F(\zeta, z) f(\zeta)\, d\sigma(\zeta) \right| \le \|f\|_{\mathbf{T}}.$$

Hence

$$\|Bf\|_{\overline{\mathbf{D}}} = \|f\|_{\mathbf{T}}, \qquad f \in C(\mathbf{T}). \tag{1-3}$$

Put $g_n(z) = z^n$ and compute

$$\begin{aligned}(Bg_n)(z) &= \int_{\mathbf{T}} F(\zeta, z) g_n(\zeta)\, d\sigma(\zeta) = \int_{\mathbf{T}} F(\zeta, z) \zeta^n\, d\sigma(\zeta) \\ &= z^n \big(1 + \tfrac{1}{2} n (1 - |z|^2)\big), \qquad z \in \mathbf{D}.\end{aligned}$$

Let $p(z) = \sum_{n=-k}^{k} c_n z^n$ be a trigonometric polynomial on $\mathbf{T}$. It follows that for every such polynomial p, we have

$$(Bp)(z) = \sum_{-k}^{k} c_n z^n \big(1 + \tfrac{1}{2} n (1 - |z|^2)\big), \qquad z \in \mathbf{D}.$$

It follows that Bp is continuous on $\overline{\mathbf{D}}$. Since the trigonometric polynomials are dense in $C(\mathbf{T})$, we can assume that $\{p_n\}_{n=1}^{\infty}$ is a sequence of such polynomials on $\mathbf{T}$ such that $\|p_n - f\|_{\mathbf{T}} \to 0$ as $n \to \infty$. It follows from (1-3) that

$$\|Bp_n - Bf\|_{\overline{\mathbf{D}}} = \|B(p_n - f)\|_{\mathbf{D}} = \|B(p_n - f)\|_{\mathbf{T}} = \|p_n - f\|_{\mathbf{T}} \to 0, \quad \text{as } n \to \infty.$$

Hence Bp_n converges uniformly to Bf on $\overline{\mathbf{D}}$. Since each Bp_n is continuous on $\overline{\mathbf{D}}$, we see that $Bf \in C(\overline{\mathbf{D}})$. □

Proposition 1.2. *Let ν be a finite Borel measure on the unit circle $\mathbf{T}$. Then the mapping $\nu \mapsto F[\nu]$, where*

$$F[\nu](z) = \int_{\mathbf{T}} F(\zeta, z)\, d\nu(\zeta), \qquad z \in \mathbf{D},$$

is injective. Moreover,

$$\int_{\mathbf{T}} F[\nu](rz)\, d\sigma(z) = \int_{\mathbf{T}} d\nu(z) = \nu(\mathbf{T}), \qquad 0 \le r < 1.$$

Proof. We shall prove that $F[\nu] = 0$ implies $\nu = 0$. For this, it suffices to verify that for every $f \in C(\mathbf{T})$ we have $\int_{\mathbf{T}} f(z)\, d\nu(z) = 0$. Note that the kernel function $F(\zeta, z)$ has the property that

$$F(\zeta, rz) = F(z, r\zeta), \qquad (\zeta, z) \in \mathbf{T} \times \mathbf{T} \text{ and } 0 \le r < 1.$$

We now use the above mentioned symmetry of $F(\zeta, z)$, Fubini's theorem, and the assumption that $F[\nu] = 0$ to obtain

$$\begin{aligned}\int_{\mathbf{T}} F[f](rz)\, d\nu(z) &= \int_{\mathbf{T}} \int_{\mathbf{T}} F(\zeta, rz) f(\zeta)\, d\sigma(\zeta)\, d\nu(z) \\ &= \int_{\mathbf{T}} \left(\int_{\mathbf{T}} F(z, r\zeta)\, d\nu(z) \right) f(\zeta)\, d\sigma(\zeta) \\ &= \int_{\mathbf{T}} F[\nu](r\zeta) f(\zeta)\, d\sigma(\zeta) = 0.\end{aligned}$$

According to Proposition 1.1, the functions $F[f](rz)$ converge uniformly on $\mathbf{T}$ to $f(z)$ as $r \to 1$. Hence

$$\int_{\mathbf{T}} f(z)\, d\nu(z) = 0, \qquad f \in C(\mathbf{T}),$$

from which it follows that $\nu = 0$. As for the remaining statement, we observe that

$$\begin{aligned}\int_{\mathbf{T}} F[\nu](rz)\, d\sigma(z) &= \int_{\mathbf{T}} \int_{\mathbf{T}} F(\zeta, rz)\, d\nu(\zeta)\, d\sigma(z) = \int_{\mathbf{T}} \left(\int_{\mathbf{T}} F(\zeta, rz)\, d\sigma(z) \right) d\nu(\zeta) \\ &= \int_{\mathbf{T}} \tfrac{1}{2}\big((1 - r^2) + (1 + r^2)\big)\, d\nu(\zeta) = \int_{\mathbf{T}} d\nu(\zeta) = \nu(\mathbf{T}).\end{aligned}$$

The proof is complete. ▫

Remark 1.3. Let u be a function continuous on $\mathbf{T}$ and biharmonic on $\mathbf{D}$. Denote the biharmonic extension of $u|_{\mathbf{T}}$, the restriction of u to $\mathbf{T}$, to the unit disk by $v = Bu$. In general, u and v need not agree inside the unit disk. This is in contrast to the case when u is harmonic inside the unit disk, and the Poisson integral of u agrees with u inside $\mathbf{D}$. Indeed, the Dirichlet problem for the operator Δ^2 has two boundary data: $u|_{\mathbf{T}}$ and $\partial_n u|_{\mathbf{T}}$.

2. Radial super-biharmonic functions

Let u be a real-valued function defined on the unit disk. We call u radial provided that it depends only on $r = |z|$, for $0 \le r < 1$. It is well known that a radial subharmonic function is an increasing convex function of $t = \log r$, for $-\infty < t < 0$ (see [7, Theorem 2.6.6]). Radial super-biharmonic functions have analogous properties, as we shall see. It turns out that the radial super-biharmonic functions are either C^2 on $]0,1]$, or they are convex functions of $\log r$ for r close to 1. We prove the following lemma:

Lemma 2.1. *Let u be a radial super-biharmonic function on $\mathbf{D}$. Then either u is a C^2-function on $]0,1]$, or there exists an r_0, $0 < r_0 < 1$, such that u is a convex function of $\log r$ for $r_0 < r < 1$.*

Proof. Let u be a super-biharmonic function which depends only on r, for $0 < r < 1$. Put $t = \log r$, then for $-\infty < t < 0$ we get

$$\Delta^2 u(e^t) = \frac{1}{4e^{2t}} \frac{d^2}{dt^2}\left(\frac{1}{4e^{2t}} \frac{d^2 u}{dt^2}\right) \ge 0, \qquad -\infty < t < 0.$$

It follows that

$$k(t) = e^{-2t} \frac{d^2 u}{dt^2}, \qquad -\infty < t < 0,$$

is a convex function of t. By the basic properties of convex functions, only the following two cases can occur.

Case 1: $\lim_{t\to 0} k(t)$ exists as a finite number. The function k being continuous on $]-\infty,0]$, we can solve the second order differential equation $d^2u(e^t)/dt^2 = e^{2t}k(t)$, and obtain that $u(e^t)$ is a C^2-function on $]-\infty,0]$.

Case 2: $\lim_{t\to 0} k(t) = +\infty$. The second derivative of u is now positive on some neighborhood of 0, say $-\delta < t < 0$, for some $\delta > 0$. Therefore, u is a convex function of t on this neighborhood. □

Lemma 2.2. *Let u be a radial biharmonic function on $\mathbf{D} \setminus \{0\}$. Then we have*

$$u(r) = a + br^2 + \big(c + dr^2\big)\log r,$$

where a, b, c, and d are constants.

Proof. As in the proof of the preceding lemma, we put $t = \log r$ and obtain

$$\Delta^2 u(e^t) = \frac{d^2}{dt^2}\left(\frac{1}{4e^{2t}} \frac{d^2 u}{dt^2}\right) = 0, \qquad -\infty < t < 0.$$

It follows that

$$\frac{d^2 u}{dt^2}(e^t) = 4e^{2t}(c_1 t + c_2),$$

where c_1 and c_2 are constants. Now, integrating twice leads to the result. □

Proposition 2.3. *Let u be a super-biharmonic function on $\mathbf{D}$. Then*

$$\tilde{u}(\xi) = \int_{\mathbf{T}} u(\xi z)\, d\sigma(z), \qquad \xi \in \mathbf{D},$$

is radial and super-biharmonic. Moreover, if u satisfies the condition (A), *then the function $\tilde{u}(r)$, $0 \le r < 1$, has a finite limit as $r \to 1$.*

Proof. It follows by a change of variables argument that $\tilde{u}$ is radial. The fact that $\tilde{u}$ is super-biharmonic follows from the super-biharmonicity of u. As for the second part of the statement, we use Lemma 2.1 to conclude that in a neighborhood $]r_0, 1]$ of 1 the function $\tilde{u}$ is either C^2, or a convex function of $\log r$. In case $\tilde{u}$ is C^2, we obtain $\lim_{r\to 1} \tilde{u}(r) = \tilde{u}(1)$, by continuity. Assume that $\tilde{u}$ is a convex function of $\log r$, for r sufficiently close to 1. Then either $\lim_{r\to 1} \tilde{u}(r) = +\infty$ or this limit is finite. The assumption (A) on u^+ rules out the first possibility, so that $\tilde{u}(r)$ approaches a finite limit as $r \to 1$. □

Proposition 2.4. *Let u be a super-biharmonic function on the unit disk which satisfies the condition* (A). *Then the family $\{u_r\}_{0\le r<1}$ is uniformly bounded in the Banach space $L^1(\mathbf{T})$, that is*

$$\sup_{0\le r<1} \int_{\mathbf{T}} \bigl|u(rz)\bigr|\, d\sigma(z) < +\infty,$$

and hence there exists a sequence $\{r_j\}_j$, $0 < r_1 < r_2 < \cdots$, with $\lim_{j\to\infty} r_j = 1$, such that $u_{r_j} d\sigma$ converges weak-star to a finite real-valued Borel measure ν on $\mathbf{T}$, as $j \to \infty$.

Proof. By Proposition 2.3,

$$\tilde{u}(r) = \int_{\mathbf{T}} u(rz)\, d\sigma(z)$$

tends to a finite limit as $r \to 1$. Since $u^-(rz) = u^+(rz) - u(rz)$, it follows from the assumption (A) that

$$\sup_{0\le r<1} \int_{\mathbf{T}} u^-(rz)\, d\sigma(z) < +\infty.$$

On the other hand,

$$\int_{\mathbf{T}} |u(rz)|\, d\sigma(z) = \int_{\mathbf{T}} u^+(rz)\, d\sigma(z) + \int_{\mathbf{T}} u^-(rz)\, d\sigma(z),$$

so that, again by (A),

$$\sup_{0\le r<1} \int_{\mathbf{T}} |u(rz)|\, d\sigma(z) < +\infty. \tag{2-1}$$

We now consider the linear functionals

$$\Lambda_r f = \int_{\mathbf{T}} u_r f \, d\sigma, \qquad 0 \le r < 1, \; f \in C(\mathbf{T}).$$

It follows from (2-1) that

$$\sup_{0 \le r < 1} \|\Lambda_r\| < +\infty. \tag{2-2}$$

Since $C(\mathbf{T})$ is a separable Banach space, it follows from (2-2) and the Banach–Alaoglu theorem that there exists a subsequence r_j with $r_j \to 1$ such that Λ_{r_j} is weak-star convergent as $j \to \infty$. This means that there exists a Borel measure ν on the unit circle such that

$$\lim_{j \to \infty} \int_{\mathbf{T}} u_{r_j} f \, d\sigma = \int_{\mathbf{T}} f \, d\nu, \qquad f \in C(\mathbf{T}).$$

The proof is complete. □

Proposition 2.5. *Let u be a super-biharmonic function on the unit disk which satisfies the condition* (A). *Let $\{r_j\}_{j=1}^{\infty}$ and ν be as in Proposition* 2.4. *Let $F[\nu]$ be the potential defined by* (0-6). *Then we have*

$$\lim_{j \to \infty} \int_{\mathbf{T}} \big(u - F[\nu]\big)(r_j z) f(z) \, d\sigma(z) = 0, \qquad f \in C(\mathbf{T}).$$

Moreover, if u satisfies the condition (B), *then we have*

$$\int_{\mathbf{T}} \big|\big(u - F[\nu]\big)(rz)\big| \, d\sigma(z) = O(1-r), \qquad \text{as } r \to 1.$$

Proof. We first observe that for every $f \in C(\mathbf{T})$ we have

$$\begin{aligned}\int_{\mathbf{T}} F[\nu](rz) f(z) \, d\sigma(z) &= \int_{\mathbf{T}} \int_{\mathbf{T}} F(\zeta, rz) \, d\nu(\zeta) f(z) \, d\sigma(z) \\ &= \int_{\mathbf{T}} \int_{\mathbf{T}} F(z, r\zeta) f(z) \, d\sigma(z) \, d\nu(\zeta) = \int_{\mathbf{T}} F[f](r\zeta) \, d\nu(\zeta).\end{aligned}$$

Here, we used Fubini's theorem together with the property $F(\zeta, rz) = F(z, r\zeta)$ of the biharmonic Poisson kernel F. By Proposition 1.1, $F[f](r\zeta) \to f(\zeta)$ uniformly on $\mathbf{T}$ as $r \to 1$. Hence

$$\lim_{r \to 1} \int_{\mathbf{T}} F[\nu](rz) f(z) \, d\sigma(z) = \int_{\mathbf{T}} f(z) \, d\nu(z), \qquad f \in C(\mathbf{T}).$$

On the other hand, ν is the weak-star limit of $u_{r_j}\, d\sigma$, so that

$$\lim_{j\to\infty}\int_{\mathbf{T}} u(r_j z) f(z)\, d\sigma(z) = \int_{\mathbf{T}} f(z)\, d\nu(z), \qquad f \in C(\mathbf{T}).$$

It follows that

$$\lim_{j\to\infty}\int_{\mathbf{T}} \big(u - F[\nu]\big)(r_j z) f(z)\, d\sigma(z) = 0, \qquad f \in C(\mathbf{T}), \tag{2-3}$$

as asserted. As for the second part of the statement, we put $w = u - F[\nu]$. Since

$$\int_{\mathbf{T}} w^+(rz)\, d\sigma(z) = \int_{\mathbf{T}} w(rz)\, d\sigma(z) + \int_{\mathbf{T}} w^-(rz)\, d\sigma(z), \tag{2-4}$$

we see that

$$0 \le \int_{\mathbf{T}} w^+(rz)\, d\sigma(z) \le \left(\int_{\mathbf{T}} w(rz)\, d\sigma(z)\right)^+ + \int_{\mathbf{T}} w^-(rz)\, d\sigma(z). \tag{2-5}$$

By assumption, we know that

$$\int_{\mathbf{T}} w^-(rz)\, d\sigma(z) = O(1-r), \qquad \text{as } r \to 1. \tag{2-6}$$

It follows from (2-3), with f identically 1, that

$$\lim_{j\to\infty}\int_{\mathbf{T}} w(r_j z)\, d\sigma(z) = 0. \tag{2-7}$$

The super-biharmonic function w satisfies the following condition:

$$\sup_{0\le r<1}\int_{\mathbf{T}} w^+(rz)\, d\sigma(z) < +\infty. \tag{2-8}$$

Indeed, since by Proposition 1.2,

$$\begin{aligned}
\int_{\mathbf{T}} w^+(rz)\, d\sigma(z) &\le \int_{\mathbf{T}} u^+(rz)\, d\sigma(z) + \int_{\mathbf{T}} \big(F[\nu]\big)^-(rz)\, d\sigma(z)\\
&\le \int_{\mathbf{T}} u^+(rz)\, d\sigma(z) + \int_{\mathbf{T}} F[\nu^-](rz)\, d\sigma(z)\\
&= \int_{\mathbf{T}} u^+(rz)\, d\sigma(z) + \nu^-(\mathbf{T}),
\end{aligned}$$

where ν^- is the negative part of the signed measure ν, the assertion (2-8) is an immediate consequence of the assumption (A). In view of (2-8), we can apply Proposition 2.3 with u replaced by w to conclude that

$$\widetilde{w}(r) = \int_{\mathbf{T}} w(rz)\, d\sigma(z), \qquad 0 \le r < 1,$$

has a finite limit as $r \to 1$. Observe that (2-7) reads $\lim_{j\to\infty} \widetilde{w}(r_j) = 0$, so that $\lim_{r\to 1} \widetilde{w}(r) = 0$. According to Lemma 2.1, the radial super-biharmonic function $\widetilde{w}(r)$ is either C^2-smooth on $]0,1]$ or a convex function of $\log r$, for r close to 1. In case $\widetilde{w}(r)$ is C^2-smooth, it follows from the Taylor expansion of $\widetilde{w}$ about 1 that

$$\widetilde{w}(r) = \int_{\mathbf{T}} w(rz)\, d\sigma(z) = O(1-r), \qquad \text{as } r \to 1. \tag{2-9}$$

It now follows from (2-4), (2-6), and (2-9) that in this case, we have

$$\int_{\mathbf{T}} w^+(rz)\, d\sigma(z) = O(1-r), \qquad \text{as } r \to 1. \tag{2-10}$$

We turn to the remaining case when $\widetilde{w}$ is a convex function of $\log r$, for r close to 1. By taking the positive part, we are in fact cutting off the part of the graph of the function which lies under the horizontal axis, and replacing that part with the constant function 0. By elementary properties of convex functions, using that $\widetilde{w}(r) \to 0$ as $r \to 1$, we see that

$$\widetilde{w}^+(r) = \left(\int_{\mathbf{T}} w(rz)\, d\sigma(z)\right)^+ = O(1-r), \qquad \text{as } r \to 1.$$

This, together with (2-5) and (2-6), implies that (2-10) holds in this case as well. Now, (2-10) together with the identity $|w| = w^+ + w^-$ and (2-6) implies that

$$\int_{\mathbf{T}} |w(rz)|\, d\sigma(z) = O(1-r), \qquad \text{as } r \to 1,$$

as claimed. The proof is complete. □

Remark 2.6. Let u be a super-biharmonic function satisfying the conditions (A) and (B). Then the measure ν obtained in Proposition 2.4 is unique. Indeed, it follows from Proposition 2.5 that

$$\lim_{r\to 1} \int_{\mathbf{T}} \big(u - F[\nu]\big)(rz) f(z)\, d\sigma(z) = 0, \qquad f \in C(\mathbf{T}).$$

3. A Riesz representation formula

Let u be a super-biharmonic function which fulfills the conditions (A) and (B). We first show that

$$\int_{\mathbf{D}} \Gamma(z,\zeta)\, d\mu(\zeta) < +\infty, \qquad z \in \mathbf{D},$$

where the positive Borel measure μ is the distributional derivative $\Delta^2 u$. We begin with the following lemma.

Lemma 3.1. *Suppose u is a super-biharmonic function satisfying the conditions* (A) *and* (B)*. Let μ be the positive Borel measure on $\mathbf{D}$ corresponding to the distributional derivative $\Delta^2 u$. Then we have*

$$\int_{\mathbf{D}} \left(1-|z|^2\right)^2 d\mu(z) < +\infty.$$

Proof. We recall the function $F[\nu]$ defined by (0-6), and put $w = u - F[\nu]$. Since the function w is super-biharmonic, it defines a positive Borel measure μ on $\mathbf{D}$ which is the distributional derivative $\Delta^2 w = \Delta^2 u$. For $0 < r < 1$, we have

$$\begin{cases} \Delta w_r(z) = r^2 (\Delta w)(rz), \\ \Delta^2 w_r(z) = r^4 (\Delta^2 w)(rz) = r^4 d\mu(rz). \end{cases}$$

We consider the expression

$$I(r) = \int_{\mathbf{D}} \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2}\, d\mu(rz), \qquad 0 < r < 1.$$

The dilation of $d\mu$ by r means that only the restriction of μ to the smaller disk $\{z : |z| < r\}$ is involved in the above integral expression. The integrand vanishes on the boundary $\mathbf{T}$ along with its normal derivative, so that by Green's formula we have

$$r^4 I(r) = \int_{\mathbf{D}} \Delta_z \left(\frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right) \Delta_z \big(w(rz)\big)\, dA(z).$$

Another application of Green's formula yields

$$\begin{aligned} r^4 I(r) = &\int_{\mathbf{D}} \Delta_z^2 \left(\frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right) w(rz)\, dA(z) \\ &- \frac{1}{2} \int_{\mathbf{T}} \Delta_z \left(\frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right) \partial_{n(z)} w(rz)\, d\sigma(z) \\ &+ \frac{1}{2} \int_{\mathbf{T}} \partial_{n(z)} \left(\Delta_z \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right) w(rz)\, d\sigma(z). \end{aligned} \tag{3-1}$$

We want to estimate the right-hand side expression in (3-1). A computation shows that

$$\Delta_z \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} = \left(1-|z|^2\right) \frac{4r^4|z|^6 + (r^4 - 11r^2)|z|^4 + (r^4 - 2r^2 + 9)|z|^2 + r^2 - 3}{\left(1-r^2|z|^2\right)^3}, \tag{3-2}$$

which takes the value 0 for $z \in \mathbf{T}$. Hence the first boundary integral in (3-1) vanishes. Using (3-2), we see that

$$\partial_{n(z)} \Delta_z \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} = \frac{12}{1-r^2}.$$

The equation (3-1) now simplifies:

$$r^4 I(r) = \int_{\mathbf{D}} \Delta_z^2 \left(\frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right) w(rz)\, dA(z) + \frac{6}{1-r^2} \int_{\mathbf{T}} w(rz)\, d\sigma(z). \tag{3-3}$$

According to Proposition 2.5, there exists a constant C such that

$$\left| \int_{\mathbf{T}} w(rz)\, d\sigma(z) \right| \le \int_{\mathbf{T}} |w(rz)|\, d\sigma(z) \le C(1-r^2), \qquad 0<r<1. \tag{3-4}$$

To estimate the first integral on the right-hand side of (3-3), we notice that

$$\frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} = 1 + (r^2+1)|z|^2 + (r^4-3r^2+3)|z|^4 + (r^2-1)^3 \frac{|z|^6}{1-r^2|z|^2}.$$

A calculation based on the above identity shows that

$$\Delta_z^2 \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} = 4(1-r^2)^3|z|^2 \left(\frac{-r^8|z|^8 + 5r^6|z|^6 - 10r^4|z|^4 + 9r^2|z|^2 + 9}{\left(1-r^2|z|^2\right)^5} \right) + 4(r^4-3r^2+3).$$

This leads to

$$\left| \Delta_z^2 \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right| \le 36 \frac{(1-r^2)^3}{\left(1-r^2|z|^2\right)^5} + 12,$$

from which it follows that

$$\begin{aligned} &\left| \int_{\mathbf{D}} \Delta_z^2 \left(\frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} \right) w(rz)\, dA(z) \right| \\ &\qquad \le 36(1-r^2)^3 \int_{\mathbf{D}} \frac{|w(rz)|}{\left(1-r^2|z|^2\right)^5}\, dA(z) + 12 \int_{\mathbf{D}} |w(rz)|\, dA(z). \end{aligned} \tag{3-5}$$

By (3-4), we have

$$\begin{aligned} &(1-r^2)^3 \int_{\mathbf{D}} \frac{|w(rz)|}{\left(1-r^2|z|^2\right)^5}\, dA(z) \\ &\qquad = (1-r^2)^3 \frac{1}{\pi} \int_0^1 \frac{s\, ds}{(1-r^2s^2)^5} \int_0^{2\pi} \left|w(rse^{i\theta})\right| d\theta \\ &\qquad \le C(1-r^2)^3 \int_0^1 \frac{2s\, ds}{(1-r^2s^2)^4} = \frac{C}{3}\left((1-r^2) + (1-r^2)^2 + 1\right) \le C. \end{aligned} \tag{3-6}$$

That the second term on the right-hand side of (3-5) remains bounded as $r \to 1$ is immediate from (3-4). Adding the terms together in (3-3), we obtain

$$r^4 I(r) = \int_{\mathbf{D}} \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2}\, d\mu(rz) \le (36C + 12C) + 6C = 54C.$$

We now let $r \to 1$ in the above estimate, and use Fatou's lemma to obtain

$$\int_{\mathbf{D}} \left(1-|z|^2\right)^2 d\mu(z) \le 54C < +\infty.$$

The proof is complete. □

We need the following lemma regarding the biharmonic Green function for the unit disk [1, Proposition 2.3]:

Lemma 3.2. *Let $\Gamma(z,\zeta)$ be the biharmonic Green function for the unit disk. Then for every $(z,\zeta)\in\mathbf{D}\times\mathbf{D}$, we have*

$$0<\frac{1}{2}\frac{\left(1-|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{|1-\overline{\zeta}z|^2}\le\Gamma(z,\zeta)\le\frac{\left(1-|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{|1-\overline{\zeta}z|^2},$$

and

$$\tfrac{1}{2}\left(1-|z|\right)^2\left(1-|\zeta|^2\right)^2\le\Gamma(z,\zeta)\le\left(1+|z|\right)^2\left(1-|\zeta|^2\right)^2.$$

Proposition 3.3. *Let μ be a positive Borel measure on $\mathbf{D}$ with*

$$\int_{\mathbf{D}}\left(1-|z|^2\right)^2d\mu(z)<+\infty.$$

Let $\Gamma(z,\zeta)$ be the biharmonic Green function for the unit disk. Then we have

$$0\le\Gamma[\mu](z)=\int_{\mathbf{D}}\Gamma(z,\zeta)\,d\mu(\zeta)<+\infty,\qquad z\in\mathbf{D}.$$

Proof. Since $\Gamma(z,\zeta)>0$ and μ is a positive measure, it follows that $\Gamma[\mu]$ is nonnegative. As for the second inequality, we use Lemma 3.2 to write

$$\Gamma(z,\zeta)\le\left(1+|z|\right)^2\left(1-|\zeta|^2\right)^2,\qquad(z,\zeta)\in\mathbf{D}\times\mathbf{D}.$$

It follows that

$$\Gamma[\mu](z)=\int_{\mathbf{D}}\Gamma(z,\zeta)\,d\mu(\zeta)\le4\int_{\mathbf{D}}\left(1-|\zeta|^2\right)^2d\mu(\zeta)<+\infty.$$

The proof is complete. □

Proposition 3.4. *Let μ and $\Gamma[\mu]$ be as in Proposition* 3.3. *Define*

$$\Gamma[\mu]_r(z)=\Gamma[\mu](rz),\qquad 0\le r<1,\ z\in\mathbf{T}.$$

Then we have

$$\left\|\frac{\Gamma[\mu]_r}{1-r}\right\|_{L^1(\mathbf{T})}\to0,\qquad\text{as } r\to1.$$

Proof. By Proposition 3.3, $\Gamma[\nu]$ is well-defined and nonnegative. Thus

$$\left\|\frac{\Gamma[\mu]_r}{1-r}\right\|_{L^1(\mathbf{T})}=(1+r)\int_{\mathbf{T}}\frac{\Gamma[\mu](rz)}{1-r^2}\,d\sigma(z).$$

Hence it is enough to verify that

$$C(r)=\int_{\mathbf{T}}\frac{\Gamma[\mu](rz)}{1-r^2}\,d\sigma(z)\to 0,\qquad \text{as } r\to 1. \tag{3-7}$$

To this end, we note that

$$C(r)=\int_{\mathbf{T}}\int_{\mathbf{D}}\frac{\Gamma(rz,\zeta)}{1-r^2}\,d\mu(\zeta)\,d\sigma(z).$$

It follows from Fubini's theorem and Lemma 3.2 that

$$\begin{aligned} C(r)&\le\int_{\mathbf{D}}\int_{\mathbf{T}}\frac{\left(1-r^2|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{\left(1-r^2\right)|1-rz\bar\zeta|^2}\,d\mu(\zeta)\,d\sigma(z)\\ &=\int_{\mathbf{D}}\left(1-|\zeta|^2\right)^2\int_{\mathbf{T}}\frac{1-r^2}{|1-rz\bar\zeta|^2}\,d\sigma(z)\,d\mu(\zeta).\end{aligned}\tag{3-8}$$

To compute the boundary integral in (3-8), we write

$$\frac{1}{|1-rz\bar\zeta|^2}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}r^{m+n}(\bar\zeta z)^m(\zeta\,\bar z)^n,\qquad (z,\zeta)\in\mathbf{T}\times\mathbf{D}.$$

Using this identity, we conclude that

$$\int_{\mathbf{T}}\frac{1-r^2}{|1-rz\bar\zeta|^2}\,d\sigma(z)=\frac{1-r^2}{1-r^2|\zeta|^2},\qquad \zeta\in\mathbf{D}. \tag{3-9}$$

It follows from (3-8) and (3-9) that

$$C(r)\le\int_{\mathbf{D}}\frac{1-r^2}{1-r^2|\zeta|^2}\left(1-|\zeta|^2\right)^2 d\mu(\zeta). \tag{3-10}$$

Since the integrand in (3-10) is nonnegative and bounded from above by the μ-integrable function $(1-|\zeta|^2)^2$, we can apply the dominated convergence theorem to obtain

$$\begin{aligned} 0\le\lim_{r\to1}C(r)&\le\lim_{r\to1}\int_{\mathbf{D}}\frac{1-r^2}{1-r^2|\zeta|^2}\left(1-|\zeta|^2\right)^2 d\mu(\zeta)\\ &=\int_{\mathbf{D}}\lim_{r\to1}\frac{1-r^2}{1-r^2|\zeta|^2}\left(1-|\zeta|^2\right)^2 d\mu(\zeta)=0.\end{aligned}$$

Hence (3-7) follows, which completes the proof of the proposition. □

We are now in a position to state the main result of this paper.

Theorem 3.5. *Let u be a super-biharmonic function satisfying the conditions* (A) *and* (B). *Then there exist two unique real-valued Borel measures ν and λ on the unit circle, and a unique positive Borel measure μ on the unit disk with the property*

$$\int_{\mathbf{D}} \left(1-|\zeta|^2\right)^2 d\mu(\zeta) < +\infty,$$

such that the following representation formula holds:

$$u(z) = \int_{\mathbf{D}} \Gamma(\zeta, z)\, d\mu(\zeta) + \int_{\mathbf{T}} F(\zeta, z)\, d\nu(\zeta) + \int_{\mathbf{T}} H(\zeta, z)\, d\lambda(\zeta), \qquad z \in \mathbf{D}.$$

Here, $\Gamma(\zeta, z)$ is the biharmonic Green function for the unit disk, $H(\zeta, z)$ is the harmonic compensator given by (0-4), *and $F(\zeta, z) = -\frac{1}{2}\partial_{n(\zeta)}\Delta_\zeta \Gamma(\zeta, z)$ is given by* (0-5).

Proof. The existence of the Borel measures μ and ν was verified earlier; μ is the distributional derivative $\Delta^2 u$ on $\mathbf{D}$, and ν is the measure obtained as a weak-star limit of the measures $u_{r_j} d\sigma$ on $\mathbf{T}$ (see Proposition 2.4). Let us recall the real-valued functions $\Gamma[\mu]$ and $F[\nu]$ given by

$$\Gamma[\mu](z) = \int_{\mathbf{D}} \Gamma(\zeta, z)\, d\mu(\zeta), \qquad \text{and} \qquad F[\nu](z) = \int_{\mathbf{T}} F(\zeta, z)\, d\nu(\zeta), \qquad z \in \mathbf{D}.$$

The proof of the theorem will proceed according to the following two steps:

Step 1: We shall first verify that the function u can be represented as

$$u(z) = \Gamma[\mu](z) + F[\nu](z) + \left(1-|z|^2\right)h(z), \qquad z \in \mathbf{D},$$

where h is a harmonic function in the unit disk. To do so, we put

$$U = u - F[\nu] - \Gamma[\mu],$$

and observe that according to the assumptions (A) and (B) and Proposition 2.5 we have

$$\int_{\mathbf{T}} \left|\left(u - F[\nu]\right)(rz)\right| d\sigma(z) = O(1-r), \qquad \text{as } r \to 1.$$

This together with Proposition 3.4 implies that

$$\|U_r\|_{L^1(\mathbf{T})} = \int_{\mathbf{T}} \left|U(rz)\right| d\sigma(z) = O(1-r), \qquad \text{as } r \to 1. \tag{3-11}$$

Since U is a real-valued biharmonic function, it follows from the Almansi representation formula for biharmornic functions (see [5, Lemma 3.1], or, for a generalization, [4]) that there exist two real-valued harmonic functions g and h such that

$$U(z) = g(z) + \left(1-|z|^2\right)h(z), \qquad z \in \mathbf{D}.$$

For an integer n, we denote by $\hat f(n)$ the nth Fourier coefficient of the function $f \in L^1(\mathbf{T})$. For $0 < r < 1$, we have

$$U(rz) = g(rz) + \big(1 - r^2\big)h(rz), \qquad z \in \mathbf{T},$$

so that

$$\begin{aligned}\int_{\mathbf{T}} \bar z^n U(rz)\,d\sigma(z) &= \int_{\mathbf{T}} \bar z^n g(rz)\,d\sigma(z) + (1-r^2)\int_{\mathbf{T}} \bar z^n h(rz)\,d\sigma(z)\\ &= \hat g_r(n) + (1-r^2)\hat h_r(n).\end{aligned}$$

Since g and h are harmonic functions in the unit disk, we can represent them by Fourier series:

$$g(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat g(n) r^{|n|} e^{in\theta} \qquad \text{and} \qquad h(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat h(n) r^{|n|} e^{in\theta}.$$

It then follows that

$$\int_{\mathbf{T}} \bar z^n U(rz)\,d\sigma(z) = r^{|n|}\,\hat g(n) + (1-r^2) r^{|n|}\,\hat h(n).$$

We now let $r \to 1$ in this equality, and use (3-11) to obtain that $\hat g(n) = 0$, for every integer n. Thus, g is identically zero. This means that we have proved that there exists a harmonic function h in the unit disk such that

$$U(z) = \big(u - F[\nu] - \Gamma[\mu]\big)(z) = (1 - |z|^2)h(z), \qquad z \in \mathbf{D}, \tag{3-12}$$

as asserted.

Step 2: We want to find a representation formula for the harmonic function h in (3-12) in terms of some measure $d\lambda$ on $\mathbf{T}$. It follows from (3-11) and (3-12) that

$$\sup_{0 \le r < 1} \int_{\mathbf{T}} \big|h(rz)\big|\,d\sigma(z) \le C < +\infty,$$

so that there exists a unique real-valued Borel measure $d\lambda$ on the unit circle such that h is its Poisson integral [8, Theorem 11.30]:

$$h(z) = P[\lambda](z) = \int_{\mathbf{T}} \frac{1 - |z|^2}{|1 - \bar\zeta z|^2}\,d\lambda(\zeta), \qquad z \in \mathbf{D}. \tag{3-13}$$

The measure $d\lambda$ is the weak-star limit of the measures $d\lambda_r = h_r d\sigma$, as $r \to 1$. It follows from (3-12) and (3-13) that

$$U(z) = \big(1 - |z|^2\big)h(z) = \int_{\mathbf{T}} \frac{\big(1 - |z|^2\big)^2}{|1 - \bar\zeta z|^2}\,d\lambda(\zeta) = \int_{\mathbf{T}} H(\zeta, z)\,d\lambda(\zeta), \qquad z \in \mathbf{D}.$$

Rewriting this identity, we obtain

$$u(z) = \Gamma[\mu](z) + F[\nu](z) + \int_{\mathbf{T}} H(\zeta, z)\,d\lambda(\zeta), \qquad z \in \mathbf{D},$$

which is the desired representation formula for u. The proof is complete. □

Corollary 3.6. *Let u be a super-biharmonic function satisfying the conditions*

$$\lim_{r \to 1} \int_{\mathbf{T}} u^+(rz)\, d\sigma(z) = 0,$$

and

$$\int_{\mathbf{T}} u^-(rz)\, d\sigma(z) = O(1-r), \qquad \text{as } r \to 1.$$

Then there exist a unique positive Borel measure μ on the unit disk and a unique finite real-valued Borel measure λ on the unit circle such that

$$u(z) = \int_{\mathbf{D}} \Gamma(\zeta, z)\, d\mu(\zeta) + \int_{\mathbf{T}} H(\zeta, z)\, d\lambda(\zeta), \qquad z \in \mathbf{D}.$$

The measure μ is the distributional derivative $\Delta^2 u$, and the measure λ is the limit of the measures

$$\frac{u(rz)}{1-r^2}\, d\sigma(z), \qquad z \in \mathbf{D},$$

as $r \to 1$, in the weak-star topology of Borel measures on the unit circle. In particular, if u is positive, then λ is positive.

Proof. Since $|u| = u^+ + u^-$, it follows from the assumptions that

$$\lim_{r \to 1} \int_{\mathbf{T}} \big|u(rz)\big|\, d\sigma(z) = 0.$$

In other words, $u_r\, d\sigma \to 0$ in the norm topology of Borel measures on the unit circle, so that the measure ν appearing in the preceding theorem is identically zero. The corollary is now an immediate consequence of Theorem 3.5. □

Corollary 3.7. *Let u be a nonnegative super-biharmonic function satisfying the condition*

$$\lim_{r \to 1} \int_{\mathbf{T}} u(rz)\, d\sigma(z) = 0.$$

Then there exist a unique positive Borel measure μ on the unit disk and a unique positive Borel measure λ on the unit circle such that

$$u(z) = \int_{\mathbf{D}} \Gamma(\zeta, z)\, d\mu(\zeta) + \int_{\mathbf{T}} H(\zeta, z)\, d\lambda(\zeta), \qquad z \in \mathbf{D}.$$

The measure μ is the distributional derivative $\Delta^2 u$, and the measure λ is the limit of the measures

$$\frac{u(rz)}{1-r^2}\, d\sigma(z), \qquad z \in \mathbf{D},$$

as $r \to 1$, in the weak-star topology of Borel measures on the unit circle.

Proof. We note that $u^+ = u$ and $u^- = 0$, so that u fulfills the conditions of the preceding corollary. □

4. Final comments

We should mention that we feel it is possible to extend the results of the present paper to higher dimensional $\mathbf{R}^n$, $n = 3, 4, 5, \ldots$, with the appropriate modifications. A complication that appears is that the positivity of the biharmonic Poisson kernel (denoted by $F(z,\zeta)$ in this paper) depends on the dimension; see [2]. However, there is a combination of the harmonic compensator and the biharmonic Poisson kernel that is positive always [2], and this should save the situation.

We also mention that it should be possible to obtain results concerning higher powers of the Laplacian. However, in this case, more boundary data are required, and the conditions to ensure a Riesz–Herglotz formula will definitely be considerably more complicated.

A final comment is that we consider the Green function for Δ^2 with Dirichlet boundary conditions primarily because of its well-known connection with the theory of the Bergman spaces [1], [6]. It may be possible to obtain some kind of Riesz–Herglotz representation also with some other set of boundary conditions, like the value and second normal derivative.

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