Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 25, 2000, 151–160

A DECOMPOSITION THEOREM FOR SOLUTIONS OF PARABOLIC EQUATIONS

N.A. Watson

University of Canterbury, Department of Mathematics
Private Bag 4800, Christchurch, New Zealand; N.Watson@math.canterbury.ac.nz

Abstract. Let L be a second order linear parabolic partial differential operator, with smooth and bounded coefficients defined on $X = \mathbf{R}^n \times]0, a[$. Let E be an open subset of X, and let K be a compact subset of E. If u is a solution of $Lu = 0$ on $E \backslash K$, we prove that there is a unique decomposition $u = v + w$, where $Lv = 0$ on E, $Lw = 0$ on $X \backslash K$, and w is zero both at infinity and on $\mathbf{R}^n \times]0, k[$, where $k = \inf\{t : K \cap (\mathbf{R}^n \times \{t\}) \neq \emptyset\}$. A more detailed decomposition is given for the case where $K \subseteq \mathbf{R}^n \times \{d\}$.

1. Introduction

Let L be a second order linear uniformly parabolic partial differential operator, in divergence form and with bounded smooth coefficients, defined on the closure of $X = \mathbf{R}^n \times]0, a[$. The central theorem of this paper states that, if E is an open subset of X, K is a compact subset of E, and u is a solution of $Lu = 0$ on $E \backslash K$, then u can be written uniquely in the form $u = v + w$, where v is a solution of $Lv = 0$ on the whole of E, w is a solution on $X \backslash K$, and w vanishes at infinity and on $\mathbf{R}^n \times]0, k[$, where $k = \inf\bigl\{t : K \cap (\mathbf{R}^n \times \{t\}) \neq \emptyset\bigr\}$. This is analogous to a classical result for harmonic functions [3, p. 172], and is new even for the heat equation.

To prove the decomposition theorem, we require a representation theorem for an arbitrary $C^{2,1}$ function, in terms of the fundamental solution of $Lu = 0$. Such a formula in terms of the fundamental solution of Laplace's equation is classical [7, p. 11]. The result below was proved for the heat equation by Smyrnélis [11]; furthermore Doob gave a less natural version for that case [7, p. 271].

Once the decomposition theorem is established, it permits an easy deduction of a general analogue of Bôcher's theorem from the particular case where E is an infinite strip.

Finally, we consider in detail the case of the decomposition where K is a subset of a characteristic hyperplane. Under minimal conditions on u, we establish that w is the integral of a signed measure against the fundamental solution of $Lu = 0$.

A typical element of X is written $p = (x, t)$ or $q = (y, s)$. Therefore the element of Lebesgue measure in $n + 1$ dimensions is written as dq, that in n

1991 Mathematics Subject Classification: Primary 35K10; Secondary 31B35, 35C15, 35K05.

dimensions as dx or dy, and that in 1 dimension as dt or ds. Where an integral is taken over the boundary of a piecewise smooth domain, the element of surface area is written as $d\sigma$, and the outward unit normal as (ν_x, ν_t). The gradient in the spatial variables is written ∇_x, and the inner product in $\mathbf{R}^n$ as $\langle \cdot, \cdot \rangle$. All measures appearing below are Radon measures.

Let $A = (a_{ij})$ be a C^∞, symmetric, $n \times n$ matrix-valued function on X such that, for some $\lambda \in]0,1[$,

$$\lambda \|\xi\|^2 \leq \langle A(x,t)\xi,\ \xi \rangle \leq \lambda^{-1} \|\xi\|^2$$

whenever $(x,t) \in X$ and $\xi \in \mathbf{R}^n$. Let

$$Lu = \sum_{i=1}^{n} D_i(a_{ij} D_j u) - D_t u = 0 \tag{1}$$

be the corresponding parabolic partial differential equation with divergence form, and let

$$L^* u = \sum_{i=1}^{n} D_i(a_{ij} D_j u) + D_t u = 0$$

be its adjoint.

Under these hypotheses on A, the fundamental solution Γ exists and satisfies

$$\Gamma(x,t;y,s) \leq \left(\frac{\beta}{4\pi(t-s)} \right)^{n/2} \exp\left(-\frac{\|x-y\|^2}{2\alpha(t-s)} \right),$$
$$\|\nabla_x \Gamma(x,t;y,s)\| \leq \beta (t-s)^{-(n+1)/2} \exp\left(-\frac{\|x-y\|^2}{2\alpha(t-s)} \right)$$

for all $(x,t), (y,s) \in \overline{X}$ such that $s < t$, where α and β are positive constants. Furthermore, for each fixed (x,t),

$$L^* \Gamma(x,t; \cdot, \cdot) = 0$$

on $\mathbf{R}^n \times]0,t[$. Details are given in [9]. We adopt the convention that $\Gamma(x,t;y,s) = 0$ whenever $t \leq s$.

Given $(x_0, t_0) \in X$ and $c > 0$, the identity

$$\left(\frac{\beta}{4\pi(t_0 - s)} \right)^{n/2} \exp\left(-\frac{\|x_0 - y\|^2}{2\alpha(t_0 - s)} \right) = (4\pi c)^{-n/2}$$

holds if and only if

$$\|x_0 - y\|^2 = n\alpha(t_0 - s) \log\left(\frac{c\beta}{t_0 - s} \right).$$

Therefore, if $c < t_0/\beta$, the sets

$$\Omega(x_0, t_0; c) = \{(y, s) : \Gamma(x_0, t_0; y, s) > (4\pi c)^{-n/2}\}$$

and

$$\Psi(x_0, t_0; c) = \{(y, s) : \Gamma(x_0, t_0; y, s) = (4\pi c)^{-n/2}\}$$

have their closures in X. Since $\Gamma(x_0, t_0; \cdot, \cdot) \in C^\infty(\mathbf{R}^n \times]0, t_0[)$, for almost every such c the set $\Psi(x_0, t_0; c)$ is a smooth regular n-dimensional manifold, by Sard's theorem [12, p. 45]. Fabes and Garofalo [8] have studied mean values of solutions of (1) over the sets Ω and Ψ. In particular, they have shown that, if

$$\mathscr{M}(u; x_0, t_0; c) = \int_{\Psi(x_0, t_0; c)} u \langle A \nabla_x \Gamma(x_0, t_0; \cdot, \cdot), \nu_x \rangle \, d\sigma,$$

then $u(x_0, t_0) = \mathscr{M}(u; x_0, t_0; c)$ whenever u is a solution of (1). In the sequel, we shall use only the form of $\mathscr{M}$ and the fact that $\mathscr{M}(1; x_0, t_0; c) = 1$.

2. The decomposition theorem

To prove the decomposition theorem, we require the following representation theorem, for an arbitrary sufficiently smooth function, in terms of the fundamental solution Γ.

Theorem 1. *Let E be a bounded open subset of X with a piecewise smooth boundary, and let $u \in C^{2,1}(\overline{E})$. Then*

$$u(x_0, t_0) = -\int_E \Gamma_0 (Lu) \, dq - \int_{\partial E} \big(\langle A(u \nabla_x \Gamma_0 - \Gamma_0 \nabla_x u), \nu_x \rangle + u \Gamma_0 \nu_t \big) \, d\sigma$$

for each $(x_0, t_0) \in E$, where $\Gamma_0 = \Gamma(x_0, t_0; \cdot, \cdot)$.

Proof. Given $(x_0, t_0) \in E$, and any $c \in]0, t_0/\beta[$ such that $\Psi(x_0, t_0; c)$ is a smooth surface, for all $\gamma \in]0, c\beta/e]$ we put

$$S(\gamma) = \{(y, s) : \|x_0 - y\|^2 < \alpha n \gamma \log(c\beta/\gamma),\ t_0 - \gamma < s < t_0\}$$

and

$$\Omega_\gamma = \Omega(x_0, t_0; c) \cup S(\gamma).$$

We choose c and γ such that $\overline{\Omega}_\gamma \subseteq E$. Then, by Green's formula for L,

$$\int_{E \backslash \overline{\Omega}_\gamma} \Gamma_0 (Lu) \, dq = \int_{\partial(E \backslash \overline{\Omega}_\gamma)} \big(\langle A(\Gamma_0 \nabla_x u - u \nabla_x \Gamma_0), \nu_x \rangle - u \Gamma_0 \nu_t \big) \, d\sigma \tag{2}$$

since $L^* \Gamma_0 = 0$ on $E \backslash \overline{\Omega}_\gamma$.

We consider the integral on the right-hand side of (2), splitting the range of integration into five pieces. First, on $\partial\Omega_\gamma \cap (\mathbf{R}^n \times \{t_0\})$ we have $\Gamma_0 = 0$ and $\nabla_x \Gamma_0 = 0$, so that this piece contributes nothing. Second, if

$$\Lambda(\gamma) = \big\{(y,s) : \|x_0 - y\|^2 = \alpha n\gamma \log(c\beta/\gamma),\ t_0 - \gamma < s < t_0\big\}$$

denotes the lateral boundary of $S(\gamma)$, then $\nu_t = 0$ on $\Lambda(\gamma)$, and

$$\int_{\Lambda(\gamma)} \langle A\Gamma_0 \nabla_x u,\ \nu_x\rangle \, d\sigma \to 0 \quad \text{as} \quad \gamma \to 0$$

because $A\nabla_x u$ and Γ_0 are bounded on $S(c\beta/e)\backslash\Omega(x_0, t_0; c)$. Furthermore, if σ_n denotes the surface area of the unit sphere in $\mathbf{R}^n$, then for any $r > 0$ we have

$$\int_{\partial B(0,r)\times]0,\gamma[} t^{-(n+1)/2} \exp\left(-\frac{\|x\|^2}{2\alpha t}\right) d\sigma = \sigma_n (2\alpha)^{(n-1)/2} \int_{r^2/2\alpha\gamma}^{\infty} s^{(n-3)/2} e^{-s}\, ds,$$

so that

$$\int_{\Lambda(\gamma)} \|\nabla_x \Gamma_0\| \, d\sigma \le \kappa \int_{(n/2)\log(c\beta/\gamma)}^{\infty} s^{(n-3)/2} e^{-s}\, ds \to 0$$

as $\gamma \to 0$, where $\kappa = \beta\sigma_n(2\alpha)^{(n-1)/2}$. It follows that

$$\int_{\Lambda(\gamma)} \langle uA\nabla_x \Gamma_0,\ \nu_x\rangle \, d\sigma \to 0 \quad \text{as} \quad \gamma \to 0,$$

so that the entire integral over $\Lambda(\gamma)$ tends to zero. Third, if

$$F(\gamma) = \partial\Omega_\gamma \cap (\mathbf{R}^n \times \{t_0 - \gamma\}),$$

then the measure of $F(\gamma)$ tends to zero as $\gamma \to 0$, and $u\Gamma_0$ is bounded on the union over all $\gamma \in]0, c\beta/e]$ of the sets $F(\gamma)$, so that

$$\int_{F(\gamma)} u\Gamma_0 \nu_t \, d\sigma \to 0 \quad \text{as} \quad \gamma \to 0.$$

Fourth, let $B(\gamma) = (\mathbf{R}^n \times]-\infty,\ t_0 - \gamma[\,) \cap \partial\Omega_\gamma$, so that $\Gamma_0 = (4\pi c)^{-n/2}$ on $B(\gamma)$. Therefore, as $\gamma \to 0$,

$$\begin{aligned} -\int_{B(\gamma)} (\langle A\nabla_x u,\ \nu_x\rangle - u\nu_t)\Gamma_0 \, d\sigma &\to (4\pi c)^{-n/2} \int_{\partial\Omega(x_0,t_0;c)} (\langle A\nabla_x u,\ \nu_x\rangle - u\nu_t)\, d\sigma \\ &= (4\pi c)^{-n/2} \int_{\Omega(x_0,t_0;c)} Lu\, dq \end{aligned}$$

by Green's formula. Furthermore, as $\gamma \to 0$,

$$\int_{B(\gamma)} u\langle A\nabla_x\Gamma_0, \nu_x\rangle \, d\sigma \to \mathscr{M}(u; x_0, t_0; c).$$

Fifth, the integral over ∂E is left unchanged. It now follows from (2) that

$$\begin{aligned}\int_{E\setminus\overline{\Omega}(x_0,t_0;c)} \Gamma_0(Lu)\, dq = \int_{\partial E} \big(\langle A(\Gamma_0\nabla_x u - u\nabla_x\Gamma_0),\ \nu_x\rangle - u\Gamma_0\nu_t\big)\, d\sigma \\ - (4\pi c)^{-n/2}\int_{\Omega(x_0,t_0;c)} Lu\, dq - \mathscr{M}(u; x_0, t_0; c).\end{aligned}$$

We now make $c \to 0$, so that $\mathscr{M}(u; x_0, t_0; c) \to u(x_0, t_0)$ because u is continuous and $\mathscr{M}(1; x_0, t_0; c) = 1$, and

$$(4\pi c)^{-n/2}\int_{\Omega(x_0,t_0;c)} Lu\, dq \to 0$$

because the integrand is bounded and the measure of $\Omega(x_0, t_0; c)$ is dominated by $c^{(n+2)/2}$. This proves the theorem.

For our present purpose, we do not need the full generality of Theorem 1, just the following consequence.

Corollary. *If $u \in C^{2,1}(X)$ and has compact support in X, then*

$$u(x_0, t_0) = -\int_X \Gamma(x_0, t_0; \cdot, \cdot) Lu\, dq$$

for each $(x_0, t_0) \in X$.

Proof. In Theorem 1, choose E to contain both (x_0, t_0) and the support of u.

We can now prove the decomposition theorem, using the method employed in [3, p. 172] to prove the corresponding result for Laplace's equation.

Theorem 2. *Let K be a compact subset of an open subset E of X, and let u satisfy $Lu = 0$ on $E\setminus K$. Then u can be written uniquely as $u = v + w$, where $Lv = 0$ on E, $Lw = 0$ on $X\setminus K$, and w is zero both at infinity and on $\mathbf{R}^n \times]0, k[$ for $k = \inf\{t : K \cap (\mathbf{R}^n \times \{t\}) \neq \emptyset\}$.*

Proof. Suppose first that E is bounded. For any set S and $r > 0$, we denote by S_r the set of all points with distance less than r from S. We choose r such that $K_r \cap (\partial E)_r = \emptyset$, and put

$$E(\rho) = E\setminus\big(K_\rho \cup (\partial E)_\rho\big)$$

for each $\rho \leq r$. Let $\phi_r \in C^\infty(X)$, have compact support in $E\backslash K$, and be equal to 1 throughout $E(r)$. Given $(x_0, t_0) \in E(r)$, we can apply the above corollary to $u\phi_r$ and obtain

$$-u(x_0, t_0) = v_r(x_0, t_0) + w_r(x_0, t_0),$$

where v_r and w_r are defined on X by

$$v_r(x,t) = \int_{(\partial E)_r} \Gamma(x,t;\cdot,\cdot) L(u\phi_r)\, dq$$

and

$$w_r(x,t) = \int_{K_r} \Gamma(x,t;\cdot,\cdot) L(u\phi_r)\, dq.$$

Differentiation under the integral sign shows that $Lv_r = 0$ and $Lw_r = 0$ outside their respective ranges of integration. Furthermore, w_r is zero at infinity and on $\mathbf{R}^n \times]0,k[$.

If $0 < s < r$, then on $E(r)$ we have $v_r + w_r = u = v_s + w_s$, so that $w_r - w_s$ is a solution of (1) on $X\backslash K_r$ that equals $v_s - v_r$ on $E(r)$, and can therefore be extended to a solution of (1) on X. Since $w_r - w_s$ is zero at infinity and on $\mathbf{R}^n \times]0,k[$, it is zero everywhere. Hence $w_r = w_s$ and $v_r = v_s$ on $E(r)$. Therefore, given $(\xi,\tau) \in E$ we can choose r such that $(\xi,\tau) \in E\backslash(\partial E)_r$, and define $v(\xi,\tau) = v_r(\xi,\tau)$ unambiguously. Similarly, if $(\xi',\tau') \in X\backslash K$ we can define $w(\xi',\tau') = w_r(\xi',\tau')$ for small r. Then $u = v + w$ as asserted, and the uniqueness follows by similar reasoning to that used to show that $w_r = w_s$ and $v_r = v_s$ above.

Now suppose that E is unbounded. For any bounded open set D such that $K \subseteq D \subseteq E$, we have the unique decomposition $u = v + w$ on $D\backslash K$, as above. Then $u - w$ is a solution of (1) on $E\backslash K$ that can be extended by v to a solution h on E. Hence $u = h + w$ as required, and the uniqueness follows as before.

3. Some consequences of the decomposition theorem

Theorem 2 enables us to easily deduce a general analogue of Bôcher's theorem from the particular case first considered by Krzyżański [10]. Subtler analogues were given by Aronson [1]. Isolated singularities of nonnegative solutions of the heat equation were characterized by Widder [16, p. 119], and those of arbitrary solutions by Chung and Kim [5].

Theorem 3. *Let E be an open subset of X, let $(y_0, s_0) \in E$, and let u be a solution of (1) on $E\backslash\{(y_0,s_0)\}$ such that u is bounded below on some cylinder $B(y_0,r)\times]s_0,t_0[$. Then u can be written uniquely in the form*

$$u = v + \kappa\Gamma(\cdot,\cdot;y_0,s_0),$$

where v is a solution of (1) on E, and $\kappa \in [0,\infty[$.

Proof. By Theorem 2, there is a unique decomposition $u = v + w$ on $E\backslash\{(y_0, s_0)\}$, where $Lv = 0$ on E, $Lw = 0$ on $X\backslash\{(y_0, s_0)\}$, and w is zero both at infinity and on $\mathbf{R}^n\times]0, s_0[$. If $B(y_0, r)\times]s_0, t_0[$ is chosen to have its closure in E, then $w = u - v$ is bounded below on that set. Hence, if $\varepsilon > 0$ and

$$h(x,t) = w(x,t) + \varepsilon \int_{B(y_0,r)} \Gamma(x,t;y,s_0)\|y - y_0\|^{-n/2}\, dy,$$

then

$$\liminf_{(x,t)\to(z,s_0+)} h(x,t) \geq 0$$

for all $z \in \mathbf{R}^n$. Since h is a solution of (1) that vanishes at infinity, it follows from the minimum principle that $h \geq 0$. Making $\varepsilon \to 0$, we deduce that $w \geq 0$. It now follows that $w = \kappa\Gamma(\cdot\,,\cdot\,; y_0, s_0)$, by [4, Theorem 3].

Using more sophisticated techniques, we can improve Theorem 3 in several directions. This requires the following result on the uniqueness of parts of a representing measure. The result holds in the more general context of [14], but here we keep to the present one.

We shall use the following terminology. A family $\mathscr{F}$ of closed balls in $\mathbf{R}^n$ is called an *abundant Vitali covering* of $\mathbf{R}^n$ if, given any $x \in \mathbf{R}^n$ and $\varepsilon > 0$, $\mathscr{F}$ contains uncountably many balls centred at x with radius less than ε.

We also use the following notation. Given any open subset D of $\mathbf{R}^{n+1}$ such that $D \cap (\mathbf{R}^n \times \{0\}) \neq \emptyset$, we put $D(0) = \{x \in \mathbf{R}^n : (x,0) \in D\}$ and $D_+ = D \cap X$.

Theorem 4. *Let u be a solution of* (1) *such that*

$$u(x,t) = \int_{\mathbf{R}^n} \Gamma(x,t;y,0)\, d\mu(y) + v(x,t)$$

for all $(x,t) \in D_+$, where μ is a signed measure concentrated on $D(0)$ and v is a solution of (1) *on D_+ with a continuous extension to 0 on $D(0) \times \{0\}$. Let $\mathscr{F}$ be an abundant Vitali covering of $\mathbf{R}^n$. If there is a signed measure ν concentrated on $D(0)$ such that*

$$\lim_{t\to 0+} \int_{A\cap V} u(x,t)\, dx = \nu(A \cap V) \tag{3}$$

whenever $A, V \in \mathscr{F}$, $V \subseteq D(0)$, and $A \cap V \neq \emptyset$, then $\mu = \nu$.

Proof. By [14, Theorem 3(i)], there is an abundant Vitali covering $\mathscr{F}_0 \subseteq \mathscr{F}$ such that $|\mu|(\partial A) = 0$ for all $A \in \mathscr{F}_0$. Given $V \in \mathscr{F}_0$ such that $V \subseteq D(0)$, put

$$w_V(x,t) = \int_V \Gamma(x,t;y,0)\, d\mu(y)$$

and

$$w_{D\setminus V}(x,t) = \int_{D(0)\setminus V} \Gamma(x,t;y,0)\, d\mu(y)$$

for all $(x,t) \in X$. Then $w_V = u - v - w_{D\setminus V}$ on D_+.

If $A \in \mathscr{F}_0$ and $A \cap V \neq \emptyset$, then $A \cap V$ is a compact subset of $D(0)$, so that

$$\lim_{t\to 0+} \int_{A\cap V} v(x,t)\, dx = 0. \tag{4}$$

Furthermore, because the boundaries of $A\cap V$ and $A\setminus V$ are both μ-null, it follows from [14, Theorem 1(i)] that

$$\lim_{t\to 0+} \int_{A\cap V} w_{D\setminus V}(x,t)\, dx = 0 = \lim_{t\to 0+} \int_{A\setminus V} w_V(x,t)\, dx. \tag{5}$$

Combining (3), (4) and (5), we obtain

$$\lim_{t\to 0+} \int_A w_V(x,t)\, dx = \lim_{t\to 0+} \int_{A\cap V} w_V(x,t)\, dx = \nu(A\cap V).$$

On the other hand, if $A \in \mathscr{F}_0$ and $A \cap V = \emptyset$, then it follows from [14, Theorem 1(i)] that

$$\lim_{t\to 0+} \int_A w_V(x,t)\, dx = 0 = \nu(A\cap V).$$

Therefore the restrictions of μ and ν to V are identical, by [14, Theorem 3(ii)].

Given any open subset U of $D(0)$, choose a sequence $\{V_k\}$ in $\mathscr{F}_0$ with union U, and put $W_1 = V_1$, $W_j = V_j \setminus \bigcup_{k=1}^{j-1} V_k$ for all $j \geq 2$. Then, by the above,

$$\mu(U) = \sum_{j=1}^{\infty} \mu(W_j) = \sum_{j=1}^{\infty} \nu(W_j) = \nu(U).$$

The result now follows from the regularity of Radon measures.

Theorem 4 enables us to consider the uniqueness of just a part of the representing measure, because we can vary $D(0)$ without altering D_+. For example, if $D_+ = X$, G is any relatively open subset of $\mathbf{R}^n$, and u has the representation

$$u(x,t) = \int_{\mathbf{R}^n} \Gamma(x,t;y,0)\, d\lambda(y)$$

for all $(x,t) \in X$, we can take $D = X \cup (G \times \{0\}) \cup (\mathbf{R}^n \times]-\infty, 0[)$, μ the restriction of λ to G, and

$$v(x,t) = \int_{\mathbf{R}^n\setminus G} \Gamma(x,t;y,0)\, d\lambda(y).$$

This technique is used in the proof of our next theorem, which generalizes [15, Theorem 5], where only the heat equation was considered and the method of proof was very different. The case of lower bounded solutions of the heat equation was discovered independently by Chung [6], who used yet another approach.

Theorem 5. *Let E be an open subset of X, and let K be a nonempty compact subset of $E\cap(\mathbf{R}^n\times\{b\})$. If u is a solution of (1) on $E\backslash K$ such that*

$$\liminf_{t\to b+}\int_U u^+(x,t)\,dx<\infty$$

for some relatively open subset U of $\mathbf{R}^n$ such that $K\subseteq U\times\{b\}$, then there exist a unique solution v of (1) on E, and a unique signed measure μ supported in $K(b)=\{x\in\mathbf{R}^n:(x,b)\in K\}$, such that

$$u(x,t)=v(x,t)+\int_{K(b)}\Gamma(x,t;y,b)\,d\mu(y)$$

for all $(x,t)\in E\backslash K$.

Proof. We may assume that $\overline{U}\times[b,d]$ is a compact subset of E, for some $d>b$. By Theorem 2, u can be written uniquely as the sum of a solution v of (1) on E, and a solution w of (1) on $X\backslash K$ that is zero both at infinity and on $\mathbf{R}^n\times]0,b[$. If $M=\max\{|v(x,t)|:x\in\overline{U},\ b\le t\le d\}$, and m_n denotes n-dimensional Lebesgue measure, then

$$\liminf_{t\to b+}\int_U w^+(x,t)\,dx\le\liminf_{t\to b+}\int_U u^+(x,t)\,dx+Mm_n(U)<\infty.$$

By the maximum principle, w is bounded outside $U\times[b,\varepsilon]$ for any $\varepsilon\in]b,d[$. Therefore, for any $\alpha>0$, the function

$$\int_{\mathbf{R}^n}\exp(-\alpha\|x\|^2)w^+(x,\cdot\,)\,dx$$

is bounded on $]\varepsilon,a[$, and there is a number N such that

$$\begin{aligned}\liminf_{t\to b+}\int_{\mathbf{R}^n}\exp(-\alpha\|x\|^2)w^+(x,t)\,dx\le N&\int_{\mathbf{R}^n\backslash U}\exp(-\alpha\|x\|^2)\,dx\\&+\liminf_{t\to b+}\int_U w^+(x,t)\,dx<\infty.\end{aligned}$$

It now follows from [13, Theorem 13, Corollary] that there is a signed measure μ on $\mathbf{R}^n$ such that

$$w(x,t)=\int_{\mathbf{R}^n}\Gamma(x,t;y,b)\,d\mu(y)$$

for all $(x,t)\in X\backslash K$. The uniqueness of such a representation is proved in [2, p. 688]. Finally, since w is continuous and zero on $(\mathbf{R}^n\times\{b\})\backslash K$, it follows from Theorem 4 (with $D(0)$ corresponding to $\mathbf{R}^n\backslash K(b)$) that $\mathbf{R}^n\backslash K(b)$ is μ-null.

References

[1] Aronson, D.G.: Isolated singularities of solutions of second order parabolic equations. - Arch. Rational Mech. Anal. 19, 1965, 231–238.

[2] Aronson, D.G.: Non-negative solutions of linear parabolic equations. - Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22, 1968, 607–694.

[3] Axler, S., P. Bourdon, and W. Ramey: Harmonic Function Theory. - Springer-Verlag, 1992.

[4] Chabrowski, J., and N.A. Watson: Properties of solutions of weakly coupled parabolic systems. - J. London Math. Soc. 23, 1981, 475–495.

[5] Chung, S.-Y., and D. Kim: Characterization of temperature functions with isolated singularity. - Math. Nachr. 168, 1994, 55–60.

[6] Chung, S.-Y.: Decomposition theorems for temperature functions with singularity. - Preprint.

[7] Doob, J.L.: Classical Potential Theory and its Probabilistic Counterpart. - Springer-Verlag, 1984.

[8] Fabes, E.B., and N. Garofalo: Mean value properties of solutions to parabolic equations with variable coefficients. - J. Math. Anal. Appl. 121, 1987, 305–316.

[9] Il'in, A.M., A.S. Kalashnikov, and O.A. Oleinik: Linear equations of the second order of parabolic type. - Uspekhi Mat. Nauk 17:3, 1962, 3–147; Russian Math. Surveys 17:3, 1962, 1–143.

[10] Krzyżański, M.: Sur les solutions non négatives de l'équation linéaire normale parabolique. - Rev. Roumaine Math. Pures Appl. 9, 1964, 393–408.

[11] Smyrnélis, E.P.: Sur les moyennes des fonctions paraboliques. - Bull. Sci. Math. 93, 1969, 163–173.

[12] Sternberg, S.: Lectures on Differential Geometry. - Prentice-Hall, 1964.

[13] Watson, N.A.: Uniqueness and representation theorems for parabolic equations. - J. London Math. Soc. 8, 1974, 311–321.

[14] Watson, N.A.: Boundary measures of solutions of partial differential equations. - Mathematika 29, 1982, 67–82.

[15] Watson, N.A.: Time-isolated singularities of temperatures. - J. Austral. Math. Soc. Ser. A 65, 1998, 416–429.

[16] Widder, D.V.: The Heat Equation. - Academic Press, 1975.

Received 21 April 1998