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GEOMETRIC CHARACTERIZATION OF HYPERELLIPTIC RIEMANN SURFACES

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Abstract. A new geometric characterization of hyperelliptic Riemann surfaces is given. It is proved that a closed Riemann surface of genus $g \geq 2$ is hyperelliptic if and only if M has a set of at least $2g-2$ simple closed geodesics which all intersect in the same point (and mutually intersect in no other point).

1. Introduction

From a classical point of view the hyperelliptic surfaces are the most simple Riemann surfaces. They can be defined by an algebraic curve

$$y^2 = F(x)$$

where $F(x)$ is a polynomial of degree $2g+1$ or $2g+2$ with distinct roots (g is the genus of the surface); the automorphism $(x,y) \to (x,-y)$ then defines the so-called hyperelliptic involution.

Equivalently, we can define that a Riemann surface M is hyperelliptic if there exists a holomorphic map $M \to P^1$ of degree 2.

Here is a more geometric, again equivalent definition. A Riemann surface M of genus g is hyperelliptic if M has an involution with exactly $2g+2$ fixed points.

Finally, hyperelliptic surfaces of genus g are characterized by the fact that the number of different Weierstrass points is minimal, namely $2g+2$ (the fixed points of the hyperelliptic involution), while on the other hand, the weight of each Weierstrass point is maximal, namely $\frac{1}{2}g(g-1)$.

For basic information on hyperelliptic surfaces the reader may consult for example Farkas and Kra [2], Miranda [3].

The main result of this paper is the following theorem (in the sequel, a *surface* will be a Riemann surface equipped with a metric of constant curvature -1).

Theorem A. *A closed surface M of genus $g \geq 2$ is hyperelliptic if and only if M contains $2g-2$ different simple closed geodesics which all intersect in the same point and mutually intersect in no other point.*

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This is an example that a local property can imply a global property of a surface. The proof will be given in Section 3. In Section 2, some results of topological properties of sets of simple closed curves are proved, in particular the following result which generalizes the fact that a closed surface has non-separating and separating simple closed geodesics.

Theorem B. *Let M be a closed surface of genus $g \geq 2$. Let k be an integer with $1 \leq k \leq 2g$. Let S_k be a set of k simple closed curves which intersect (transversally) in the same point such that the elements of S_k have no further intersection point. Then there is only one equivalence class of sets S_k if and only if k is even (S_k and S'_k are in the same equivalence class if M has a homeomorphism mapping the homotopy classes of the elements of S_k to those of S'_k).*

2. On sets of simple closed curves

Definition. (i) A *surface* is a Riemann surface equipped with a metric of constant curvature -1.

(ii) A (g,n)*-surface* is a surface of genus g with n boundary components which are simple closed geodesics, called *boundary geodesics*.

(iii) A *star set* is a set S_k of k simple closed curves on a surface M which all intersect (transversally) in the same point such that among the elements of S_k there are no further intersection points. S_k is called a *geodesic* star set if the k simple closed curves are simple closed geodesics.

(iv) Let M and M' be two (g,n)-surfaces. Let F be a set of simple closed curves on M, let F' be a set of simple closed curves on M'. Then F and F' are called *equivalent* if there exists a homeomorphism ϕ from M to M' which maps the homotopy classes of the elements of F to those of F'. This relation gives rise to *equivalence classes* of sets of simple closed curves on M.

(v) Let M be a surface and u a simple closed geodesic of M. Then u is called *separating* if $M \setminus u$ has two connected components, otherwise, u is called *non-separating.*

Lemma 1. *Let M be a $(g,2)$-surface with boundary geodesics u and v, $u \neq v$. Let t be a simple geodesic in M relying u and v. Then M has a unique simple closed geodesic z and a unique embedded $(0,3)$-surface Y such that u, v, z are the boundary geodesics of Y and t lies in Y.*

Proof. Obvious. □

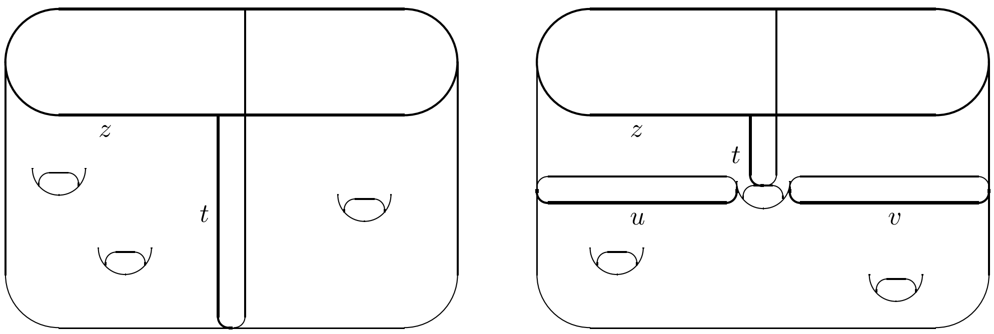

Figure 1. On the left hand side, a $(g,1)$-surface M with a simple geodesic t such that $M\setminus t$ is not connected. On the right hand side, $M\setminus t$ is connected.

Lemma 2. *Let $g\geq 2$ and let M be a $(g,1)$-surface with boundary geodesic z. Let t be a simple geodesic in M, starting and ending on z. Then either $M\setminus t$ is not connected or $M\setminus t$ is connected and has a unique pair (u,v) of simple closed geodesics and a unique embedded $(0,3)$-surface Y such that z,u,v are the boundary geodesics of Y and t lies in Y. Moreover, both cases are possible for every M.*

Proof. Obvious by Figure 1. □

Theorem 1. *Let M be a closed surface of genus $g\geq 2$. Let k be an integer with $1\leq k\leq 2g$. Then M has only one equivalence class of star sets S_k if and only if k is even.*

Proof. (i) Recall that two (g,n)-surfaces are homeomorphic (by the classification of compact surfaces).

(ii) The case $k=1$ is obvious since M has separating and non-separating simple closed curves. So assume that $k\geq 2$. It follows that all elements of S_k are non-separating. We therefore may assume that the elements of S_k are geodesic loops. Let $u\in S_k$. Cut M along u, the result is a surface M_1 of genus $g-1$ with two boundary components (in the proof of Theorem 1 we allow that the boundary components are only piecewise geodesic). Let $v\in S_k$, $v\neq u$. Then, by Lemma 1, $M_1\setminus v$ is a surface M_2 of genus $g-1$ with one boundary component. By (i), this proves the theorem for $k=2$. Assume that $k\geq 3$. Let $w\in S_k\setminus\{u,v\}$. If $k=3$, then, by Lemma 2, $M_3=M_2\setminus w$ is connected or not connected which proves the theorem for $k=3$ (Lemma 2 proves this if the genus of M_2 is at least 2; if M_2 has genus 1, then the fact remains true since the boundary component of M_2 is only piecewise geodesic). If $k>3$, then M_3 must be connected and is therefore (by Lemma 2) a surface of genus $g-2$ with two boundary components.

(iii) The general case follows by induction. □

Corollary 1. *Let M be a closed surface of genus g. Let k be an integer with $1 \leq k \leq 2g$. Let $C_k = \{u_1, u_2, \ldots, u_k\}$ be a set of k simple closed curves of M such that u_i and u_{i+1} intersect exactly once, $i = 1, \ldots, k-1$, and such that u_i and u_j are disjoint if $j \notin \{i-1, i, i+1\}$. Then M has only one equivalence class of sets C_k if and only if k is even.*

Proof. The proof is completely analogous to the proof of Theorem 1.

3. Hyperelliptic surfaces

Definition. (i) An *involution* $\phi \neq \mathrm{id}$ on a surface M is an isometry with $\phi^2 = \mathrm{id}$ where id is the identity.

(ii) A (g,n)-surface M is *hyperelliptic* if M has an involution with exactly $2g+2$ fixed points if n is even, and with exactly $2g+1$ fixed points if n is odd. Such an involution will be called a *hyperelliptic involution*.

Remarks. (i) It is well known that for closed surfaces the above given definition is equivalent to the usual one.

(ii) Let M be a hyperelliptic (g,n)-surface. It is easy to show that M has a unique hyperelliptic involution if $n \leq 2$. However, if $n \geq 3$, then a hyperelliptic involution is not unique, in general.

(iii) In the cases $(g,n) = (1,1)$ and $(g,n) = (2,0)$ all (g,n)-surfaces are hyperelliptic. If all boundary geodesics have equal length, then also in the cases $(g,n) \in \{(0,3),(0,4),(1,2)\}$ all (g,n)-surfaces are hyperelliptic.

(iv) Let M be a hyperelliptic (g,n)-surface and ψ a hyperelliptic involution. If n is odd, then $\psi(u) = u$ for a unique boundary geodesic of M. If $n \geq 2$ is even, then $\psi(u) \neq u$ for all boundary geodesics of M.

Theorem 2. *Let M be a closed surface of genus g. Then M is hyperelliptic if and only if M has a geodesic star set S_k with $k \geq 2g-2$.*

Proof. (i) Let M be hyperelliptic. Let A_i, $i = 1, \ldots, 2g+2$, be the fixed points of the hyperelliptic involution ψ of M. Let $A = A_1$. Let h_1 be a simple geodesic from A to A_2. Then $u_1 = h_1 \cup \psi(h_1)$ is a simple closed non-separating geodesic of M. Cut M along u_1, the result is a $(g-1,2)$-surface M_1 with boundary geodesics $u_1(L)$ and $u_1(R)$. Let $A(L)$ be the copy of A on $u_1(L)$ and let $A_2(R)$ be the copy of A_2 on $u_1(R)$. Then M_1 has a simple geodesic h_2 from $A(L)$ to $A_2(R)$. It follows that $u_2 = h_2 \cup \psi(h_2)$ is a simple closed geodesic of M which intersects u_1 only in A. Cut M_1 along h_2, the result is a surface M_2 of genus $g-1$ with one boundary component b, see Lemma 1. It follows that M_2 has a simple geodesic h_3 between a copy of A on b and a copy of A_3 on b such that $M_2 \setminus h_3$ is connected (this is possible by Lemma 2). It follows that $u_3 = h_3 \cup \psi(h_3)$ is a simple closed geodesic of M and that $\{u_1, u_2, u_3\}$ is a geodesic star set of M. Repeating this argument we can construct a geodesic star set of order $2g+1$ on M.

(ii) Assume now that M has a geodesic star set S_k with $k = 2g-2$. It follows by Theorem 1 and its proof that M has an embedded $(g-1,1)$-surface N which contains all elements of S_k. Denote by z the boundary geodesic of N. In M, z is separating and is also the boundary geodesic of an embedded $(1,1)$-surface X. As noted above, X is hyperelliptic with a unique hyperelliptic involution ψ_X. In order to prove that M is hyperelliptic, it is therefore sufficient to show that N is hyperelliptic since N then has a unique hyperelliptic involution ψ_N which acts on z in the same manner as ψ_X.

We now prove that N is hyperelliptic by induction with respect to g. If $g = 2$, then N is hyperelliptic since all $(1,1)$-surfaces are hyperelliptic. Let u and v be two different elements of S_k and let $S_{k-2} = S_k \setminus \{u, v\}$. By Theorem 1, N has an embedded $(g-2,1)$-surface P which contains all elements of S_{k-2}. By induction, P is hyperelliptic with a unique hyperelliptic involution ψ_P. Denote by z' the boundary geodesic of P. Let $S_{k-1} = S_{k-2} \cup \{u\}$. Again by Theorem 1 and its proof it follows that M has an embedded $(g-2,2)$-surface R containing S_{k-1}; denote by x and y the two boundary geodesics of R. Denote by Y the, in M, embedded $(0,3)$-surface with boundary geodesics z', x, y. Then Y is separated by u into two connected components, denote them by $Y(x)$ and $Y(y)$. Remember that P is hyperelliptic and that u passes through a fixed point of ψ_P (namely, the intersection point of the elements of S_k). It follows that the two angles in the two intersection points of u with z' are equal and that $z' \cap Y(x)$ and $z' \cap Y(y)$ have equal length. This implies by hyperbolic trigonometry (see for example [1]) that $Y(x)$ and $Y(y)$ are isometric. In particular, x and y have the same length. Therefore, Y has a hyperelliptic involution ψ_Y which interchanges x and y. This implies that ψ_P extends to a hyperelliptic involution ψ_R on R (since the action on z' is the same for ψ_P and for ψ_Y).

Let $T = N \setminus R$. T is an embedded $(0,3)$-surface with boundary geodesics z, x, y. Since x and y have the same length, T has a hyperelliptic involution ψ_T which interchanges x and y. It remains thus to prove that the action of ψ_T and ψ_R is the same on $x \cup y$. This again follows by the argument that v passes through a fixed point of ψ_R and therefore intersects x and y by the same angle and such that $\psi_T(v \cap T) = v \cap T$ (by Theorem 1 and its proof v intersects x and y only once). This proves that the involutions ψ_R and ψ_T induce a hyperelliptic involution on N. □

Corollary 2. (i) *Let M be a $(g,1)$-surface. Then M is hyperelliptic if and only if M has a geodesic star set S_k with $k = 2g$.*

(ii) *Let M be a $(g,2)$-surface. Then M is hyperelliptic if and only if M has a geodesic star set S_k with $k = 2g+1$.*

Proof. The proof of Theorem 2 applies to this situation. □

Corollary 3. *All closed surfaces of genus* 2 *are hyperelliptic.*

Proof. This was shown during the proof of Theorem 2 using the fact that all $(1,1)$-surfaces are hyperelliptic. After the proof of Theorem 2, we could also argue that all closed surfaces have a star set S_k with $k = 2$. □

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