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# REMOVABLE SINGULARITIES FOR $H^p$ SPACES OF ANALYTIC FUNCTIONS, $0<p<1$

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*Dedicated to Lars Inge Hedberg on the occasion of his 65th birthday.*

**Abstract.** In this paper we study removable singularities for Hardy $H^p$ spaces of analytic functions on general domains, mainly for $0<p<1$. For each $p<1$ we prove that there is a self-similar linear Cantor set with Hausdorff dimension greater than $0.4p$ removable for $H^p$, thereby obtaining the first removable sets with positive Hausdorff dimension for $0<p<1$. (Cf. the author's older result that a set $E$ removable for $H^p$, $0<p<1$, must satisfy $\dim E \le p$.) We use this to extend some results earlier proved for $1\le p<\infty$ to $0<p<\infty$ or $\frac{1}{2}\le p<\infty$.

## 1. Introduction

Let $\mathbf{S}=\mathbf{C}\cup\{\infty\}$ be the Riemann sphere and $A(\Omega)=\{f : f \text{ is analytic in } \Omega\}$.

**Definition 1.1.** For $0<p<\infty$ and a domain $\Omega\subset\mathbf{S}$ let

$$\begin{aligned} H^p(\Omega) &= \{f\in A(\Omega) : |f|^p \le u \text{ in } \Omega \text{ for some harmonic function } u\},\\ H^\infty(\Omega) &= \{f\in A(\Omega) : \sup_{z\in\Omega}|f(z)|<\infty\}. \end{aligned}$$

Let further, $\|f\|_{H^p(\Omega)}=u(a)^{1/p}$ for $f\in H^p(\Omega)$, where $u$ is the (pointwise) least harmonic majorant of $|f|^p$ in $\Omega$, and $a\in\Omega$ is the *norming-point.* Let also $\|f\|_{H^\infty(\Omega)}=\sup_{z\in\Omega}|f(z)|$.

For $1\le p\le\infty$, $H^p(\Omega)$ is a Banach space, for $0<p<1$, $H^p(\Omega)$ is a quasi-Banach space.

**Definition 1.2.** Let $0<p\le\infty$, $\Omega\subset\mathbf{S}$ be a domain and $E\subset\Omega$ be relatively closed in $\Omega$ such that $\Omega\setminus E$ is also a domain. Then $E$ is *weakly removable* for $H^p(\Omega\setminus E)$ if $H^p(\Omega\setminus E)\subset A(\Omega)$. Further, $E$ is *strongly removable* for $H^p(\Omega\setminus E)$ if $H^p(\Omega\setminus E)=H^p(\Omega)$ (as sets).

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It is clear that if $E$ is strongly removable for $H^p(\Omega \setminus E)$, then it is also weakly removable for $H^p(\Omega \setminus E)$. If $E$ is compact more is true. Theorem 4.5 in Björn [4], essentially due to Hejhal, shows that if $K$ is compact and weakly removable for one domain $\Omega \supset K$, then $K$ is strongly removable for all domains $\Omega \supset K$. Hence, for compact $K$ we just say that $K$ is removable for $H^p$, without specifying weak/strong or the domain.

For $H^\infty$ it is easy to see that weak and strong removability is the same for all sets, not only for compact sets. Removable singularities for $H^\infty$ have been studied by many people for more than a century, we will only use it for comparison.

We want to recall that, if $E$ is weakly removable for $H^p(\Omega \setminus E)$ and $q > p$, then $H^q(\Omega \setminus E) \subset H^p(\Omega \setminus E) \subset A(\Omega)$, and hence, $E$ is weakly removable for $H^q(\Omega \setminus E)$.

In Björn [1], [2], [4] weakly removable sets were called "I-removable" or "removable in the first sense", and strongly removable sets were called "II-removable" or "removable in the second sense". In the earlier literature the two types of removability were not treated simultaneously. However, most of the earlier papers dealt only with the compact case.

The following characterization for weakly removable sets was given in Björn [4].

**Theorem 1.3.** *Let $0 < p \leq \infty$. Let $\Omega \subset \mathbf{S}$ be a domain and $E \subset \Omega$ be such that $\Omega \setminus E$ is also a domain. Then $E$ is weakly removable for $H^p(\Omega \setminus E)$ if and only if $E$ can be written as a countable union of well-separated compact sets $K_1$, $K_2, \dots$, removable for $H^p$, where by well-separated we mean that $\operatorname{dist}\bigl(K_k, \bigcup_{j=1, j\neq k}^\infty K_j\bigr) > 0$ for $k = 1, 2, \dots$.*

Since it is well known that weakly removable sets are totally disconnected (it follows from the same property for $p = \infty$), the condition of well-separatedness on the compact sets $K_k$ can be equivalently stated by saying that $K_k$ should be pairwise disjoint compact sets such that the sets $\bigcup_{j=k}^\infty K_j$ are relatively closed subsets of $\Omega$ for $k = 1, 2, \dots$. The theorem was stated and proved in this form as Theorem 4.10 in Björn [4].

This theorem shows that a set $E$ weakly removable for $H^p(\Omega \setminus E)$ for some domain $\Omega \supset E$ with $\Omega \setminus E$ being a domain, is weakly removable for all domains $\Omega \supset E$, with $\Omega \setminus E$ being a domain. Thus we usually just say that a set $E$ is weakly removable for $H^p$ without specifying the domain.

The first one to study removable singularities for $H^p$ spaces, $p < \infty$, was Parreau [19], [20] in the early 1950s, and soon afterwards Rudin [21]. In the late 1960s Heins [11] and Yamashita [23], [24] wrote some papers. In the 1970s important contributions were made by Hejhal [12], [13], Kobayashi [15], [16] and Hasumi [9].

Removability is a conformal invariant. The major motivation at this time was to find out if different $p$ give different removable singularities, and hence different conformal invariants.

Hejhal [12], [13] made important partial solutions to this problem. Kobayashi [15], [16] refined Hejhal's methods finally obtaining that, if $0 < p < q \leq \infty$ and $q \geq 1$, then there is a compact set removable for $H^q$ but not removable for $H^p$. At about the same time as Kobayashi obtained his result, Hasumi [9] obtained the same result without the restriction $q \geq 1$ in an even more general form; we refer to Hasumi [9] or Björn [4] for the exact formulation. Hasumi used a method completely different from the method used by Hejhal and Kobayashi.

In the 1980s Järvi [14] wrote a paper on the higher dimensional case, Conway–Dudziak–Straube [7] wrote a paper on the related problem of isometrically removable singularities including the higher dimensional case and Øksendal [18] wrote a paper on singularities lying on curves. After that only the author seems to have studied the problem.

We refer to Björn [4] for a more detailed survey.

Most of the above papers and results were only obtained for $p \geq 1$, which is not the major concern of this paper.

Relatively closed subsets of zero logarithmic capacity are always strongly removable for all $H^p$, $0 < p \leq \infty$, see Conway–Dudziak–Straube [7]. The above mentioned result by Hasumi is another of the few results available for $p < 1$. It is easy to see (by looking at the construction) that all the sets Hasumi constructs have zero Hausdorff dimension. Of course, we can use Theorem 1.3 to obtain more removable sets, but still all of these sets have zero Hausdorff dimension.

Hasumi's examples show that there are zero-dimensional compact sets not removable for any $H^p$, $0 < p < \infty$. This is in great contrast to Painlevé's theorem that all sets with zero one-dimensional Hausdorff measure are removable for $H^\infty$.

Kobayashi showed that there are zero-dimensional compact subsets of $\mathbf{R}$ not removable for any $H^p$, $0 < p < 1$. This is obtained by the proof of Lemma 2 in Kobayashi [16], even if it is not mentioned explicitly in the paper. Again, this is in great contrast to the result that any subset of $\mathbf{R}$ with zero one-dimensional Hausdorff measure is removable for $H^1$, see Björn [5] for a historical account of this and related results.

Theorem 4.17 in Björn [4] shows that every (weakly) removable set $E$ for $H^p$, $0 < p \leq 1$, has $\dim E \leq p$. The result was stated in terms of Riesz capacities, we refer to Björn [4] for the exact statement.

This is essentially all that is known about removable singularities for $H^p$, $0 < p < 1$, before this paper. The main result of this paper is the following theorem, which will be proved in Section 2.

**Theorem 1.4.** *Let $0 < p < 1$. Then there exists $\alpha$ such that the linear self-similar Cantor set $C_\alpha$, defined in Definition 2.1, is removable for $H^p$ and $\dim C_\alpha > 0.4p$. If $p \leq 0.65$, then $\alpha$ can be chosen so that $\dim C_\alpha > 0.5p$, and if $p \leq 0.4$, $\alpha$ can be chosen so that $\dim C_\alpha > 0.6p$.*

The method used in Section 2 is partially similar to the method that was used in Björn [1, Section 8], to obtain removability results for planar Cantor sets for $H^p$, $1 \le p < \infty$. More precisely, Proposition 2.2 and Lemmas 2.5 and 2.7 are modifications of results in Björn [1, Section 8], to linear Cantor sets.

In Section 3 we use Theorem 1.4 to extend several results given in Björn [4] for $1 \le p < \infty$, to $0 < p < \infty$ or $\frac{1}{2} \le p < \infty$. In Section 4 we draw some conclusions and make three conjectures.

We conclude this section with some notation that we will need. Let $D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$, $\mathbf{D} = D(0,1)$ and $\delta D(z_0, r) = D(z_0, \delta r)$. Let further dim denote the Hausdorff dimension and $\omega(E; \Omega, z_0)$ denote the harmonic measure of the set $E \subset \partial\Omega$ for the domain $\Omega$ evaluated at $z_0 \in \overline{\Omega}$.

By a domain we mean a non-empty open connected set. Because of the uniqueness theorem for analytic functions we will not distinguish between restrictions and extensions of analytic functions.

## 2. Removability for Cantor sets

Henry Smith [22, Sections 15–16], used some Cantor-type sets in 1875. It was not until 1883 that Georg Cantor [6, p. 590] (p. 407 in Acta Math.), used them. Despite this we will join the mathematical tradition and call these sets Cantor sets.

**Definition 2.1.** Let $0 < \alpha < \frac{1}{2}$ be fixed. Let $C^{(0)} = [0,1]$ and let $C^{(1)}$ be the set remaining after having removed the open middle part of length $1 - 2\alpha$ from the interval. Continue in this way by letting $C^{(m+1)}$ be the set remaining after having removed the open middle parts of length $(1 - 2\alpha)\alpha^m$ from each of the intervals in $C^{(m)}$. The set $C^{(m)}$ consists of $2^m$ disjoint closed intervals of length $\alpha^m$, call them $I_j^{(m)}$, $j = 1, \dots, 2^m$, numbered from the left. Finally let $C_\alpha = \bigcap_{m=0}^\infty C^{(m)}$.

Then $C_\alpha$ is a (linear self-similar) *Cantor set* in $\mathbf{R} \subset \mathbf{C}$.

The set $C^{(m)}$ is composed of intervals. For us it will be advantageous to work with discs rather than intervals. For this reason we here give an alternative characterization of $C_\alpha$.

**Proposition 2.2.** *Let $0 < \alpha < \frac{1}{2}$. Let $\overline{D}_j^{(m)}$ be the smallest closed disc containing $I_j^{(m)}$, thus having diameter $\alpha^m$. Then*

$$C_\alpha = \bigcap_{m=0}^{\infty} \bigcup_{j=1}^{2^m} \overline{D}_j^{(m)}.$$

*Proof.* It is immediate that

$$C_\alpha = \bigcap_{m=0}^{\infty} \bigcup_{j=1}^{2^m} I_j^{(m)} \subset \bigcap_{m=0}^{\infty} \bigcup_{j=1}^{2^m} \overline{D}_j^{(m)}.$$

For the converse, let $z \in \bigcap_{m=0}^{\infty} \bigcup_{j=1}^{2^m} \overline{D}_j^{(m)}$. For each $m$ there exists $j_m$ such that $z \in \overline{D}_{j_m}^{(m)}$ and thus

$$\operatorname{dist}(z, C^{(m)}) \leq \operatorname{dist}(z, I_{j_m}^{(m)}) \leq \tfrac{1}{2}\alpha^m \to 0, \quad \text{as } m \to \infty.$$

Hence $\operatorname{dist}(z, C_\alpha) = 0$ and since $C_\alpha$ is closed we have $z \in C_\alpha$. □

**Lemma 2.3.** *Let $C_\alpha$ be as above, then $\dim C_\alpha = -\log 2/\log \alpha$.*

A proof of this fact can be found in many places, e.g. Section 4.10 in Mattila [17].

**2.1. Removability results.** Our main tool is the following result.

**Proposition 2.4.** *Let $0<\alpha<\frac{1}{2}$ be fixed and let $\overline{D}_j^{(m)}$ be as defined in Proposition 2.2. Let $\delta>1$ be so small that the discs $\{\delta^2 \overline{D}_j^{(m)}\}_{j=1}^{2^m}$ are disjoint. Let $E_j^{(m)} = \delta \overline{D}_j^{(m)}$,*

$$\Omega_m = \mathbf{S} \setminus \bigcup_{j=1}^{2^m} E_j^{(m)} \quad \text{and} \quad \omega_m = \omega(\,\cdot\,; \Omega_m, \infty).$$

*Assume that there is $M>0$ such that*

$$\omega_m(\partial E_j^{(m)}) \geq M^m \quad \text{for all } m \geq 0,\ 1 \leq j \leq 2^m.$$

*(It follows that $M \leq \frac{1}{2}$.) If*

$$p > \frac{\log M}{\log \alpha},$$

*then $C_\alpha$ is removable for $H^p$.*

*Proof.* For $p \geq 1$ it follows from, e.g., Theorem 4.1 in Björn [5] that $C_\alpha$ is always removable for $H^p$ without any further condition on $p$. We restrict ourselves to the case $p<1$, and hence $M>\alpha$, for the rest of this proof.

Let $f \in H^p(\mathbf{S}\setminus C_\alpha)$ be arbitrary with $f(\infty)=0$ and $\|f\|_{H^p(\mathbf{S}\setminus C_\alpha)} \leq 1$, where we use the norming-point $\infty$. We are going to prove that $f'(\infty) := \lim_{z\to\infty} z\bigl(f(z)-f(\infty)\bigr) = 0$ from which it follows, using Proposition 5.1 in Björn [4], that $C_\alpha$ is removable for $H^p$.

Let $U$ be the least harmonic majorant of $|f|^p$ in $\mathbf{S}\setminus C_\alpha$. Let also $u_j^{(m)} = \min\{U(z) : z \in \partial E_j^{(m)}\}$. Using Harnack's inequality in the annulus $\delta^2 D_j^{(m)} \setminus \overline{D}_j^{(m)}$, we see that there exists a constant $C$ independent of $j$ and $m$ such that

$$|f(z)|^p \leq U(z) \leq C u_j^{(m)} \quad \text{for all } m \geq 0,\ 1 \leq j \leq 2^m \text{ and } z \in \partial E_j^{(m)}.$$

Using this and Cauchy's theorem we obtain

$$(1)\qquad \begin{aligned} |f'(\infty)| &= \frac{1}{2\pi}\left|\int_{\partial\Omega_m} f(z)\,dz\right| \le \frac{1}{2\pi}\sum_{j=1}^{2^m}\int_{\partial E_j^{(m)}} |f(z)|\,ds \\ &\le C_1\sum_{j=1}^{2^m}\int_{\partial E_j^{(m)}} (u_j^{(m)})^{1/p}\,ds = C_2\alpha^m\sum_{j=1}^{2^m}(u_j^{(m)})^{1/p}, \end{aligned}$$

where $C_1$ and $C_2$ are constants independent of $m$, and $ds$ denotes arc length. From the quasi-norm condition on $f$ we see that

$$1 \ge U(\infty) = \int_{\partial\Omega_m} U(z)\omega_m(dz) \ge \sum_{j=1}^{2^m} u_j^{(m)}\omega_m(\partial E_j^{(m)}).$$

Now we will need the assumption that $p<1$. We get

$$\sum_{j=1}^{2^m}(u_j^{(m)})^{1/p} \le \left(\sum_{j=1}^{2^m} u_j^{(m)}\right)^{1/p} \le \frac{\left(\sum_{j=1}^{2^m} u_j^{(m)}\omega_m(\partial E_j^{(m)})\right)^{1/p}}{\min_{1\le j\le 2^m}\omega_m(\partial E_j^{(m)})^{1/p}} \le \frac{1}{M^{m/p}}.$$

Combining this with (1) gives us that

$$|f'(\infty)| \le C_2\left(\frac{\alpha}{M^{1/p}}\right)^m.$$

The condition on $p$ in the statement of the theorem is equivalent to $\alpha/M^{1/p}<1$, and thus the right-hand side tends to $0$, as $m\to\infty$, and $f'(\infty)=0$. □

Next we have a lemma that enables us to find a suitable value for the constant $M$ in Proposition 2.4.

**Lemma 2.5.** *Let $0<\alpha<\frac{1}{2}$ be fixed and let $E_j^{(m)}$, $\Omega_m$, $\omega_m$ and $\delta>1$ be as in Proposition 2.4. Let*

$$\begin{aligned} D &= D(0,r), \quad r = \frac{\alpha\delta}{2-3\alpha\delta}, \\ D_j &= D\left(\frac{(-1)^j(\alpha-\alpha^2)}{2-3\alpha\delta}, r_0\right), \quad r_0 = \alpha r, \\ u(z) &= \omega(\partial D;\mathbf{D}\setminus\overline{D},z) = \frac{\log|z|}{\log r}, \\ u_0(z) &= \omega\big(\partial D_0;\mathbf{D}\setminus(\overline{D}_0\cup\overline{D}_1),z\big) \end{aligned}$$

*for $z \in \mathbf{D} \setminus \overline{D}$ and $j=0,1$. Let $\tilde{r}$ satisfy $r<\tilde{r}<1$ and assume that*

$$u_0(z) \geq Mu(z) \quad \text{for } |z|=\tilde{r}.$$

*Then*

$$\omega_{m+1}(\partial E_{2j-k}^{(m+1)}) \geq M\omega_m(\partial E_j^{(m)}) \quad \text{for all } m \geq 0,\ 1 \leq j \leq 2^m \text{ and } k=0,1.$$

*Proof.* Let $\Omega$ be any domain containing $\mathbf{D} \cup \{\infty\}$ and let $D' = D(0,\tilde{r})$, $\Gamma = \partial D'$ and

$$\mu(\,\cdot\,;z) = \int_{\partial \mathbf{D}} \omega(\,\cdot\,;\Omega \setminus \overline{D}', \zeta)\omega(d\zeta; \mathbf{D} \setminus \overline{D}, z) \quad \text{for } z \in \mathbf{D} \setminus \overline{D}.$$

Using the definition, harmonicity and boundary values of the harmonic measure we get, for $z_0 \in \mathbf{D} \setminus \overline{D}$,

$$\begin{aligned}
\omega(\partial D;\Omega\setminus\overline{D}, z_0) &= \int_{\partial(\mathbf{D}\setminus\overline{D})} \omega(\partial D;\Omega\setminus\overline{D},\zeta)\omega(d\zeta;\mathbf{D}\setminus\overline{D},z_0)\\
&= \int_{\partial D} \omega(d\zeta;\mathbf{D}\setminus\overline{D},z_0)\\
&\quad + \int_{\partial\mathbf{D}}\int_{\partial(\Omega\setminus\overline{D}')} \omega(\partial D;\Omega\setminus\overline{D},z_1)\omega(dz_1;\Omega\setminus\overline{D}',\zeta)\omega(d\zeta;\mathbf{D}\setminus\overline{D},z_0)\\
&= u(z_0) + \int_{\partial\mathbf{D}}\int_{\Gamma} \omega(\partial D;\Omega\setminus\overline{D},z_1)\omega(dz_1;\Omega\setminus\overline{D}',\zeta)\omega(d\zeta;\mathbf{D}\setminus\overline{D},z_0)\\
&= u(z_0) + \int_{\Gamma} \omega(\partial D;\Omega\setminus\overline{D},z_1)\mu(dz_1;z_0).
\end{aligned}$$

By repeated use of this formula we get, for $z_0 \in \mathbf{D} \setminus \overline{D}$,

$$\begin{aligned}
\omega(\partial D;\Omega \setminus \overline{D}, z_0) &= u(z_0) + \int_{\Gamma} \omega(\partial D;\Omega \setminus \overline{D}, z_1)\mu(dz_1;z_0)\\
&= u(z_0) + \int_{\Gamma} u(z_1)\mu(dz_1;z_0)\\
&\quad + \int_{\Gamma}\int_{\Gamma} \omega(\partial D;\Omega \setminus \overline{D}, z_2)\mu(dz_2;z_1)\mu(dz_1;z_0)\\
&= \cdots = u(z_0) + \sum_{k=1}^{\infty} \int_{\Gamma} \cdots \int_{\Gamma} u(z_k)\mu(dz_k;z_{k-1})\cdots\mu(dz_1;z_0).
\end{aligned} \tag{2}$$

As $\Gamma \subset \mathbf{D} \setminus \overline{D}$ is compact there exists $C$ such that

$$\omega(\partial\mathbf{D};\mathbf{D} \setminus \overline{D}, z) \leq C < 1 \quad \text{for } |z|=\tilde{r}.$$

It follows that

$$\mu(\Gamma; z) \le C < 1 \quad \text{for } |z| = \tilde{r}.$$

This justifies the last equality in (2) and shows that the infinite sum converges.

Let now

$$\mu_0(\,\cdot\,; z) = \int_{\partial \mathbf{D}} \omega(\,\cdot\,; \Omega \setminus \overline{D}', \zeta)\omega(d\zeta; \mathbf{D} \setminus (\overline{D}_0 \cup \overline{D}_1), z) \quad \text{for } z \in \mathbf{D} \setminus \overline{D}.$$

Carleman's principle gives directly that

$$\mu_0(\,\cdot\,; z) \ge \mu(\,\cdot\,; z) \quad \text{for } z \in \mathbf{D} \setminus \overline{D}.$$

Hence, for $|z_0| = \tilde{r}$,

$$\begin{aligned}
\omega\bigl(\partial D_0; \Omega \setminus (\overline{D}_0 \cup \overline{D}_1), z_0\bigr) &= u_0(z_0) \\
&\quad + \sum_{k=1}^{\infty} \int_{\Gamma} \cdots \int_{\Gamma} u_0(z_k)\mu_0(dz_k; z_{k-1}) \cdots \mu_0(dz_1; z_0) \\
&\ge Mu(z_0) \\
&\quad + \sum_{k=1}^{\infty} \int_{\Gamma} \cdots \int_{\Gamma} Mu(z_k)\mu(dz_k; z_{k-1}) \cdots \mu(dz_1; z_0) \\
&= M\omega(\partial D; \Omega \setminus \overline{D}, z_0),
\end{aligned}$$

where the first equality is obtained in exactly the same way as (2) above. Thus

$$\begin{aligned}
\omega\bigl(\partial D_0; \Omega \setminus (\overline{D}_0 \cup \overline{D}_1), \infty\bigr) &= \int_{\Gamma} \omega\bigl(\partial D_0; \Omega \setminus (\overline{D}_0 \cup \overline{D}_1), z\bigr)\omega(dz; \Omega \setminus \overline{D}', \infty) \\
&\ge \int_{\Gamma} M\omega(\partial D; \Omega \setminus \overline{D}, z)\omega(dz; \Omega \setminus \overline{D}', \infty) \\
&= M\omega(\partial D; \Omega \setminus \overline{D}, \infty).
\end{aligned}$$

And by properly rescaling and rotating the above situation we get

$$\begin{aligned}
\omega_{m+1}(\partial E_{2j-k}^{(m+1)}) = \omega(\partial E_{2j-k}^{(m+1)}; \Omega_{m+1}, \infty) &\ge M\omega(\partial E_j^{(m)}; \Omega_{m+1} \setminus E_j^{(m)}, \infty) \\
&\ge M\omega(\partial E_j^{(m)}; \Omega_m, \infty) = M\omega_m(\partial E_j^{(m)})
\end{aligned}$$

for $m \ge 0$, $1 \le j \le 2^m$ and $k = 0, 1$. The second inequality follows from Carleman's principle. □

We want to determine $M$ so that we can use Lemma 2.5. In order to do that we need to estimate how small $u_0(z)$ gets for $|z| = \tilde{r}$.

**Lemma 2.6.** *Let $u_0$ and $\tilde r$ be as in Lemma 2.5. Then*

$$\min_{|z|=\tilde r} u_0(z) = u_0(-\tilde r).$$

*Proof.* Let $0 \le \beta \le \pi$ and let $\varphi(re^{i\theta}) = re^{i(2\beta-\theta)}$ be the orthogonal reflection in the line $\{z \in \mathbf{C} : \arg z = \beta\}$. Let $v = u_0 \circ \varphi$ and

$$\Omega = \{z \in \mathbf{D} : \beta < \arg z < \beta + \pi\} \setminus \big(\varphi(\overline{D}_0) \cup \overline{D}_1\big).$$

Both $u_0$ and $v$ are defined and harmonic in $\Omega$. For $z \in \partial\varphi(D_0) \cap \partial\Omega$ we have $u_0(z) \le 1 = v(z)$. For $z \in \partial D_1 \cap \partial\Omega$ we have $u_0(z) = 0 \le v(z)$. On the rest of $\partial\Omega$, $u_0$ and $v$ have the same boundary values. Hence, by the maximum principle, $u_0(z) \le v(z)$ for all $z \in \Omega$. In particular, $u_0(-\tilde r) \le v(-\tilde r) = u_0\big(\tilde r e^{i(2\beta-\pi)}\big)$. Since $\beta \in [0,\pi]$ was arbitrary the result follows. □

We also want to choose $\tilde r$ optimally. We do not need to prove that the choice we will make is optimal, but as it is very easy to do we prove optimality.

**Lemma 2.7.** *Let $u$, $u_0$ and $r$ be as in Lemma 2.5. Then the function $M$ defined by*

$$M(\tilde r) = \min_{|z|=\tilde r} \frac{u_0(z)}{u(z)} \quad \text{for } r < \tilde r < 1$$

*is non-decreasing.*

*Proof.* Let $r < r_1 < r_2 < 1$, $D' = D(0, r_1)$ and $|z| = r_2$. Then

$$u_0(z) = \int_{\partial D'} u_0(\zeta)\omega(d\zeta; \mathbf{D}\setminus\overline{D}', z) \ge \int_{\partial D'} M(r_1)u(\zeta)\omega(d\zeta; \mathbf{D}\setminus\overline{D}', z) = M(r_1)u(z). \ \square$$

We are now ready to combine Proposition 2.4 with Lemmas 2.5 and 2.6 using the optimal choice of $\tilde r$ we found in Lemma 2.7.

**Theorem 2.8.** *Let $0 < \alpha < \frac{1}{2}$ be fixed, and let $u^{(\delta)}$ and $u_0^{(\delta)}$ be defined as $u$ and $u_0$ in Lemma 2.5 for a given $\delta \ge 1$. Let*

$$\widetilde{M} = \frac{{u_0^{(1)}}'(-1)}{{u^{(1)}}'(-1)},$$

*where the derivatives should be understood as the partial derivatives along the $\mathbf{R}$-axis. If*

$$p > \frac{\log \widetilde{M}}{\log \alpha},$$

*then $C_\alpha$ is removable for $H^p$.*

*Proof.* It follows directly from the expression for $u$ given in the statement of Lemma 2.5 that ${u^{(\delta)}}'(-1)$ is non-zero and depends continuously on $\delta$. It is not hard to argue that also ${u_0^{(\delta)}}'(-1)$ is non-zero and depends continuously on $\delta$. Using l'Hopital's rule, Lemma 2.6 and the fact that $u$ is constant on circles with centre at the origin, we obtain

$$\widetilde{M} = \lim_{\delta\to 1+} \frac{{u_0^{(\delta)}}'(-1)}{{u^{(\delta)}}'(-1)} = \lim_{\delta\to 1+} \lim_{\tilde{r}\to 1-} \frac{u_0^{(\delta)}(-\tilde{r})}{u^{(\delta)}(-\tilde{r})} = \lim_{\delta\to 1+} \lim_{\tilde{r}\to 1-} \min_{|z|=\tilde{r}} \frac{u_0^{(\delta)}(z)}{u^{(\delta)}(z)}.$$

Thus we can find $M>0$, $\delta>1$ and $\tilde{r}$ with $r<\tilde{r}<1$ such that

$$u_0^{(\delta)}(z) \geq M u^{(\delta)}(z) \text{ for } |z|=\tilde{r} \quad \text{and} \quad p > \frac{\log M}{\log \alpha}.$$

It now follows from Proposition 2.4 and Lemma 2.5 that $C_\alpha$ is removable for $H^p$. □

**2.2. Estimating** $u_0$. We are able to calculate ${u^{(1)}}'(-1)$ exactly, but for ${u_0^{(1)}}'(-1)$ we have to use estimates. We need an underestimate for $\widetilde{M}$ and hence an underestimate for ${u_0^{(1)}}'(-1)$. We will use the following lemma.

**Lemma 2.9.** *Let $0<\alpha<\frac{1}{2}$ be fixed and let $r$, $r_0$, $D_0$ and $u_0$ be as defined in Lemma 2.5 for $\delta=1$. Let also*

$$\begin{aligned}
v(z) &= \omega(\partial D_0; \mathbf{D}\setminus \overline{D}_0, z) \quad \textit{for } z\in \mathbf{D}\setminus\overline{D}_0,\\
a &= r-2r_0 = \frac{\alpha(1-2\alpha)}{2-3\alpha},\\
b &= \frac{1+ar-\sqrt{(1-a^2)(1-r^2)}}{a+r},\\
A &= v(-a),\\
B &= v(-r),\\
\varphi(z) &= \frac{z-b}{1-bz} \quad \textit{for } z\in\mathbf{D}.
\end{aligned}$$

*Then*

$$u_0'(-1) > \frac{A(1+b)^2-(1-b)^2}{(1-b^2)(1-AB)\log\varphi(r)}.$$

*Proof.* We have $D_0\cap\mathbf{R}=(a,r)$ and it can be easily checked that $\varphi(a) = -\varphi(r)$, hence $\varphi(D_0)=D(0,\varphi(r))$ and $b$ is the hyperbolic centre of $D_0$.

Since $\varphi$ maps $\mathbf{D}\setminus\overline{D}_0$ conformally onto $\mathbf{D}\setminus\overline{D}(0,\varphi(r))$, we have

$$v(z) = \frac{\log|\varphi(z)|}{\log\varphi(r)}.$$

Let $D_1$ be as in Lemma 2.5 and

$$\tilde{v}(z) = \frac{v(z) - Av(-z)}{1 - AB} \quad \text{for } z \in \mathbf{D} \setminus (\overline{D}_0 \cup \overline{D}_1).$$

The constants $A$ and $B$ were chosen so that $B \le v(z) \le A$ for $z \in \overline{D}_1$. It follows that $\tilde{v}(z) \le 1$ for $z \in \partial D_0$, $\tilde{v}(z) \le 0$ for $z \in \partial D_1$, $\tilde{v}(z) = 0$ for $z \in \partial\mathbf{D}$ and as $\tilde{v}$ is harmonic in $\mathbf{D} \setminus (\overline{D}_0 \cup \overline{D}_1)$, we obtain

$$\tilde{v}(z) \le u_0(z) \quad \text{for } z \in \mathbf{D} \setminus (\overline{D}_0 \cup \overline{D}_1).$$

Since $\tilde{v}(a) < 1 = u_0(a)$, $\tilde{v} \not\equiv u_0$. Let $r < \tilde{r} < 1$. By the maximum principle there exists $C < 1$ such that $\tilde{v}(z) \le Cu_0(z)$ for $|z| = \tilde{r}$. It follows, again by the maximum principle, that the inequality holds for $\tilde{r} \le |z| < 1$. Since $u_0(-1) = \tilde{v}(-1) = 0$, we get

$$u_0'(-1) > \tilde{v}'(-1) = \frac{v'(-1) + Av'(1)}{1 - AB}.$$

Using that

$$v'(-1) = -\frac{\varphi'(-1)}{\log\varphi(r)} = -\frac{1 - b^2}{(1+b)^2 \log\varphi(r)} = -\frac{(1-b)^2}{(1-b^2)\log\varphi(r)}$$

and

$$v'(1) = \frac{\varphi'(1)}{\log\varphi(r)} = \frac{1 - b^2}{(1-b)^2 \log\varphi(r)} = \frac{(1+b)^2}{(1-b^2)\log\varphi(r)}$$

the result follows. □

**Corollary 2.10.** *Let $0 < \alpha < \frac{1}{2}$ be fixed, and let $A$, $B$, $b$, $r$ and $\varphi$ be as given in Lemma* 2.9. *Let also*

$$M' = \frac{\big((1-b)^2 - A(1+b)^2\big)\log r}{(1-b^2)(1-AB)\log\varphi(r)}.$$

*If*

$$p \ge \frac{\log M'}{\log\alpha},$$

*then $C_\alpha$ is removable for $H^p$.*

*Proof.* Let $u$ be as given in Lemma 2.5 for $\delta = 1$, then $u'(-1) = -1/\log r$. The corollary thus follows from Theorem 2.8 and Lemma 2.9. □

**2.3. Explicit values for removability.** We are now ready to prove Theorem 1.4.

*Proof.* The following table lists, for various values of $\alpha$, an underestimate for $\dim C_\alpha$ and an overestimate for $p$ given by Corollary 2.10 such that $C_\alpha$ is removable for $H^p$, and hence for all larger $p$.

| $\alpha$ | $\dim C_\alpha$ | $p$ |
|---|---|---|
| 0.0010 | 0.1002 | 0.1505 |
| 0.0020 | 0.1114 | 0.1667 |
| 0.0038 | 0.1243 | 0.1855 |
| 0.0067 | 0.1384 | 0.2067 |
| 0.0110 | 0.1536 | 0.2305 |
| 0.0166 | 0.1690 | 0.2558 |
| 0.0230 | 0.1836 | 0.2813 |
| 0.0296 | 0.1968 | 0.3056 |
| 0.0359 | 0.2082 | 0.3279 |
| 0.0413 | 0.2174 | 0.3466 |
| 0.0458 | 0.2247 | 0.3622 |
| 0.0493 | 0.2302 | 0.3742 |
| 0.0520 | 0.2343 | 0.3836 |
| 0.0539 | 0.2372 | 0.3902 |

| $\alpha$ | $\dim C_\alpha$ | $p$ |
|---|---|---|
| 0.0553 | 0.2393 | 0.3950 |
| 0.0563 | 0.2408 | 0.3985 |
| 0.0793 | 0.2734 | 0.4815 |
| 0.0959 | 0.2956 | 0.5464 |
| 0.1065 | 0.3094 | 0.5911 |
| 0.1127 | 0.3174 | 0.6185 |
| 0.1162 | 0.3219 | 0.6345 |
| 0.1181 | 0.3244 | 0.6433 |
| 0.1192 | 0.3258 | 0.6484 |
| 0.1505 | 0.3659 | 0.8140 |
| 0.1662 | 0.3861 | 0.9144 |
| 0.1733 | 0.3954 | 0.9649 |
| 0.1764 | 0.3994 | 0.9880 |
| 0.1777 | 0.4011 | 0.9980 |

For $0.1505 \le p < 1$ pick $\alpha$ in the table as large as possible to that $C_\alpha$ is removable for $H^p$. It is easy to see that $\dim C_\alpha > 0.4p$. It is also easy to see that for $0.1505 \le p \le 0.65$, $\dim C_\alpha > 0.5p$ and for $0.1505 \le p \le 0.4$, $\dim C_\alpha > 0.6p$.

For small $p$ we need to estimate $M'$, from Corollary 2.10, further. This part of the proof works for $p \le 0.164$.

Let $0 < \alpha \le 0.001$ be fixed. Let $A$, $B$, $a$, $b$, $r$, $r_0$ and $\varphi$ be as in Lemma 2.9. We know that $0 < a < b < r \le \frac{1}{1997}$. Moreover, the hyperbolic distance between $a$ and $b$ is the same as the hyperbolic distance between $b$ and $r$, by the choice of $b$. Hence, the Euclidean distance between $a$ and $b$ is larger than the Euclidean distance between $b$ and $r$. Thus

$$r - b < r_0 < b - a.$$

We next want to prove that $0.476 < B < A < \frac{1}{2} < \log r / \log \varphi(r)$. The inequality $B < A$ is clear; see the proof of Lemma 2.9. We have, using that $r \le \frac{1}{1997}$,

$$|\varphi(-a)| = \frac{a+b}{1+ab} > \frac{2a}{1+r^2} = \frac{2\alpha(1-2\alpha)}{(2-3\alpha)(1+r^2)} > 0.99\alpha,$$

and

$$\varphi(r)=\frac{r-b}{1-br}<\frac{r_0}{1-r^2}=\frac{\alpha^2}{(2-3\alpha)(1-r^2)}<0.51\alpha^2<(0.99\alpha)^2. \tag{3}$$

Thus

$$A=\frac{\log|\varphi(-a)|}{\log\varphi(r)}<\frac{\log 0.99\alpha}{\log(0.99\alpha)^2}=\frac{1}{2}.$$

Next, we have

$$\varphi(r)=-\varphi(a)=\frac{b-a}{1-ab}>r_0>\frac{\alpha^2}{2}, \tag{4}$$

and therefore

$$\frac{\log r}{\log\varphi(r)}>\frac{\log(\alpha/\sqrt{2})}{\log\frac{1}{2}\alpha^2}=\frac{1}{2}.$$

Finally

$$|\varphi(-r)|=\frac{b+r}{1+br}<2r=\frac{2\alpha}{2-3\alpha}<1.0016\alpha,$$

which together with (4) gives

$$\begin{aligned} B=\frac{\log|\varphi(-r)|}{\log\varphi(r)}&>\frac{\log 1.0016\alpha}{\log\frac{1}{2}\alpha^2}=\frac{1+\log 1.0016/\log\alpha}{2-\log 2/\log\alpha}\\ &\geq\frac{1+\log 1.0016/\log 0.001}{2-\log 2/\log 0.001}>0.476. \end{aligned} \tag{5}$$

Thus we have proved

$$0.476<B<A<\frac{1}{2}<\frac{\log r}{\log\varphi(r)}.$$

Next we have

$$(1-b)^2-A(1+b)^2>(1-b)^2-\tfrac{1}{2}(1+b)^2>\tfrac{1}{2}-3b>\tfrac{1}{2}-3r>0.4984.$$

Hence, letting $M'$ be as defined in Corollary 2.10, we have

$$M'>\frac{0.4984\cdot\frac{1}{2}}{1-0.476^2}>0.322.$$

So for $p\leq 0.164$, if we let $\alpha=0.322^{1/p}<0.001$, then $C_\alpha$ is removable for $H^p$ and

$$\frac{\dim C_\alpha}{p}=\frac{-\log 2}{\log 0.322}>0.6116. \square$$

**Remark.** It can be seen from the table that $C_{0.0038}$ is removable for $H^{0.1855}$ and that $\dim C_{0.0038}>0.67\cdot 0.1855$. In fact, with a more detailed table one can show that for $0.18\leq p\leq 0.20$ there is $\alpha$ such that $C_\alpha$ is removable for $H^p$ and $\dim C_\alpha>0.67p$. We refrain from giving such a table here.

From (5) we see that $\lim_{\alpha\to 0+} B = \frac{1}{2}$, hence also $\lim_{\alpha\to 0+} A = \frac{1}{2}$. Using (3) we also have

$$\frac{1}{2} < \frac{\log r}{\log \varphi(r)} < \frac{\log \frac{1}{2}\alpha}{\log 0.51\alpha^2},$$

and thus so $\lim_{\alpha\to 0+} \log r/\log\varphi(r) = \frac{1}{2}$. Hence, $\lim_{\alpha\to 0+} M' = \frac{1}{3}$. If we let $p = \log M'/\log\alpha$, we have $\lim_{\alpha\to 0+} \dim C_\alpha/p = \log 2/\log 3 = 0.6309\ldots$.

We conclude that the largest quotient $\dim C_\alpha/p$ given by Corollary 2.10 is not obtained for very small $p$, but seems to be obtained for $p \approx 0.19$.

## 3. Properties of removable sets

In this section we use Theorem 1.4 to extend Propositions 4.12 and 5.11 in Björn [4] to $0 < p < \infty$, and Propositions 4.11, 4.14 and 5.10 in Björn [4] to $\frac{1}{2} \le p < \infty$. In Björn [4] they were all proved for $1 \le p < \infty$.

**Proposition 3.1.** *Let $0 < p < \infty$. Then there exists a domain $\Omega \subset \mathbf{C}$ and a set $E \subset \Omega$, with $\Omega \setminus E$ being a domain, such that $E$ is weakly but not strongly removable for $H^p(\Omega \setminus E)$.*

**Remark.** As was already remarked in the introduction, this theorem is false for $p = \infty$. In fact, all the results in this section are false for $p = \infty$. This is easy to see except for, perhaps, Corollary 3.2.

*Proof.* This was proved for $1 \le p < \infty$ in Proposition 4.12 in Björn [4]. We therefore assume that $0 < p < 1$ here.

Let $\alpha$ be given by Theorem 1.4 so that $C_\alpha$ is removable for $H^p$. Let $A = C_\alpha - \frac{1}{2}$, then $A$ is a set with the properties (a)–(e) in the proof of Lemma 2 in Kobayashi [16]. Let $B = \bigcup_{k=-\infty}^{\infty}(A + 2k)$, $h(z) = 1/z$, $E = h(B)$ and $F = E \cup \{0\}$. By the proof of Lemma 2 in Kobayashi [16] the compact set $F$ is not removable for $H^p(\mathbf{S} \setminus F)$.

Let $\Omega = \mathbf{C}\setminus\{0\}$. As finite sets are removable, $H^p(\Omega) = H^p(\mathbf{S}) \neq H^p(\mathbf{S}\setminus F) = H^p(\Omega \setminus E)$, i.e. $E$ is not strongly removable for $H^p(\Omega \setminus E)$.

However, let $f \in H^p(\Omega \setminus E)$. By conformal invariance $f \circ h \in H^p(\Omega \setminus B)$. Since $C_\alpha$ is removable for $H^p$ we can continue $f \circ h$ analytically to $A + 2k$ for each $k$, so $f \circ h \in A(\Omega)$, and hence $f \in A(\Omega)$. Thus $E$ is weakly removable for $H^p(\Omega \setminus E)$. □

**Corollary 3.2.** *Let $0 < p < \infty$. Then there exists a countable family of disjoint compact sets $K_1$, $K_2, \ldots$, such that $K_j$ is removable for $H^p$ for all $j = 1, 2, \ldots$, but $\bigcup_{j=1}^{\infty} K_j$ is a compact set not removable for $H^p$.*

*Furthermore, there exists a sequence of compact sets $K'_1 \subset K'_2 \subset \cdots$, removable for $H^p$, but with $\bigcup_{j=1}^{\infty} K'_j$ being a compact set not removable for $H^p$.*

*Proof.* By the proof of Proposition 3.1, if $0 < p < 1$, and Proposition 4.12 in Björn [4], if $1 \le p < \infty$, one can choose the domain $\Omega = \mathbf{C}\setminus\{0\}$ in Proposition 3.1. Let also $E$ be as given by Proposition 3.1.

As $E$ is weakly removable, we can use Theorem 1.3 to write $E$ as a union of disjoint compact sets $K_2$, $K_3, \dots$, removable for $H^p$. Let $K_1 = \{0,\infty\}$ and $K = \bigcup_{j=1}^\infty K_j$. If $K$ were removable for $H^p$ we would have $H^p(\Omega \setminus E) = H^p(\mathbf{S}\setminus K) = H^p(\mathbf{S}) = H^p(\Omega)$, contradicting the fact that $E$ was not strongly removable for $H^p(\Omega\setminus E)$. This concludes the first part.

The second part is obtained by letting $K'_k = \bigcup_{j=1}^k K_j$, $k=1,2,\dots$, since $K'_k$ is removable for $H^p$ by Theorem 1.3. □

**Remarks.** A direct consequence is that Proposition 5.11 in Björn [4] is true for $0<p<\infty$.

This corollary also shows that the condition of well-separatedness in Theorem 1.3 is needed for all $0<p<\infty$. Furthermore, it shows that Proposition 4.8 in Björn [4] cannot be extended to countable unions for any $0<p<\infty$.

For $p=\infty$ it is well known that the first part of Corollary 3.2 is false, see Exercise 1.6, p. 12, in Garnett [8]. A direct consequence is that the condition of well-separatedness in Theorem 1.3 is not necessary for $p=\infty$.

The second part of Corollary 3.2 is equivalent to the first part, and hence is also false for $p=\infty$. We leave it as an exercise for the interested reader to prove the equivalence.

**Proposition 3.3.** *Let $\frac12 \le p < \infty$. Then there exist compact sets $K_1$ and $K_2$ removable for $H^p$ such that $K_1\cup K_2$ is not removable for $H^p$.*

**Remark.** A direct consequence is that Proposition 5.10 in Björn [4] is true for $\frac12 \le p<\infty$.

*Proof.* This was proved for $p=1$ in Example 1 in Hejhal [13]; the proof was generalized to $1\le p<\infty$ by Proposition 4.11 in Björn [4]. We therefore assume that $\frac12\le p<1$ here.

Let $A$, $B$ and $F$ be as in the proof of Proposition 3.1. Let $K_1 = \{z\in F : z\ge 0\}$ and $K_2=\{z\in F: z\le 0\}$. Then $F=K_1\cup K_2$ is not removable for $H^p$, by the proof of Proposition 3.1.

Let $f\in H^p(\mathbf{S}\setminus K_1)$ and let $u$ be a positive harmonic majorant of $|f|^p$ in $\mathbf{S}\setminus K_1$. In the same way as in the proof of Proposition 3.1 we see that $f$ can be continued analytically to $\mathbf{S}\setminus\{0\}$. Thus we can write $f$ as a Laurent series around the origin,

$$f(z) = \sum_{n=0}^\infty \frac{a_n}{z^n}, \quad z\ne 0.$$

Let $r_k = 1/(2k+1)$ and $m_k = [6\log k]+27$, $k=1,2,\dots$, then $\operatorname{dist}(r_k,K_1) = r_k/(4k+3)$. Fix $k\ge 1$ and let $z_j = r_k e^{i\pi(1-1/2\pi)^j}$, $j=0,1,\dots,m_k-1$ and $z_{m_k}=r_k$. Then $|z_j - z_{j+1}| < \frac12 r_k(1-1/2\pi)^j \le \frac12 \operatorname{dist}(z_j,K_1)$, $j=0,1,\dots,m_k-2$, as $\operatorname{dist}(z_j,K_1)=r_k$, if $(1-1/2\pi)^j\ge\frac12$, and $\operatorname{dist}(z_j,K_1)\ge r_k\sin\pi(1-1/2\pi)^j$, otherwise. Using that $|z_{m_k-1}-z_{m_k}|<r_k\pi(1-1/2\pi)^{m_k-1}$, we have

$$\begin{aligned}\operatorname{dist}(z_{m_k-1},K_1) &\ge \operatorname{dist}(r_k,K_1)-|z_{m_k-1}-z_{m_k}| \ge \frac{r_k}{7k}-|z_{m_k-1}-z_{m_k}|\\ &> \left(\frac{1}{7k\pi(1-1/2\pi)^{m_k-1}}-1\right)|z_{m_k-1}-z_{m_k}| > 2|z_{m_k-1}-z_{m_k}|,\end{aligned}$$

by the choice of $m_k$.

By Harnack's inequality there exists a constant $C>1$ such that if $v$ is a positive harmonic function in $\mathbf{D}$ and $\zeta_1,\zeta_2\in\overline{D}(0,\frac12)$, then $v(\zeta_1)<Cv(\zeta_2)$. We obtain $u(r_k)<Cu(z_{m_k-1})<\cdots<C^{m_k}u(-r_k)$. Obviously, we also obtain $u(z)<C^{m_k}u(-r_k)$ for all $|z|=r_k$.

Furthermore, $u(-r_{2k})<Cu(-r_k)$ and hence $u(-r_{2^l})<C^lu(-r_1)$. Moreover, $m_{2^l}<6l+27\le 33l$, if $l\ge1$. So for $|z|=r_{2^l}$, $l\ge1$, we have $u(z)<C^{l+m_{2^l}}u(-r_1)<C^{34l}u(-r_1)$.

Cauchy's theorem gives, with $ds$ denoting arc length,

$$\begin{aligned}|a_n| &\le \frac{1}{2\pi}\int_{|z|=r_{2^l}}|z|^{n-1}|f(z)|\,ds \le \frac{1}{2\pi}\int_{|z|=r_{2^l}}|z|^{n-1}u(z)^{1/p}\,ds\\ &< C^{34l/p}r_{2^l}^n u(-r_1)^{1/p} < (C^{34/p}2^{-n})^l u(-r_1)^{1/p}\to 0, \quad \text{as } l\to\infty,\end{aligned}$$

if $n$ is so large that $C^{34/p}2^{-n}<1$. Hence, the Laurent series expansion of $f$ is a finite sum and $f$ does not have an essential singularity. It is now easy to see that if there existed a non-constant $f\in H^p(\mathbf{S}\setminus K_1)$, then also $g\in H^p(\mathbf{S}\setminus K_1)$, where $g(z)=1/z$.

Let $\varphi(w)=-(1-w)^2/(1+w)^2$, a conformal mapping from $\mathbf{D}$ to $\mathbf{S}\setminus[0,\infty]$. Then

$$\begin{aligned}\int_{-\pi}^{\pi}|g\circ\varphi(e^{i\theta})|^p\,d\theta &= \int_{-\pi}^{\pi}\left(\frac{(1+e^{i\theta})(1+e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\right)^p d\theta = \int_{-\pi}^{\pi}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^p d\theta\\ &\ge \int_0^{\pi/2}\frac{1}{\theta^{2p}}\,d\theta=\infty.\end{aligned}$$

Therefore, $g\circ\varphi\notin H^p(\mathbf{D})$, and by conformal invariance $g\notin H^p(\mathbf{S}\setminus[0,\infty])\supset H^p(\mathbf{S}\setminus K_1)$. Hence, $H^p(\mathbf{S}\setminus K_1)=\{f: f \text{ constant}\}$ and $K_1$ is removable for $H^p$. By conformal invariance also $K_2$ is removable for $H^p$. □

**Remark.** The proof actually shows that there exist compact sets $K_1$ and $K_2$ removable for $H^{1/2}$, but with $K_1\cup K_2$ not removable for any $H^p$, $0<p<1$, and with $K_1\cap K_2=\{0\}$.

**Corollary 3.4.** *Let $\frac12\le p<\infty$. Then there exists a compact set $K$ removable for $H^p$, a domain $\Omega\subset\mathbf{S}$ and a set $E\subset\Omega$, with $\Omega\setminus E$ being a domain, such that $E\subset K$ and $E$ is not strongly removable for $H^p(\Omega\setminus E)$.*

**Remarks.** This shows that subsets of strongly removable sets do not have to be strongly removable. This is different from the situation for weak removability, cf. Proposition 4.13 in Björn [4].

This result was proved for $1 \le p < \infty$ in Björn [4], Proposition 4.14. The same proof can now be used here; we repeat it for completeness.

*Proof.* Let $K_1$ and $K_2$ be as in Proposition 3.3. Let $K = K_1$, $\Omega = \mathbf{S} \setminus K_2$ and $E = K_1 \setminus K_2$. By the assumption in Proposition 3.3, $K_1$ and $K_2$ are removable for $H^p$. On the other hand, if $E$ were strongly removable for $H^p(\Omega \setminus E)$, we would have

$$H^p\big(\mathbf{S} \setminus (K_1 \cup K_2)\big) = H^p(\Omega \setminus E) = H^p(\Omega) = H^p(\mathbf{S} \setminus K_2) = H^p(\mathbf{S}),$$

and $K_1 \cup K_2$ would be removable for $H^p$, contradicting Proposition 3.3. □

**Proposition 3.5.** *Let $0 < p < \infty$. Then there exist domains $\Omega_1, \Omega_2 \subset \mathbf{S}$ and a set $E \subset \Omega_1 \cap \Omega_2$, with $\Omega_1 \setminus E$, $\Omega_2 \setminus E$ being domains, such that $E$ is strongly removable for $H^p(\Omega_1 \setminus E)$, but $E$ is not strongly removable for $H^p(\Omega_2 \setminus E)$.*

**Remarks.** This shows that strong removability is domain dependent, which is different from the situation for weak removability, cf. Theorem 1.3 and the remark following it.

This result was proved for $1 \le p < \infty$ in Björn [4], the remark following Proposition 4.14. The same proof can be based on Proposition 3.3 proving the result for $\frac{1}{2} \le p < \infty$. We here give a different proof for $0 < p < 1$.

*Proof.* The result was proved for $1 \le p < \infty$ in Björn [4], the remark following Proposition 4.14. Therefore assume that $0 < p < 1$ is fixed. Let also $N$ be a positive integer such that $Np \ge 1$.

Let $\alpha$ and $F$ be as in the proof of Proposition 3.1, let $K = \{z \in F : z \ge 0\}$ and $E = K \setminus \{0\}$. Let $K_k = \bigcup_{j=0}^{k} e^{ij\pi/N} K$, $k = 0, 1, \dots, N$. We have

$$H^p(\mathbf{S}) \subset H^p(\mathbf{S} \setminus K_0) \subset H^p(\mathbf{S} \setminus K_1) \subset \dots \subset H^p(\mathbf{S} \setminus K_N). \tag{6}$$

By the proof of Lemma 2 in Kobayashi [16], $H^p(\mathbf{S}) \subsetneq H^p(\mathbf{S} \setminus F) \subset H^p(\mathbf{S} \setminus K_N)$, so at least one of the inclusions in (6) is proper. We shall show that at least one is non-proper, i.e. an equality.

Let $f \in H^p(\mathbf{S} \setminus K_N)$ and let $u$ be a positive harmonic majorant of $|f|^p$ in $\mathbf{S} \setminus K_N$. In the same way as in the proof of Proposition 3.1 we see that $f$ can be analytically continued to $\mathbf{S} \setminus \{0\}$, and we can write $f$ as a Laurent series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad z \ne 0,$$

as in the proof of Proposition 3.3.

Let $C>1$ be the constant from Harnack's inequality used in the proof of Proposition 3.3 and let $r_k=1/(2k+1)$. By the same argument as in Proposition 3.3 we see that if $|z|=r_k$ and $j\pi/2N\le\arg z\le(j+1)\pi/2N$, then $u(z)<C^{6\log k+27}u(r_ke^{ij\pi/2N})$ for $k=1,2,\dots$, and $j=0,1,\dots,2N-1$. If $|z|=r_k$, $k\ge1$ and $-\pi\le\arg z\le0$, then $u(z)<C^{6\log k+27}u(-ir_k)$. Combining this we see that $u(z)<C^{(2N+1)(6\log k+27)}u(-ir_k)$ for $|z|=r_k$.

We also have $u(-ir_{2k})<Cu(-ir_k)$ and hence $u(-ir_{2^l})<C^lu(-ir_1)$. So for $|z|=r_{2^l}$, $l\ge1$, we have $u(z)<C^{l+(2N+1)(6l+27)}u(-ir_1)<C^{34(2N+1)l}u(-ir_1)$.

Cauchy's theorem gives

$$\begin{aligned}|a_n|&\le\frac{1}{2\pi}\int_{|z|=r_{2^l}}|z|^{n-1}|f(z)|\,ds\le\frac{1}{2\pi}\int_{|z|=r_{2^l}}|z|^{n-1}u(z)^{1/p}\,ds\\&<C^{34(2N+1)l/p}r_{2^l}^n u(-ir_1)^{1/p}<(C^{34(2N+1)/p}2^{-n})^lu(-ir_1)^{1/p}\to0,\end{aligned}$$

as $l\to\infty$, if $n$ is so large that $C^{34(2N+1)/p}2^{-n}<1$. Hence, the Laurent series expansion of $f$ is a finite sum and $f$ does not have an essential singularity.

By Theorem 10 in Hejhal [13], or the more general Theorem 5.3 in Björn [5], $K_N$ is removable for $H^{Np}$. Hence, $z\mapsto|z|^{-Np}$ does not have a harmonic majorant in $\mathbf{S}\setminus K_N$, so if $f(z)=1/z^N$, then $f\notin H^p(\mathbf{S}\setminus K_N)$. We thus conclude that $\dim H^p(\mathbf{S}\setminus K_N)\le N$ and, as $\dim H^p(\mathbf{S})=1$, at least two of the inclusions in (6) are non-proper.

We have proved that there are $j$ and $k$ such that $H^p(\mathbf{S}\setminus K_{j-1})\ne H^p(\mathbf{S}\setminus K_j)$ and $H^p(\mathbf{S}\setminus K_{k-1})=H^p(\mathbf{S}\setminus K_k)$. Then, by symmetry and conformal invariance $E$ is not strongly removable for $H^p(\mathbf{S}\setminus K_j)$, but $E$ is strongly removable for $H^p(\mathbf{S}\setminus K_k)$. □

## 4. Conclusions and conjectures

In Björn [3] it was asked if there are sets of positive Hausdorff dimension removable for $H^p$, $p<1$. Theorem 1.4 has now answered Problem 2c in Björn [3] in the affirmative.

Theorem 1.4 clearly does not give the exact values for removability of linear self-similar Cantor sets, we have had to make many estimates throughout the proof. It is possible that $C_\alpha$ is removable for $H^p$ if $\dim C_\alpha<p$.

If we let $d(p)=\sup\{\dim K: K \text{ is removable for } H^p\}$, we now know that

$$\begin{aligned}0.6p<d(p)&\le p, & 0<p&\le0.4,\\0.5p<d(p)&\le p, & 0.4<p&\le0.65,\\0.4p<d(p)&\le p, & 0.65<p&<1,\\d(p)&=1, & 1\le p&\le\infty,\end{aligned}$$

and that $d$ is a non-decreasing function. The most natural choice for $d(p)$ seems to be $\min\{1,p\}$.

**Conjecture 4.1.** Let $0 < p < 1$, then there exists a compact set $K$ removable for $H^p$ with $\dim K = p$.

This was asked as Problem 2b in Björn [3].

If this is true, then only Problems 1 and 2a in Björn [3] remain. I have no idea of whether they are true or not, but I would like to formulate them here as one combined important problem.

**Problem 4.2.** Let $0 < p < \infty$ and $d = \min\{1, p\}$. Does there exist a compact set $K$ removable for $H^p$ with positive $d$-dimensional Hausdorff measure?

For $p = \infty$ it is well known that this is true.

In Section 3 we were not able to obtain Proposition 3.3 for $0 < p < \frac{1}{2}$. I believe that this is not due to lack of removable sets, only due to lack of examples of removable sets. I therefore would like to make the following conjecture.

**Conjecture 4.3.** Let $0 < p < \frac{1}{2}$, then there exist two compact sets $K_1$ and $K_2$ removable for $H^p$ with $K_1 \cup K_2$ not removable for $H^p$.

Corollary 3.4 would of course follow for $0 < p < \frac{1}{2}$. I would also like to make the following conjecture.

**Conjecture 4.4.** Let $0 < p < 1$ and $0 < \alpha < \beta < \frac{1}{2}$. If $C_\beta$ is removable for $H^p$, then $C_\alpha$ is removable for $H^p$.

It is easy to believe this conjecture, but since $C_\alpha \not\subset C_\beta$, unless $\alpha = \beta/2^k$, $k = 1, 2, \ldots$, it seems harder to prove. Some partial results can be obtained from the results in Section 2, e.g. if $\alpha^{0.6116} \le \beta \le 0.001$ and $C_\beta$ is removable for $H^p$, then $p \ge \dim C_\beta \ge \dim C_\alpha / 0.6116$, and $C_\alpha$ is removable for $H^p$, by the proof of Theorem 1.4.

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