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# CANONICAL CONJUGATIONS AT FIXED POINTS OTHER THAN THE DENJOY–WOLFF POINT

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**Abstract.** Let $\phi$ be an analytic map of the unit disk $\mathbf{D}$ such that $\phi(\mathbf{D}) \subset \mathbf{D}$. By Schwarz's lemma, the map $\phi$ has at most one fixed point in $\mathbf{D}$ (except for $\phi(z) = z$). On the other hand, there might be many boundary fixed points, i.e. points $\zeta \in \partial\mathbf{D}$ where $\phi$ has non-tangential limit $\zeta$ and where the derivative of $\phi$ has a finite non-tangential limit, which we write as $\phi'(\zeta)$. One fixed point of $\phi$ plays a special role: the Denjoy–Wolff point. It is the unique fixed point $\omega$ of $\phi$ where $|\phi'(\omega)| \le 1$. If $\phi$ does have a fixed point in $\mathbf{D}$, that point is $\omega$, otherwise $\omega \in \partial\mathbf{D}$.

Since the 1880's, until as recently as the 1980's, various conjugations have been introduced to study the iteration of $\phi$ near $\omega$. In this paper, we produce conjugations to study $\phi$ near boundary fixed points $\zeta$ different from $\omega$. Our method uses some results of Cowen and Pommerenke on inequalities for multipliers at boundary fixed points.

## 1. Introduction

Let $\phi$ be an analytic map of the disk such that $\phi(\mathbf{D}) \subset \mathbf{D}$. If $z \in \overline{\mathbf{D}}$ is a fixed point for $\phi$, we call $\phi'(z)$ the *multiplier* at $z$. Note that if $z \in \partial\mathbf{D}$, then $\phi(z)$ and $\phi'(z)$ are defined as non-tangential limits, i.e. we do not require any smoothness across the boundary.

Since $\phi$ is a self-map of the disk, multipliers at boundary fixed points are always positive. Aside for the case when $\phi$ is an elliptic automorphism, the following holds.

**Theorem 1.1** (Denjoy–Wolff). *There exist a unique point $\omega \in \overline{\mathbf{D}}$ such that the iterates $\phi_n$ converge to $\omega$ uniformly on compact subsets of the disk. Moreover, $\omega$ is the only fixed point of $\phi$ satisfying $|\phi'(\omega)| \le 1$.*

The special point $\omega$ is called the Denjoy–Wolff point of $\phi$. By Schwarz's lemma, $\phi$ has at most one fixed point in $\mathbf{D}$ and if so, such a fixed point must be the Denjoy–Wolff point. Hence, every fixed point $\zeta$ which is different from $\omega$ must lie on $\partial\mathbf{D}$, and if its multiplier $\phi'(\zeta)$ is finite, then $\phi'(\zeta) > 1$. For this reason, we call such points *boundary repelling fixed points* (BRFP).

Conjugations are a powerful tool to study the iterates of $\phi$ near $\omega$. In 1884, G. Kœnigs [Koe] proved that if $\omega \in \mathbf{D}$ and $\phi'(\omega) \neq 0$, then there exists an analytic

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map $\sigma$ defined on $\mathbf{D}$ such that $\sigma\circ\phi=\phi'(\omega)\sigma$. In 1904, L. Böttcher [Bo] showed that if $\omega\in\mathbf{D}$ and $\phi'(\omega)=0$, then there exists an analytic map $\sigma$ defined in a small neighborhood of $\omega$, such that $\sigma\circ\phi=\sigma^n$, where $n$ is the smallest integer such that $\phi^{(n)}(\omega)\neq 0$. In 1931, G. Valiron [Va] studied the case when $\omega$ is on the unit circle $\partial\mathbf{D}$ and the multiplier $c$ at $\omega$ is strictly less than 1. In this case there is an analytic map $\sigma$ defined on $\mathbf{D}$ with $\operatorname{Re}\sigma(z)>0$ and $\sigma\circ\phi=(1/c)\sigma$. More recently, Pommerenke [Po1], and Baker and Pommerenke [BP] showed that when $\omega\in\partial\mathbf{D}$ and $c=1$ there is an analytic map $\sigma$ defined on $\mathbf{D}$ such that $\sigma\circ\phi=\sigma+1$.

In this paper we prove the existence of conjugations, in order to better understand the dynamics near a BRFP.

**Theorem 1.2** (Main theorem). *Suppose $\phi$ is an analytic map of $\mathbf{D}$ such that $\phi(\mathbf{D})\subset\mathbf{D}$. Assume that 1 is a BRFP for $\phi$ with multiplier $1<A<\infty$. Let $a=(A-1)/(A+1)$ and $\eta(z)=(z-a)/(1-az)$. Then there is an analytic map $\psi$ of $\mathbf{D}$, with $\psi(\mathbf{D})\subset\mathbf{D}$, which has non-tangential limit 1 at 1, such that:*

$$\psi\circ\eta(z)=\phi\circ\psi(z) \tag{1.1}$$

*for every $z\in\mathbf{D}$, and which is semi-conformal at 1, i.e.,*

$$\operatorname{Arg}\frac{1-\psi(z)}{1-z}\longrightarrow 0 \tag{1.2}$$

*as $z$ tends to 1 non-tangentially.*

*Moreover, $\psi$ is unique up to precomposition with a Möbius transformation $M$ which commutes with $\eta$, i.e., $M$ is $\pm(z-b)/(1-bz)$ for some $b\in(-1,1)$.*

**Remark 1.3.** Theorem 1.2 still holds if $\eta$ is replaced by any other Möbius transformation of the unit disk, which fixes 1 and whose derivative at 1 is equal to $A$. Moreover, the term "semi-conformal" is not standard in the literature. For instance [Po2] uses "isogonal".

In the classical proofs, the conjugating maps are obtained by considering the sequence of iterates $\{\phi_n\}$ followed by an appropriate normalization. For instance, in the case when $\omega\in\partial\mathbf{D}$, [Po1] and [BP], the normalization is done with respect to the sequence $\{\phi_n(0)\}_{n=0}^{\infty}$, which by the Denjoy–Wolff theorem tends to $\omega$.

To prove Theorem 1.2 we use the same approach. We consider the sequence of iterates $\{\phi_n\}$. However, now $\phi_n(z)$ tends to escape from the BRFP. So, the normalization has to be done before the iteration. The main difficulty, which does not arise in the classical contexts, is to first prove the existence of a backward iteration sequence.

A *backward iteration* sequence for $\phi$ is a sequence $\{w_n\}_{n=0}^{\infty}\subset\mathbf{D}$ such that $\phi(w_n)=w_{n-1}$ for $n=1,2,3,\dots$. In general, backward iteration sequences may not exist, e.g. $\phi(z)=\frac{1}{2}z$ has none.

**Lemma 1.4** (Main lemma). *Suppose that $\phi$ is an analytic map of $\mathbf{D}$ such that $\phi(\mathbf{D}) \subset \mathbf{D}$, and assume that $1$ is a BRFP for $\phi$, with multiplier $1 < A < \infty$. Let $a = (A-1)/(A+1)$. Then there exists a backward iteration sequence $\{w_n\}_{n=0}^{\infty}$ tending to $1$, and such that*

$$\left| \frac{w_{n+1} - w_n}{1 - \overline{w_n} w_{n+1}} \right| \leq a \tag{1.3}$$

*for all $n$.*

This lemma will be proved in Section 3. In Sections 2 and 4, we will deduce the existence part of Theorem 1.2 from Lemma 1.4. In Section 5 we prove the uniqueness part of Theorem 1.2. Finally, in Section 6 we formulate some questions for future investigations.

Although the proof of Theorem 1.2 revolves around the construction of a backward iteration sequence, once the theorem is proved we find that there are many such sequences.

**Corollary 1.5.** *Suppose $\phi$ is an analytic map of the disk such that $\phi(\mathbf{D}) \subset \mathbf{D}$, and assume that $1$ is a BRFP for $\phi$. Then, given any angle $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, there is a backward iteration sequence which tends to $1$ and is asymptotically tangent to the segment $[1, 1 - e^{i\theta}]$.*

*Proof.* Let $z_n = A^n e^{-i\theta}$ for $n = 0, 1, 2, 3, \ldots$. Then $c_n = (z_n - 1)/(z_n + 1)$ is a backward iteration sequence for $\eta$ in $\mathbf{D}$, which converges to $1$ and is tangent to the segment $[1, 1 - e^{i\theta}]$. By Theorem 1.2, $\psi(c_n)$ is a backward iteration sequence for $\phi$, which also converges to $1$ and is tangent to the segment $[1, 1 - e^{i\theta}]$, because $\psi$ is semi-conformal at $1$. □

**Remark 1.6.** A consequence of Corollary 1.5 is that the hypothesis of smoothness across $\partial\mathbf{D}$ can be removed in Theorem 7.23 of [CM].

**Remark 1.7.** Note that, in Theorem 1.2, $\phi$ is precomposed with the conjugation, while in the classical results mentioned above it is the conjugating function which is precomposed with $\phi$. D. Bargmann pointed out that these conjugating functions are known as Poincaré maps in the complex dynamics literature; see for instance [Al, p. 118].

**Remark 1.8.** As shown in Section 5. of [PC], one can construct examples where the conjugation $\psi$ does not have an angular derivative at $1$, i.e., even though (1.2) holds, $|1 - \psi(r)|/(1 - r)$ tends to infinity as $r$ tends to $1$.

## 2. Existence of conjugations

In the following, we let

$$M_w(z) = \frac{z-w}{1-\overline{w}z}.$$

Recall that the pseudo-hyperbolic distance between two points $z, w \in \mathbf{D}$ is defined to be

$$\delta(z,w) = |M_w(z)|.$$

Also, for $k = 1, 2, 3, \ldots$, let

$$a_k = (A^k - 1)/(A^k + 1)$$

where $A = \phi'(1)$, and set

$$\eta_k(z) = M_{a_k}(z).$$

Note that $\eta_k$ is the $k$th iterate of $\eta = \eta_1$, and $a = a_1$.

In this section we will show that in order to obtain a conjugation for the BRFP at 1 it is enough to construct a backward iteration sequence $\{w_n\}_{n=0}^{\infty}$ converging to 1 and whose steps in the pseudo-hyperbolic distance are bounded by $a$, i.e., we assume Lemma 1.4. Condition (1.3) forces $w_n$ to tend to 1 asymptotically radially and hence have pseudo-hyperbolic steps whose lengths tend to $a$.

**Lemma 2.1.** *If $\{w_n\}_{n=0}^{\infty}$ is a backward iteration sequence for $\phi$, which tends to 1, and which satisfies (1.3), then $w_n$ tends to 1 asymptotically radially, i.e.,*

$$\lim_{n\to\infty} \frac{1-|w_n|}{|1-w_n|} = 1. \tag{2.1}$$

**Corollary 2.2.** *If $\{w_n\}_{n=0}^{\infty}$ is as in Lemma 2.1, consider the Möbius transformations $\tau_n(z) = M_{-w_n}(z)$. Then,*

(1) $\tau_{n+k}^{-1} \circ \tau_n \longrightarrow \eta_k$, *uniformly on compact sets of the disk, as $n$ tends to infinity.*

(2) $\tau_{n+1}^{-1} \circ \eta^{-1} \circ \tau_n(z) \longrightarrow z$, *uniformly on compact sets of the disk, as $n$ tends to infinity.*

The proof of Lemma 2.1 and Corollary 2.2 will be given later in Section 4.

*Proof of Theorem* 1.2 (existence). Consider the normal family $\{\phi_n \circ \tau_n\}$ and let $\psi$ be one of its normal limits. Fix $z \in \mathbf{D}$, then

$$\delta\big(\phi_n \circ \tau_n(z), \phi_{n+1} \circ \tau_{n+1}(z)\big) \to 0 \tag{2.2}$$

as $n$ tends to infinity. In fact, by Schwarz's lemma,

$$\delta\big(\phi_n \circ \tau_n(z), \phi_{n+1} \circ \tau_{n+1}(z)\big) \le \delta\big(\tau_n(z), \phi \circ \tau_{n+1}(z)\big).$$

Also, the right-hand side is less than

$$\delta\big(\tau_n(z), \phi \circ \eta^{-1} \circ \tau_n(z)\big) + \delta\big(\eta^{-1} \circ \tau_n(z), \tau_{n+1}(z)\big).$$

As $n$ tends to infinity the first term tends to zero, because $\tau_n(z)$ tends to 1 non-tangentially and $\phi \circ \eta^{-1}$ has angular derivative equal to one at 1. The second term also tends to zero, by the second statement of Corollary 2.2.

It follows from (2.2) that if a subsequence $\phi_{n_k} \circ \tau_{n_k}$ converges to $\psi$ uniformly on compact subsets of the disk, then $\phi_{n_k+1} \circ \tau_{n_k+1}$ also converges to $\psi$. But

$$\phi_{n_k+1} \circ \tau_{n_k+1} = \phi \circ \phi_{n_k} \circ \tau_{n_k} \circ \tau_{n_k}^{-1} \circ \tau_{n_k+1}.$$

Hence, by the first statement of Corollary 2.2,

$$\psi = \phi \circ \psi \circ \eta^{-1}. \tag{2.3}$$

We are left to show that $\psi$ fixes 1 and is semi-conformal there. Note that at each point $a_k$ the sequence $\phi_n \circ \tau_n$ actually converges to $w_k$, since

$$\begin{aligned}\delta\big(\phi_n \circ \tau_n(a_k), w_k\big) &= \delta\big(\phi_n \circ \tau_n(a_k), \phi_n(w_{k+n})\big) \le \delta\big(a_k, \tau_n^{-1} \circ \tau_{n+k}(0)\big)\\ &= \delta\big(\eta_k^{-1}(0), \tau_n^{-1} \circ \tau_{n+k}(0)\big) \to 0\end{aligned}$$

as $n$ tends to infinity, by the first statement of Corollary 2.2. Thus

$$\psi(a_k) = w_k \qquad \text{for } k = 1, 2, 3, \dots .$$

It follows that the sequence

$$g_n(z) = \tau_n^{-1} \circ \psi \circ \eta_n^{-1}(z) \tag{2.4}$$

converges to $z$ uniformly on compact subsets of the disk. In fact, $g_n(0) = 0$ for $n = 1, 2, 3, \dots$ . On the other hand,

$$g_n(a_1) = \tau_n^{-1}(w_{n+1}) = \tau_n^{-1}\big(\tau_{n+1}(0)\big) \to \eta_1(0) = a_1$$

as $n$ tends to infinity. Hence any normal limit of the $g_n$'s must be a self-map of the disk which fixes 0 and $a_1$, i.e., it is the identity map.

We deduce that $\psi([a_k, a_{k+1}]) = \tau_k\big(g_k([0, a])\big)$ is a curve joining $w_k$ to $w_{k+1}$ which looks more and more like a hyperbolic geodesic. Hence, $\psi$ has radial (and therefore also non-tangential) limit 1 at 1.

More generally, fix a compact set $K$ in $\mathbf{D}$. Then the hyperbolic distance between $g_n(z)$ and $z$ tends to zero uniformly on $K$, and therefore so does the hyperbolic distance between $\psi\big(\eta_n^{-1}(z)\big)$ and $\tau_n(z)$. It follows from this that

$\operatorname{Arg}\bigl(1-\psi(\eta_n^{-1}(z))\bigr) - \operatorname{Arg}\bigl(1-\tau_n(z)\bigr)$ also tends to zero uniformly in $K$. A calculation using the fact that $w_n$ tends to 1 asymptotically radially shows that both $\operatorname{Arg}\bigl(1-\tau_n(z)\bigr)$ and $\operatorname{Arg}\bigl(1-\eta_n^{-1}(z)\bigr)$ tend to $\operatorname{Arg}[(1-z)/(1+z)]$. Thus $\operatorname{Arg}\bigl[\bigl(1-\psi(\eta_n^{-1}(z))\bigr)/\bigl(1-\eta_n^{-1}(z)\bigr)\bigr]$ tends to zero uniformly in $K$, i.e., (1.2) is proved.

Alternatively, we could have differentiated $g_n$ and found that the argument of $\psi'$ has non-tangential limit 0 at 1. This is always equivalent to (1.2), see for instance Theorem 2 of [Ya] (the author of [Ya] uses the words semi-conformal and conformal differently from us).

Finally, one might ask whether the sequence $\{\phi_n \circ \tau_n\}$ is actually convergent. This is indeed the case. Let $\psi$ be one of its normal limits as above. Then, using (2.3), Schwarz's lemma, and (2.4), we obtain

$$\begin{aligned}\delta\bigl(\phi_n\circ\tau_n(z),\psi(z)\bigr) &= \delta\bigl(\phi_n\circ\tau_n(z),\psi\circ\eta_n\circ\eta_n^{-1}(z)\bigr)\\ &= \delta\bigl(\phi_n\circ\tau_n(z),\phi_n\circ\psi\circ\eta_n^{-1}(z)\bigr)\\ &\le \delta\bigl(\tau_n(z),\psi\circ\eta_n^{-1}(z)\bigr) = \delta\bigl(z,g_n(z)\bigr)\to 0\end{aligned}$$

as $n$ tends to infinity. □

## 3. Main construction

In this section we construct a backward iteration sequence converging to 1. First we need to recall results of Julia and Carathéodory.

**Theorem 3.1** (Julia–Carathéodory). *Suppose $\phi$ is an analytic map of the disk with $\phi(\mathbf{D})\subset\mathbf{D}$, and $\zeta,\xi\in\partial\mathbf{D}$. If there is a sequence $\{p_n\}\subset\mathbf{D}$ such that $p_n\to\zeta$, $\phi(p_n)\to\xi$, and*

$$(3.1)\qquad \frac{1-|\phi(p_n)|}{1-|p_n|}\to A<\infty,$$

*then*

(a) $A>0$.
(b) *For every horodisk $H$ at $\zeta$, i.e., $H$ is a disk internally tangent to $\partial\mathbf{D}$ at $\zeta$, and if $M(z)=\xi\overline{\zeta}(z-a\zeta)/(1-a\overline{\zeta}z)$, with $a=(A-1)/(A+1)$, we have $\phi(H)\subset M(H)$.*
(c) *$\phi(z)\to\xi$ as $z\to\zeta$ non-tangentially.*
(d) *$\phi'(z)\to\phi'(\zeta)$ as $z\to\zeta$ non-tangentially, and $|\phi'(\zeta)|\le A$.*

For a proof of Theorem 3.1 see [Sh, Chapter 4].

**Corollary 3.2.** *Suppose that $\phi$ is an analytic map of $\mathbf{D}$ with $\phi(\mathbf{D})\subset\mathbf{D}$, and $\zeta\in\partial\mathbf{D}$ is not the Denjoy–Wolff point. Suppose further that there is a sequence $\{p_n\}\subset\mathbf{D}$ such that*

(1) $p_n\to\zeta$,
(2) $\lim_{n\to\infty}\delta\bigl(p_n,\phi(p_n)\bigr)\le a<1$.

*Then $\zeta$ is a BRFP with multiplier $\phi'(\zeta)\le(1+a)/(1-a)=A$.*

*Proof.* Condition (2) implies that $\phi(p_n)$ tends to $\zeta$ as well. Let $\Delta$ be the closed hyperbolic disk centered at $p_n$ of radius $\delta_n = \delta(p_n, \phi(p_n))$. The point in $\Delta$ which is closest to the origin has modulus $(|p_n| - \delta_n)/(1 - |p_n|\delta_n)$. So, since $\phi(p_n) \in \Delta$,

$$1 - |\phi(p_n)| \leq \frac{1 + \delta_n}{1 - |p_n|\delta_n}(1 - |p_n|).$$

Hence, a subsequence of $\{p_n\}$ satisfies (3.1). Corollary 3.2 then follows from (c) and (d) in Theorem 3.1. □

Before starting our construction, we need to recall one more result. For convenience, below we reproduce Theorem 4.1 of [CP].

**Theorem 3.3** (Cowen–Pommerenke). *Let $\phi$ be analytic with $\phi(\mathbf{D}) \subset \mathbf{D}$. Let $\omega$ be the Denjoy–Wolff point and $z_1, z_2, \ldots, z_n$ be BRFP's. Then,*

(1) *If $\omega \in \mathbf{D}$,*

$$\sum_{j=1}^{n} \frac{1}{\phi'(z_j) - 1} \leq \operatorname{Re}\left(\frac{1 + \phi'(\omega)}{1 - \phi'(\omega)}\right).$$

(2) *If $\omega \in \partial\mathbf{D}$ and $0 < \phi'(\omega) < 1$,*

$$\sum_{j=1}^{n} \frac{1}{\phi'(z_j) - 1} \leq \frac{\phi'(\omega)}{1 - \phi'(\omega)}.$$

(3) *If $\omega \in \partial\mathbf{D}$ and $\phi'(\omega) = 1$,*

$$\sum_{j=1}^{n} \frac{|\omega - z_j|^2}{\phi'(z_j) - 1} \leq \operatorname{Re}\left(\frac{\omega}{\phi(0)} - 1\right).$$

*Proof of Lemma* 1.4. Assume 1 is a BRFP for $\phi$. Let $D$ be a small closed disk centered at 1 which does not contain the Denjoy–Wolff point of $\phi$. It follows from Theorem 3.3 that $D$ contains at most finitely many BRFP's of $\phi$ with multiplier less or equal to $A$. Hence, by reducing the radius of $D$, if necessary, we can assume without loss of generality that $D$ does not contain any BRFP's of $\phi$ with multiplier less or equal to $A$, aside from 1.

Let $n_0$ be the smallest integer for which $a_n \in D$ and set $r_k = a_{n_0+k}$. Also call $J$ the arc $\partial D \cap \mathbf{D}$. For each $k$, let $\gamma_k$ be the straight segment connecting $r_k$ and $\phi(r_k)$. By the Denjoy–Wolff theorem the sequence $\{\phi_n(r_k)\}_n$ converges to the Denjoy–Wolff point $\omega$ and therefore eventually leaves the disk $D$. Since $\bigcup_{j=0}^{n-1} \phi_j(\gamma_k)$ is a path connecting $r_k$ to $\phi_n(r_k)$ there is a smallest integer $n_k$ such that $\phi_{n_k}(\gamma_k)$ intersects the arc $J$.

Note that each $r_k$ determines a horodisk $H_k$ at 1, i.e., the disk whose diameter is $[r_k, 1]$. Moreover, Theorem 3.1 implies that $\phi(H_{k+1}) \subset \eta(H_{k+1}) = H_k$.

Therefore, $\phi_j(\gamma_k)$ cannot intersect $J$ for $j=1,2,\dots,k$. Hence, we conclude that $n_k > k$.

We claim that the sequence $\{\phi_{n_k}(r_k)\}_k$ is compact in $\mathbf{D}$. In fact, suppose not. Then we can extract a subsequence $\{y_{k'}\}$ whose modulus tends to $1$. Since $\phi$ has finite angular derivative $A$ at $1$,

$$\delta\big(r_k,\phi(r_k)\big)\longrightarrow a=\frac{A-1}{A+1}$$

as $k$ tends to infinity. By Schwarz's lemma, and the way $n_k$ was chosen,

$$\delta(y_{k'},J)\le\delta\big(r_{k'},\phi(r_{k'})\big).$$

Hence, $\{y_{k'}\}$ can only accumulate at the end-points of $J$ on $\partial\mathbf{D}$. Suppose that $b$ is an end-point of $J$ and that $\{y_{k'}\}$ tends to $b$. Then, by Schwarz's lemma again,

$$\liminf_{k'\to\infty}\delta\big(y_{k'},\phi(y_{k'})\big)\le\lim\delta\big(r_{k'},\phi(r_{k'})\big)=a$$

and Corollary 3.2 implies that $b$ must be a BRFP of $\phi$ with multiplier less or equal to $A$. This is a contradiction, by our choice of the disk $D$.

So there is an infinite set $I_0$ of integers such that $\{\phi_{n_k}(r_k)\}_{k\in I_0}$ converges to a point $w_0$ of $\mathbf{D}$. Fix $j\ge 1$ and suppose that we can extract an infinite set of integers $I_j$ from $I_{j-1}$ so that the sequence $\{\phi_{n_k-j}(r_k)\}_{k\in I_j}$ converges to a point $w_j$. Consider the sequence $S=\{\phi_{n_k-(j+1)}(r_k)\}_{k\in I_j}$. Since $n_k$ tends to infinity it is eventually greater than $j+1$ and so $S$ is non-trivial. Moreover, since

$$\delta\big(\phi_{n_k-(j+1)}(r_k),\phi_{n_k-j}(r_k)\big)\le\delta\big(r_k,\phi(r_k)\big)\to a$$

as $k$ tends to infinity, $S$ is compact as well and we can extract an infinite set of integers $I_{j+1}$ from $I_j$ so that $\{\phi_{n_k-(j+1)}(r_k)\}_{k\in I_{j+1}}$ converges to a point $w_{j+1}$. By induction, we form a sequence $\{w_j\}_{j=0}^{\infty}$.

Since points of the form $\phi_{n_k-j}(r_k)$ with $n_k>j$ are always contained in $D$, we conclude that every $w_j$ is also contained in $D$. In addition,

$$\lim_{k\in I_{j+1}}\phi\big(\phi_{n_k-(j+1)}(r_k)\big)=w_j.$$

So by continuity of $\phi$, $\phi(w_{j+1})=w_j$. Finally,

$$\delta(w_{j+1},w_j)=\lim_{k\in I_{j+1}}\delta\big(\phi_{n_k-(j+1)}(r_k),\phi_{n_k-j}(r_k)\big)\le\lim_{k\in I_{j+1}}\delta\big(r_k,\phi(r_k)\big)=a.$$

That is to say, we constructed a backward iteration sequence which satisfies (1.3).

To finish the construction, we only need to show that the $w_j$'s converge to $1$. But assume that a subsequence $w_{j'}$ converges to $\zeta\ne 1$. First assume that $\zeta\in\mathbf{D}$. Then $K=\{w_{j'}\}$ is compact in $\mathbf{D}$. By the Denjoy–Wolff theorem, there is $N$ such that $\phi_n(K)\cap K=\emptyset$, for every $n\ge N$. But this yields a contradiction, because $w_{j'}$ is a subsequence of a backward iteration sequence. So assume that $\zeta\in\partial\mathbf{D}\cap D\setminus\{1\}$. Since $\delta\big(w_{j'},\phi(w_{j'})\big)$ is bounded by $a$, Corollary 3.2 implies that $\zeta$ is a BRFP for $\phi$ with multiplier less than $A$. But this is a contradiction, by our choice of $D$. □

## 4. Proof of technical lemmas

We start by proving Corollary 2.2.

*Proof of Corollary* 2.2. It follows from (2.1) that

$$\frac{1-w_n}{1-\overline{w_n}} \longrightarrow 1 \tag{4.1}$$

as $n \to \infty$. Also, for $k = 1, 2, 3, \ldots$, $\phi_k$ has angular derivative equal to $A^k$, hence

$$\frac{1-w_n}{1-w_{n+k}} \longrightarrow A^k \tag{4.2}$$

as $n$ tends to infinity. Using (4.1) and (4.2), one sees that

$$c_n = \frac{1-\overline{w_n}w_{n+k}}{1-\overline{w_{n+k}}w_n} \longrightarrow 1 \tag{4.3}$$

and

$$b_n = \frac{w_{n+k}-w_n}{1-\overline{w_n}w_{n+k}} \longrightarrow a_k \tag{4.4}$$

as $n$ tends to $\infty$. Now, a computation shows that

$$\tau_{n+k}^{-1} \circ \tau_n(z) = c_n M_{b_n}(z).$$

Thus by (4.3) and (4.4), the first statement of Corollary 2.2 holds.

To see the second statement, write $\tau_{n+1}^{-1} \circ \eta^{-1} \circ \tau_n(z)$ as

$$\frac{1-aw_{n+1}}{1-a\overline{w_{n+1}}} \frac{1-\eta(w_{n+1})\overline{w_n}}{1-\overline{\eta(w_{n+1})}w_n} M_{\tau_n^{-1}\circ\eta(w_{n+1})}(z).$$

Then, by (4.3) and (4.4),

$$\frac{1-aw_{n+1}}{1-a\overline{w_{n+1}}} \frac{1-\eta(w_{n+1})\overline{w_n}}{1-\overline{\eta(w_{n+1})}w_n} \to 1$$

and

$$\tau_n^{-1} \circ \eta(w_{n+1}) \to 0$$

as $n$ tends to infinity. □

*Proof of Lemma* 2.1. First recall that, since $w_k$ tends to 1 and $\phi$ has angular derivative $A$ at 1, by Theorem 3.1,

$$\liminf_{k\to\infty} \frac{1-|w_k|}{1-|w_{k+1}|} \geq A.$$

On the other hand, because of (1.3), by the same argument as in the proof of Corollary 3.2

$$\frac{1-|w_k|}{1-|w_{k+1}|} \leq \frac{1+a}{1-|w_{k+1}|a}.$$

Taking the limsup of both sides, we obtain:

$$\frac{1-|w_k|}{1-|w_{k+1}|} \longrightarrow A \qquad \text{as } k\to\infty. \tag{4.5}$$

By (1.3), after some rewriting,

$$\frac{A}{(A+1)^2} \leq \frac{(1-|w_k|)(1-|w_{k+1}|)}{|1-\overline{w_k}w_{k+1}|^2}.$$

Using (1.3) again to majorize the right-hand side and then dividing both sides by

$$\frac{(|w_{k+1}|-|w_k|)^2}{(1-|w_k|)(1-|w_{k+1}|)},$$

we obtain after some algebra,

$$\frac{A}{(A-1)^2}\left(1-\frac{1-|w_{k+1}|}{1-|w_k|}\right)\left(\frac{1-|w_k|}{1-|w_{k+1}|}-1\right) \leq \left(\frac{|w_{k+1}|-|w_k|}{|w_{k+1}-w_k|}\right)^2.$$

Taking square roots and then the liminf of both sides using (4.5), we obtain

$$\liminf_{k\to\infty} \frac{|w_{k+1}|-|w_k|}{|w_{k+1}-w_k|} \geq 1,$$

i.e., for every $\varepsilon > 0$ there is $n_0$ such that for $k \geq n_0$:

$$|w_{k+1}|-|w_k| \geq (1-\varepsilon)|w_{k+1}-w_k|.$$

Thus for $m > n > n_0$:

$$\begin{aligned}|w_m|-|w_n| &= \sum_{k=n}^{m-1}(|w_{k+1}|-|w_k|) \geq (1-\varepsilon)\sum_{k=n}^{m-1}|w_{k+1}-w_k| \\ &\geq (1-\varepsilon)\left|\sum_{k=n}^{m-1}(w_{k+1}-w_k)\right| = (1-\varepsilon)|w_m-w_n|.\end{aligned}$$

Letting $m$ tend to infinity, we see that

$$1-|w_n| \geq (1-\varepsilon)|1-w_n|$$

for all $n > n_0$, i.e.,

$$\lim_{n\to\infty}\frac{1-|w_n|}{|1-w_n|} = 1$$

which says that $w_n$ tends to 1 asymptotically radially. □

## 5. Uniqueness of conjugations

We follow a line of reasoning similar to [CM, p. 70]. Given a simply connected region $\Omega \subset \mathbf{D}$, we say that $\Omega$ has an *inner tangent* at $1$, if for every angular opening $\theta \in (0, \frac{1}{2}\pi)$ and there is $r > 0$ such that the cone

$$\Gamma(\theta, r) = \{z \in \mathbf{D} : |\arg(1-z)| < \theta,\ |z-1| < r\}$$

is contained in $\Omega$.

**Lemma 5.1.** *Suppose $\psi$ is analytic in $\mathbf{D}$, has non-tangential limit $1$ at $1$ and is semi-conformal at $1$, i.e., (1.2) holds. Then, there is a simply connected region $\Omega \subset \mathbf{D}$ with an inner tangent at $1$, such that $\psi$ is one-to-one on $\Omega$ and $\psi(\Omega)$ also has an inner tangent at $1$.*

Assume Lemma 5.1 for the moment (the proof is given below).

*Proof of Theorem* 1.2 (uniqueness). Suppose that $\psi_1$ and $\psi_2$ are two conjugations satisfying (1.1) and (1.2). Let $\Omega_1$ and $\Omega_2$ be the corresponding sets given by Lemma 5.1. Given $z \in \mathbf{D}$ and a small disk $z \in \Delta \subset \overline{\Delta} \subset \mathbf{D}$ there is $N = N(z)$ such that for all $n \geq N$, $\eta_n^{-1}(\Delta) \in \Omega_1$, and $\psi_1\big(\eta_n^{-1}(\Delta)\big) \in \psi_2(\Omega_2)$. Thus we can define $\beta(z) = \eta_n \circ \psi_2^{-1} \circ \psi_1 \circ \eta_n^{-1}(z)$. One checks that this definition does not depend on $n \geq N$, hence it yields an analytic one-to-one function near $z$. Then $\beta$ can be analytically continued along any path in $\mathbf{D}$, and it gives rise to a one-to-one analytic function on $\mathbf{D}$. Finally, $\beta$ is also onto $\mathbf{D}$, so $\beta$ is a Möbius transformation. Moreover, $\beta \circ \eta^{-1}$ equals $\eta_n \circ \psi_2^{-1} \circ \psi_1 \circ \eta_n^{-1} \circ \eta^{-1}$ for some $n$, and this equals $\eta^{-1} \circ \beta$. So $\beta$ commutes with $\eta$. Finally $\psi_1 = \psi_2 \circ \beta$, hence the uniqueness part of Theorem 1.2 is proved. □

*Proof of Lemma* 5.1. By Theorem 2 of [Ya], the argument of $\psi'$ has non-tangential limit $0$ at $1$. Choose a sequence $\theta_n \downarrow 0$, and for each $n$ find $r_n$ so that $|\arg\big(\psi'(z)\big)| < 1$ for every $z \in \Gamma(\theta_n, r_n)$. Also the $r_n$'s can be chosen small enough so that the convex hull $\Omega$ determined by the points $\{\partial\Gamma(\theta_n, r_n) \cap \partial\Gamma(\theta_{n+1}, r_{n+1})\}_n$ is a convex domain with an inner tangent at $1$ and $|\arg\big(\psi'(z)\big)| < 1$ for every $z \in \Omega$. Then by Proposition 1.10 of [Po2], $\psi$ is one-to-one on $\Omega$. Finally, to see that $\psi(\Omega)$ has an inner tangent at $1$, use that fact that the image of the ray $[1, 1-e^{i\theta}]$ for a given $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, is asymptotic to the same ray $[1, 1-e^{i\theta}]$. □

## 6. Question on the growth of Kœnigs maps

Here we assume that the Denjoy–Wolff point is at the origin and the multiplier $\lambda = \phi'(0)$ is not zero. Then Kœnigs produces an analytic map $\sigma$ defined on $\mathbf{D}$ such that $\sigma \circ \phi = \lambda\sigma$. The growth of the map $\sigma$ is of interest because $\sigma$ is a formal eigenfunction of the composition operator $C_\phi(f) = f \circ \phi$. The following result can also be deduced from a sharper theorem in [Bou].

**Proposition 6.1.** *Suppose $\phi$, $\lambda$ and $\sigma$ are as above. If $\phi$ has a BRFP with multiplier $A$, then for $p > (\log A)/\log(1/|\lambda|)$,*

$$\sup_{|z|=r} |\sigma(z)|^p(1-r) \to \infty$$

*as $r \uparrow 1$.*

*Proof.* Let $a_n = (A^n - 1)/(A^n + 1)$, let $\psi$ be a conjugation satisfying (1.1), and set $w_n = \psi(a_n)$. Then $\{w_n\}_{n=0}^{\infty}$ is a backward iteration sequence for $\phi$. Moreover, $\psi$ can be chosen so that $\sigma(w_0) \neq 0$ (precompose $\psi$ with a Möbius transformation which commutes with $\eta$). Then,

$$\sigma(w_0) = \sigma\big(\phi_n(w_n)\big) = \lambda^n \sigma(w_n).$$

Also, since $\psi(\mathbf{D}) \subset \mathbf{D}$, by Schwarz's lemma and some rewriting, we always have

$$\liminf_{n\to\infty} \frac{1-|w_n|}{1-|a_n|} \geq \frac{1+|\psi(0)|}{1-|\psi(0)|}.$$

Thus,

$$\begin{aligned}|\sigma(w_n)|^p(1-|w_n|) &= |\sigma(w_0)|^p|\lambda|^{-np}(1-|a_n|)\frac{1-|w_n|}{1-|a_n|}\\ &\geq C\exp\left(n\log\frac{1}{|\lambda|}\left(p - \frac{\log A}{\log(1/|\lambda|)}\right)\right)\end{aligned}$$

for some constant $C > 0$. So Proposition 6.1 is proved. □

Under the extra hypothesis that $\phi$ is one-to-one, it follows from Corollary 3.4 of [PC] that a converse of Proposition 6.1 holds. Namely,

$$\sup_{|z|=r} |\sigma(z)|^p(1-r) \to \infty, \qquad \text{for some } p > 0, \tag{6.1}$$

if and only if some iterate $\phi_N$ of $\phi$ has a BRFP.

**Question 6.2.** Is this true without the assumption of univalence?

P. Bourdon showed in [Bou] that (6.1) is always equivalent to the fact that there exists a sequence $\zeta_n \in \partial\mathbf{D}$, for which $|\phi_n(\zeta_n)| = 1$ and $|\phi_n'(\zeta_n)| \leq M^n$ for some constant $M > 1$. It follows, for instance, that if $\phi$ is a finite Blaschke product, then (6.1) holds. On the other hand, in this case the set of BRFP's of the iterates is actually dense in $\partial\mathbf{D}$. It would be interesting to characterize those inner functions for which the set of BRFP's of the iterates is dense in $\partial\mathbf{D}$.

**Question 6.3.** Suppose $\phi$ is inner. Is (6.1) equivalent to the fact that the set of BRFP's of the iterates of $\phi$ is dense in $\partial\mathbf{D}$?

**References**


[Al] Alexander, D.: A History of Complex Dynamics. - Vieweg, 1994.

[BP] Baker, I.N., and Ch. Pommerenke: On the iteration of analytic functions in a halfplane II. - J. London Math. Soc. (2) 20, 1979, 255–258.

[Bo] Böttcher, L.E.: The principal laws of convergence of iterates and their applications to analysis. - Izv. Kazan. Fiz.-Mat. Obshch. 14, 1904, 155–234 (Russian).

[Bou] Bourdon, P.: Essential angular derivatives and maximal growth of Kœnigs eigenfunctions. - Preprint.

[CM] Cowen, C., and B. MacCluer: Composition operators on spaces of analytic functions. - CRC Press, 1995.

[CP] Cowen, C., and Ch. Pommerenke: Inequalities for the angular derivative of an analytic function in the unit disk. - J. London Math. Soc. (2) 26, 1982, 271–289.

[Koe] Kœnigs, G.: Recherches sur les intégrales de certaines équations fonctionnelles. - Ann. Sci. École Norm. Sup. (3) 1, 1884, Supplément 3–41.

[PC] Poggi-Corradini, P.: Angular derivatives at boundary fixed points for self-maps of the disk. - Proc. Amer. Math. Soc. 126, 1998, 1697–1708.

[Po1] Pommerenke, Ch.: On the iteration of analytic functions in a halfplane, I. - J. London Math. Soc. (2) 19, 1979, 439–447.

[Po2] Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. - Springer-Verlag, 1992.

[Sh] Shapiro, J.: Composition Operators and Classical Function Theory. - Springer-Verlag, 1993.

[Va] Valiron, G.: Sur l'iteration des fonctions holomorphes dans un demi-plan. - Bull. Sci. Math. (2) 55, 1931, 105–128.

[Ya] Yamashita, S.: Conformality and semiconformality of a function holomorphic in the disk. - Trans. Amer. Math. Soc. 245, 1978, 119–138.


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