Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 25, 2000, 3–30

# GROWTH OF OPERATOR VALUED MEROMORPHIC FUNCTIONS

**Olavi Nevanlinna**

Helsinki University of Technology, Institute of Mathematics
P.O. Box 1100, FIN-02015 HUT, Finland; Olavi.Nevanlinna@hut.fi

**Abstract.** We discuss two characteristic functions, $T_\infty$ and $T_1$, to measure the growth of operator valued meromorphic functions. The smaller one, $T_\infty$, is based on the operator norm, while $T_1$ works within finitely trace class meromorphic functions and is preserved in inversion. Applications to perturbation analysis of linear operators are given.

## 1. Introduction

If $f$ denotes a meromorphic function satisfying a normalization condition $f(0) = 1$, then the first main theorem in the value distribution theory, [14], [15], says in particular that

$$T(r,f) = T\left(r, \frac{1}{f}\right). \tag{1.1}$$

Here $T$ is the Nevanlinna characteristic function which measures the size of $f$ in the disc $|z| \le r$. The theory has since 1925 been generalized in many ways but not to functions taking values in a space of linear operators, where, after all, an inversion is available. The obvious way of generalizing the characteristic function $T$ to vector valued meromorphic functions goes by replacing the absolute value by the norm. This leads to a characteristic function, let us call it $T_\infty$, which codes in a natural way many properties of such meromorphic functions but supports no analogy to (1.1). In fact, if for example our meromorphic function is simply $I - zA$ where $I$ is the identity and $A$ is a bounded operator, then knowing the growth of $T_\infty(r, I - zA)$ does not tell us how its inverse $(I - zA)^{-1}$ grows as a meromorphic function—or whether it is meromorphic at all.

In numerical analysis we meet questions of the following nature. If $A$ is a bounded linear operator in a Hilbert space, how small can $\|p(A)\|$ be if $p$ is a monic polynomial of given degree, or how fast this decays with the degree, see [9], [11]. The usual tool is the holomorphic functional calculus in which you would treat the resolvent as an analytic function outside the spectrum and write

$$p(A) = \frac{1}{2\pi i} \int_\Gamma p(\lambda)(\lambda I - A)^{-1}\, d\lambda. \tag{1.2}$$

---

1991 Mathematics Subject Classification: Primary 30G30, 47A10, 47B10.

Here $\Gamma$ would surround the spectrum within a suitable distance so that the resolvent would be of moderate size along it. However, often the answers to this type of questions are not sensitive to low rank perturbations but this we cannot conclude directly from (1.2) as the spectrum, (and hence a suitable path $\Gamma$) can be a rather wildly behaving function in low rank perturbations. Think now the resolvent as a meromorphic function rather than an analytic one outside the spectrum. Multiply it by a function $\chi(\lambda) = 1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \cdots$, vanishing at the poles such that the product $\chi(\lambda)(\lambda I - A)^{-1}$ is entire in the variable $1/\lambda$. We show in this paper that the growth of $T_\infty$ as $|\lambda| \to 0$ of the resolvent is insensitive in low rank perturbations. Knowing the growth allows one to estimate $\|\chi(\lambda)(\lambda I - A)^{-1}\|$ without knowledge of the actual locations of possible poles. Finally, instead of (1.2) we would use the fact that the coefficients $p_k(A)$ in the expansion

$$\chi(\lambda)(\lambda I - A)^{-1} = \sum p_k(A)\lambda^{-1-k}$$

satisfy

$$p_k(A) = \frac{1}{2\pi i}\int_{|\lambda|=r} \lambda^k \chi(\lambda)(\lambda I - A)^{-1}\, d\lambda$$

and therefore they decay with a speed determined by the growth of $\chi(\lambda)(\lambda I - A)^{-1}$.

Another place where it is useful to think the resolvent as a meromorphic function is in estimating the powers of an operator. There are well-known results in finite dimensional spaces of the form that provided a resolvent growth condition outside the unit circle is satisfied then the powers are bounded by a constant which depends on the dimension. In finite dimensional spaces resolvents are rational functions and the dimension gives an upper bound for their degree. If we measure the resolvent as a meromorphic function using $T_\infty$ we obtain dimension free results in Hilbert spaces, see [10], [13].

The full power of the first main theorem is obtained by applying (1.1) to $f-a$, with a complex constant $a$. After all, $f$ and $f-a$ are "large" at same places. This gave a quantitative meaning for a value to be exceptional in the Picard sense. Now, a simple analogy of this with operators is the following, see Examples 5.2.7, 5.2.8 in [9], and Example 4.3 below. A solution operator to a second order differential equation with pure initial value conditions is a quasinilpotent Volterra operator while the "same" differential equation with two point boundary conditions gives a self-adjoint Fredholm operator. The analogy with initial conditions corresponding the exceptional case is obvious. Observe that the change of boundary conditions is just a rank-1 perturbation.

In [12] we proposed a characteristic function, let us call it here $T_1$, for square matrices with meromorphic elements which satisfies the analogy of (1.1):

$$T_1(r, F) = T_1(r, F^{-1}) \tag{1.3}$$

provided $F(0) = I$. The starting point was the following observation: writing

$$\log|\det F| = \sum \log^+ \sigma_j(F) - \sum \log^+ \sigma_j(F^{-1})$$

where $\sigma_j$ denote the singular values, provides the analogy of the key formula $\log|f| = \log^+|f| - \log^+(1/|f|)$.

In this paper we extend $T_1$ for functions of the form $I - F$ where $F$ is *finitely* $\mathscr{S}_1$*-meromorphic*, that is, functions which take values in the trace class with the extra property that the principal parts of poles are of finite rank each. Now, these have inverses $(I - F)^{-1}$ in the same class and the analogy of (1.3) holds.

We can use $T_1$ in perturbation analysis as follows. Suppose $F^{-1}$ and $G$ are meromorphic, then

$$T_\infty(r, (F+G)^{-1}) \leq T_\infty(r, F^{-1}) + T_\infty\big(r, (I + F^{-1}G)^{-1}\big).$$

If now $F$ and $G$ are such that $F^{-1}G$ is finitely $\mathscr{S}_1$-meromorphic, then

$$T_\infty\big(r, (I + F^{-1}G)^{-1}\big) \leq T_1\big(r, (I + F^{-1}G)^{-1}\big) = T_1(r, I + F^{-1}G) + C$$

where $C$ depends on $F^{-1}G$ at the origin.

The basic concepts are presented in Section 2, Section 3 contains the identity for inversion, while in Section 4 we present some applications to perturbation theory. Finally, the concepts are applied to the resolvent in Section 5.

## 2. Basic definitions

In this paper $H$ is a separable complex infinite dimensional Hilbert space. We discuss two different extensions of the theory of scalar meromorphic functions to operator valued functions. The first one reduces to the scalar theory when applied to operators of the form $fI$ where $f$ is a meromorphic scalar function and $I$ the identity operator. The second one agrees with the scalar theory when applied to operators of the form $\operatorname{diag}(f, 1, 1, 1, \ldots)$.

Let

$$F\colon z \mapsto F(z) = \sum_{-m}^{\infty} A_j (z - z_0)^j \tag{2.1}$$

be such that it has expansions of this form around every $z_0$ in $|z_0| < R \leq \infty$. Here $A_j$ are bounded linear operators in $H$ and $\sum_{-m}^{\infty} \|A_j\| \eta^j < \infty$ for some $\eta > 0$, and $A_{-m}$ is nontrivial.

**Definition 2.1.** Let $F$ be a meromorphic operator valued function as above. If $-m < 0$, then $F$ has a pole at $z_0$ of order $m$, otherwise $F$ is analytic at $z_0$. Denote $m(z_0) := \max\{m, 0\}$ and define

$$n_\infty(r, F) := \sum_{|b| \le r} m(b). \tag{2.2}$$

Thus $n_\infty$ counts the poles together with their orders. We then define as usual

$$N_\infty(r, F) := \int_0^r \frac{n_\infty(t, F) - n_\infty(0, F)}{t}\, dt + n_\infty(0, F) \log r. \tag{2.3}$$

Also, we set

$$m_\infty(r, F) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \|F(re^{i\phi})\|\, d\phi \tag{2.4}$$

and

$$T_\infty(r, F) := m_\infty(r, F) + N_\infty(r, F). \tag{2.5}$$

Finally, we denote

$$M_\infty(r, F) := \sup_{|z|=r} \|F(e^{i\phi})\|. \tag{2.6}$$

We collect the main properties of $T_\infty$ into the following two theorems.

**Theorem 2.1.** *Let $F$, $G$ and $F_i$ be meromorphic for $|z| < R \le \infty$. Then $T_\infty(r, F)$ is a nonnegative and nondecreasing function in $r$ for $0 \le r < R$ which is convex in the variable $\log r$. The following inequalities hold*

$$T_\infty(r, FG) \le T_\infty(r, F) + T_\infty(r, G), \tag{2.7}$$

$$T_\infty\left(r, \sum_1^k F_i\right) \le \sum_1^k T_\infty(r, F_i) + \log k. \tag{2.8}$$

*Proof.* The proof is as for the scalar case. See any recent text book on the topic, or compare with the proof of Theorem 2.4. The important fact is that $\log^+ \|F\|$ is subharmonic whenever $F$ is analytic [1].

**Theorem 2.2.** *If $F$ is analytic for $|z| < R$ and $0 < r < \theta r < R$, then*

$$T_\infty(r, F) \le \log^+ M_\infty(r, F) \le \frac{\theta + 1}{\theta - 1} T_\infty(\theta r, F). \tag{2.9}$$

*Proof.* Again, since $\log^+ \|F\|$ is subharmonic if $F$ is analytic, the claim follows using the Poisson–Jensen formula as in the scalar case.

If $f$ is a scalar meromorphic function (with $f(0) = 1$) then by the first main theorem

$$T(r, f) = T\left(r, \frac{1}{f}\right).$$

It is clear that we cannot have an identity in the operator valued case for $T_\infty$. For example, if

$$F\colon z \mapsto I - zA$$

where $A$ is a bounded operator, then $T_\infty(r, F) = \log^+ r + \mathscr{O}(1)$ but the inverse $(I - zA)^{-1}$ need not even be meromorphic in the whole plane.

**Example 2.1.** Let in particular $H := l_2$ and $A := \operatorname{diag}(\alpha_j)$ where $\{\alpha_j\}$ is a decreasing sequence of positive numbers. Now $(I - zA)^{-1}$ has poles at $1/\alpha_j$ and it is meromorphic in the whole plane if and only if $\lim \alpha_j = 0$. That is, if and only if $A$ is compact. Observe that

$$m_\infty\bigl(r, (I - zA)^{-1}\bigr) = \mathscr{O}(1)$$

no matter whether $A$ is compact. All growth in $T_\infty\bigl(r, (I - zA)^{-1}\bigr)$ comes from counting the poles (this is true for all self-adjoint operators, see Theorem 5.5). In fact

$$T_\infty\bigl(r, (I - zA)^{-1}\bigr) = \sum \log^+(\alpha_j r) + \mathscr{O}(1).$$

Observe that here the eigenvalues $\alpha_j$ are also the singular values of $A$. If we like to conclude from the growth of $I - zA$ the growth for its inverse we have to see more growth in it. We do this by including all the singular values into the characteristic function, and not just the largest one, the norm.

**Definition 2.2.** The singular values $\sigma_j(B)$ of a bounded operator $B$ are

$$\sigma_j(B) := \inf_{\operatorname{rank}(B_j) < j} \|B - B_j\|. \tag{2.10}$$

Also, we set

$$\sigma_\infty(B) := \lim_{j \to \infty} \sigma_j(B). \tag{2.11}$$

**Remark 2.1.** It is well known that $K$ is compact if and only if $\sigma_\infty(K) = 0$.

**Lemma 2.1.** *If $K$ is compact, then*

$$\sigma_\infty(I - K) = 1 \tag{2.12}$$

*and in particular $\sigma_j(I - K) \geq 1$ for all $j$.*

*Proof.* Since $K$ is compact, there exists a sequence of finite rank operators, $\{A_j\}$ converging to $K$ and thus

$$\sigma_\infty(I-K) = \lim_{i\to\infty} \sigma_i(I-K) \le \liminf \|I-K+A_j\| \le 1.$$

If the inequality would be proper, then for some $k$, $\sigma_k(I-K) < 1$ and further, there would be an operator $K_k$ of rank less than $k$ such that

$$\|I-K-K_k\| < 1.$$

Taking large enough $j$ we would then have

$$\|I-A_j-K_k\| + \|K-A_j\| < 1.$$

But $A_j+K_k$ is finite dimensional and its distance to identity cannot be less than $1$.

**Definition 2.3.** A compact operator $K$ is in the Schatten class $\mathscr{S}_p$, if

$$\|K\|_p := \Big(\sum_j \sigma_j(K)^p\Big)^{1/p} < \infty.$$

**Lemma 2.2.** *If $K \in \mathscr{S}_1$, then*

$$\sum_j \log^+ \sigma_j(I-K) \le \|K\|_1. \tag{2.13}$$

*Proof.* The claim follows from $\sigma_j(I-K) \le 1+\sigma_j(K)$ and $\log^+\big(1+\sigma_j(K)\big) \le \sigma_j(K)$.

**Definition 2.4.** Let $A$ be a bounded linear operator. We define [12] the *total logarithmic size* of $A$ by

$$s(A) := \sum_j \log^+ \sigma_j(A) \tag{2.14}$$

whenever finite.

**Lemma 2.3.** *If $A$ and $B$ are compact, then for all $k$*

$$\sum_1^k |\sigma_j(I+A+B) - \sigma_j(I+A)| \le \sum_1^k \sigma_j(B). \tag{2.15}$$

*Proof.* Fix any $k$. If $B$ is bounded one can define $|\!|\!|B|\!|\!|_k := \sum_{j=1}^k \sigma_j(B)$. These are sometimes called Ky Fan norms. In finite dimensional spaces they have the following property. Let $\Sigma(B) := \operatorname{diag}\bigl(\sigma_j(B)\bigr)$ where the singular values are listed decreasingly. Then

$$|||\Sigma(A) - \Sigma(B)|||_k \le |||A - B|||_k,$$

see [6, p. 448]. So, (2.15) holds in finite dimensional spaces. Since $A$ and $B$ are compact we can approximate them within any $\varepsilon > 0$ by finite rank operators $A_m$ and $B_n$. But then we can take a large enough finite dimensional subspace $H_0$ such that $A_m$ and $B_n$ are invariant in it, and vanish in the orthogonal complement. We can then add extra dimensions to it, so many as needed to guarantee that the $k$ largest singular values of $I + A_m$ and of $I + A_m + B_n$, when restricted to that subspace, are all $\ge 1$. This is possible by Lemma 2.1. But then the result in finite dimensional spaces can be used and we have

$$\begin{aligned}\sum_1^k |\sigma_j(I+A+B) - \sigma_j(I+A)| &\le \sum_1^k |\sigma_j(I+A_m+B_n) - \sigma_j(I+A_m)| + 3k\varepsilon \\ &\le \sum_1^k \sigma_j(B) + 4k\varepsilon.\end{aligned}$$

**Continuity Lemma 2.4.** *If $A,\ B \in \mathscr{S}_1$, then*

$$|s(I+A) - s(I+B)| \le \|A - B\|_1. \tag{2.16}$$

*Proof.* If $a$ and $b$ are nonnegative numbers then

$$|\log(1+a) - \log(1+b)| \le |a-b|.$$

But $\sigma_j(I+A),\ \sigma_j(I+B) \ge 1$ so that

$$|\log^+ \sigma_j(I+A) - \log^+ \sigma_j(I+B)| \le |\sigma_j(I+A) - \sigma_j(I+B)|.$$

The conclusion now follows from Lemma 2.3.

**Theorem 2.3.** *Suppose $F\colon z \mapsto F(z)$ is an analytic $\mathscr{S}_1$-valued function for $|z - z_0| < R_0$. Then the function $u$:*

$$z \mapsto u(z) := s\bigl(I - F(z)\bigr) = \sum_j \log^+ \sigma_j\bigl(I - F(z)\bigr)$$

*is continuous and subharmonic for $|z - z_0| < R_0$.*

To prove this we first state some lemmas.

**Approximation Lemma 2.5.** *If $F$ is an analytic $\mathscr{S}_1$-valued function in $|z-z_0|<R_0$, then $F$ can be approximated in $\mathscr{S}_1$ by finite rank polynomials uniformly in discs $|z-z_0|\le\eta<R_0$.*

*Proof.* Let the Taylor coefficients of $F$ at $z_0$ be $A_j$. Then in particular

$$\sum_{n+1}^{\infty}\|A_j\|_1\eta^j\to 0 \qquad \text{as } n\to\infty.$$

Let $A_j^{(n)}$ be a finite rank approximation to $A_j$ such that

$$\|A_j-A_j^{(n)}\|_1<\frac{1}{n}\frac{1}{j!}.$$

Set

$$F_n(z):=\sum_{j=0}^{n}A_j^{(n)}(z-z_0)^j.$$

Then for $|z-z_0|<\eta$

$$\|F(z)-F_n(z)\|_1\le\sum_{j=0}^{n}\frac{1}{n}\frac{1}{j!}\eta^j+\sum_{n+1}^{\infty}\|A_j\|_1\eta^j\le\frac{1}{n}e^{\eta}+\sum_{n+1}^{\infty}\|A_j\|_1\eta^j\to 0$$

as $n\to\infty$.

**Lemma 2.6.** *The function $\sum\log^+\sigma_j\big(F(z)\big)$ is subharmonic for analytic $F$ in finite dimensional spaces.*

*Proof.* This is in [2].

*Proof of Theorem* 2.3. We start by approximating $F$ by a finite rank polynomial $F_n$ in a neighborhood of a point $z_0$. Then there exists a subspace, say $H_n$ of dimension $d\le 2\,\mathrm{rank}(F_n)$ such that $F_n$ is invariant in $H_n$ and vanishes in the orthogonal complement. Now, at most $d$ singular values of $I-F_n(z)$ can be different from $1$. Denote by $A_n(z)$ the finite dimensional operator obtained by restricting $I-F_n(z)$ to $H_n$. Then by Lemma 2.6

$$z\mapsto\sum\log^+\sigma_j\big(A_n(z)\big)$$

is subharmonic. It is continuous as $A_n$ is a polynomial and the singular values are continuous. By construction, however

$$\sum\log^+\sigma_j\big(A_n(z)\big)=\sum\log^+\sigma_j\big(I-F_n(z)\big).$$

But combining the Approximation and Continuity lemmas we conclude that

$$z\mapsto\sum\log^+\sigma_j\big(I-F(z)\big)=s\big(I-F(z)\big)$$

is the uniform limit of subharmonic continuous functions and therefore itself also subharmonic and continuous.

Theorem 2.3 suggests that we could look at the following quantities.

**Definition 2.5.** If $F$ is an analytic $\mathscr{S}_1$-valued function then we set

$$M_1(r, I-F) := \sup_{|z|=r} \prod_1^\infty \sigma_j\big(I - F(z)\big) \tag{2.17}$$

and

$$m_1(r, I-F) := \frac{1}{2\pi}\int_{-\pi}^{\pi} s\big(I - F(re^{i\phi})\big)\, d\phi. \tag{2.18}$$

Recall, that in (2.17) $\sigma_j\big(I - F(z)\big) \geq 1$ by Lemma 2.1.

**Example 2.2.** An operator valued function can be entire in the uniform norm but have a finite domain if considered as an analytic function taking values in $\mathscr{S}_1$. In fact, let $A_j$ be a diagonal operator with positive decaying diagonal elements $\alpha_{j,k}$ such that

$$\sum_{k=1}^\infty \alpha_{j,k} = 1$$

while $\alpha_{j,1} = 1/j!$. Then $\|A_j\| = 1/j!$ while $\|A_j\|_1 = 1$ and if $F(z) := \sum_j A_j z^j$ then

$$\|F(r)\|_1 = \frac{r}{1-r}$$

while

$$\|F(r)\| = e^r - 1.$$

For $1 + x \leq e$ we have $x/(e-1) \leq \log(1+x) \leq x$ which then gives

$$\frac{1}{(e-1)(1-r)} \leq \log M_1(r, I+F) \leq \frac{1}{1-r}$$

while

$$\log M_\infty(r, I+F) = r.$$

What still remains, is to specify what we here mean by a pole and how to count their multiplicities. As long as $F$ is analytic as an $\mathscr{S}_1$-valued function, either $I - F(z)$ has an inverse

$$\big(I - F(z)\big)^{-1} = I + F(z)\big(I - F(z)\big)^{-1}$$

with

$$F(z)\big(I - F(z)\big)^{-1} \in \mathscr{S}_1,$$

or 1 is an eigenvalue of $F(b)$ with a finite dimensional eigenspace. In particular, at such a point $b$, $F(z)\big(I - F(z)\big)^{-1}$ has an expansion of the form

$$\sum_{-m}^\infty B_j (z-b)^j$$

where the operators $B_j$ are all in $\mathscr{S}_1$ and of finite rank when $j < 0$.

**Example 2.3.** Let $F(z) := \operatorname{diag}\bigl(\alpha_j/(1-z)\bigr)$ where $\{\alpha_j\}$ is a decreasing sequence of positive numbers so that their sum is finite. Then $F$ is clearly a meromorphic $\mathscr{S}_1$-valued function in the whole plane. However, the inverse

$$(I-F)^{-1} = I + \operatorname{diag}\left(\frac{\alpha_j}{1-z-\alpha_j}\right)$$

has an accumulation point of poles at $1$ and we see that it is essential to assume the principal parts to be of finite rank.

**Definition 2.6.** We say that a meromorphic operator valued function $F$ is *finitely $\mathscr{S}_1$-meromorphic* in $|z| < R_1 \le \infty$, for short $F \in \mathscr{F}_1(R_1)$, if the Laurent-series (2.1) converges in $\mathscr{S}_1$, and the coefficients $A_j$ in the *principal part*

$$\sum_{-m}^{-1} A_j (z-z_0)^j$$

are of finite rank, at every pole $z_0$.

In [16] there are some related concepts. In particular, compact operator valued functions such that the principal parts are of finite rank are there called essentially meromorphic.

**Lemma 2.7.** *If $F, G \in \mathscr{F}_1(R)$, then*

$$FG \in \mathscr{F}_1(R), \tag{2.19a}$$

$$F+G \in \mathscr{F}_1(R), \tag{2.19b}$$

*and*

$$F(I-F)^{-1} \in \mathscr{F}_1(R). \tag{2.19c}$$

*If $A$ is an analytic operator valued function in $|z| < R$, then also*

$$AF \quad \text{and} \quad FA \in \mathscr{F}_1(R). \tag{2.19d}$$

*Proof.* All claims are easy. Recall that for linear operators $A, B$: $\|AB\|_1 \le \|A\|\|B\|_1$.

**Definition 2.7.** If $F \in \mathscr{F}_1(R)$ then set

$$\mu(z_0) := \limsup_{z \to z_0} \frac{s\bigl(I - F(z)\bigr)}{\log \dfrac{1}{|z-z_0|}}. \tag{2.20}$$

**Lemma 2.8.** *If $F \in \mathscr{F}_1(R)$, then at poles $b$, $\mu(b)$ is a positive integer, depending only on the principal part $\sum_{-m}^{-1} A_j (z-b)^j$.*

*Proof.* Write

$$F(z) =: \sum_{-m}^{-1} A_j(z-b)^j + G(z)$$

with $G$ analytic near $b$. Then by the Continuity Lemma

$$\big|s\big(I - F(z)\big) - s\big(I - F(z) + G(z)\big)\big| \le \|G(z)\|_1.$$

But $\|G\|_1$ is bounded near $b$ and has thus no effect on $\mu(b)$. Now

$$F(z) - G(z) = \sum_{-m}^{-1} A_j(z-b)^j$$

is of finite rank and therefore $(I - F + G)$ only has a finite number, say $n$, singular values which are not identically 1. As for any $A$, $|\sigma_j(I-A) - \sigma_j(A)| \le 1$, we have

$$\big|s\big(I - F(z) + G(z)\big) - s\big(F(z) - G(z)\big)\big| \le n$$

which shows that $\mu(b)$ depends only on the principal part.

We can now assume without restricting the generality that $b = 0$ and that $F(z) = \sum_{-m}^{-1} A_j(z-b)^j$ is a $d \times d$ matrix. Let $\lambda_j(z)$ denote the eigenvalues of $F(z)^*F(z)$, ordered decreasingly. These are nonnegative and their square roots are the singular values. The characteristic polynomial can be "expanded by diagonal elements":

$$\det\big(\lambda I - F(z)^*F(z)\big) = \lambda^d - b_1(z)\lambda^{d-1} + \cdots + (-1)^d b_d(z)$$

where $b_1 = \sum \lambda_j$, $b_2 = \sum_{i \ne j} \lambda_i\lambda_j$ etc. As all eigenvalues are nonnegative, the functions $b_j$ are nonnegative as well. Also, if $b_k = 0$ then $b_j = 0$ for $j = k+1, \ldots, d$. The coefficients $b_j$ are sums of all principal minors of order $j$ in $\det(F^*F)$. These are determinants of $j \times j$ submatrices which in turn are of the form $F_j^*F_j$ where each $F_j$ is a $d \times j$ matrix consisting of $j$ columns of $F$. Let $I_j$ denote a selection of $j$ rows from a matrix so that $F_j(I_j)$ denotes a $j \times j$ submatrix of $F_j$. The Cauchy–Binet Theorem allows us to conclude that then

$$b_j = \sum |\det F_j(I_j)|^2$$

where the summation is over all $j \times j$ minors $F_j(I_j)$ of $F$. But determinants are meromorphic and therefore there exists $c_j > 0$ and an integer $m_j$ such that

$$b_j(z) = c_j\big(1 + o(1)\big)r^{2m_j}$$

as $|z| = r \to 0$. Consider now $b_1 = \sum_1^d \lambda_j$. As the eigenvalues are numbered decreasingly we have

$$\frac{1}{d} b_1 \le \lambda_1 \le b_1$$

which further implies

$$\frac{c_1}{d} \le \liminf_{z\to 0} \frac{\lambda_1(z)}{r^{2m_1}} \le \limsup_{z\to 0} \frac{\lambda_1(z)}{r^{2m_1}} \le c_1.$$

For the coefficient $b_2$ we have in the same way

$$\lambda_1\lambda_2 \le b_2 \le \binom{d}{2} \lambda_1\lambda_2.$$

This implies

$$\frac{c_2}{c_1\binom{d}{2}} \le \liminf_{z\to 0} \frac{\lambda_2(z)}{r^{2(m_2-m_1)}} \le \limsup_{z\to 0} \frac{\lambda_2(z)}{r^{2(m_2-m_1)}} \le \frac{c_2 d}{c_1}.$$

Continuing this way we see that if $\lambda_j$ is not identically $0$, then there exists constants $a_j > 0$ and an integer $k_j$ such that

$$a_j^2 \le \liminf_{z\to 0} \frac{\lambda_j(z)}{r^{2k_j}} \le \limsup_{z\to 0} \frac{\lambda_j(z)}{r^{2k_j}} \le \frac{1}{a_j^2}.$$

Taking the logarithm and dividing by $2$ gives

$$\begin{aligned} \log a_j &\le \liminf_{z\to 0} \left[ \log \sigma_j\big(F(z)\big) + k_j \log \frac{1}{r} \right] \\ &\le \limsup_{z\to 0} \left[ \log \sigma_j\big(F(z)\big) + k_j \log \frac{1}{r} \right] \le \log \frac{1}{a_j}. \end{aligned}$$

Since the eigenvalues were ordered decreasingly there is a $J$ such that $k_j < 0$ for $j \le J$. Summing over $j$ then gives

$$\begin{aligned} \alpha &\le \liminf_{z\to 0} \left[ \sum \log^+ \sigma_j\big(F(z)\big) + \sum_{j=1}^J k_j \log \frac{1}{r} \right] \\ &\le \limsup_{z\to 0} \left[ \sum \log^+ \sigma_j\big(F(z)\big) + \sum_{j=1}^J k_j \log \frac{1}{r} \right] \le \beta \end{aligned}$$

where $\alpha := \sum_{j=1}^J \log a_j$ and $\beta := \sum_{j=1}^J \log(1/a_j)$. Thus, in particular, $\mu(0) := -\sum_{j=1}^J k_j$ is an integer.

**Lemma 2.9.** *If $F \in \mathscr{F}_1(R)$, then*

$$\mu(z_0) = \lim_{z \to z_0} \frac{s\big(I - F(z)\big)}{\log \dfrac{1}{|z - z_0|}}, \tag{2.21}$$

*and there are constants $\alpha$ and $\beta$ such that*

$$\begin{aligned}\alpha &\le \liminf_{z \to z_0} \left[ s\big(I - F(z)\big) - \mu(z_0) \log \frac{1}{|z - z_0|} \right] \\ &\le \limsup_{z \to z_0} \left[ s\big(I - F(z)\big) - \mu(z_0) \log \frac{1}{|z - z_0|} \right] \le \beta.\end{aligned}$$

*Proof.* These inequalities were actually derived while proving the previous lemma. The limit in (2.21) is then obtained by dividing with $\log\big(1/(|z - z_0|)\big)$.

**Definition 2.8.** If $F \in \mathscr{F}_1(R_1)$, then for $|z| \le R < R_1$ set (when $z$ is not a pole)

$$u_R(z) := s\big(I - F(z)\big) + \sum_{|b| \le R} \mu(b) \log \left| \frac{R(z - b)}{R^2 - \bar{b}z} \right|. \tag{2.22}$$

At poles $b$ define $u_R(b) := \limsup_{z \to b} u_R(z)$.

**Lemma 2.10.** *$u_R$ is subharmonic in $|z| \le R$ and equals $s(I - F)$ on $|z| = R$.*

*Proof.* By Lemma 2.9 $u_R$ is bounded near poles and as it is a sum of subharmonic and harmonic functions except at poles we conclude that $u_R$ is subharmonic also at poles.

It is now natural to count the multiplicities as follows.

**Definition 2.9.** If $F \in \mathscr{F}_1(R_1)$, then for $r < R_1$ denote

$$n_1(r, I - F) := \sum_{|b| \le r} \mu(b). \tag{2.23}$$

Likewise,

$$N_1(r, I - F) := \int_0^r \frac{n_1(t, I - F) - n_1(0, I - F)}{t}\, dt + n_1(0, I - F) \log r. \tag{2.24}$$

As $m_1$ was given already in Definition 2.5 we can finally set

$$T_1(r, I - F) := m_1(r, I - F) + N_1(r, I - F).$$

**Lemma 2.11.** *If $F \in \mathscr{F}_1(R_1)$, then in the notation above, for $r \le R < R_1$*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_R(re^{i\phi})\,d\phi = T_1(r, I-F) - N_1(R, I-F). \tag{2.25}$$

*Proof.* This is a direct calculation, based on

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|a - e^{i\phi}|\,d\phi = \log^+|a|.$$

The following theorem summarizes the main properties of the characteristic function $T_1$.

**Theorem 2.4.** *If $F$ is finitely $\mathscr{S}_1$-meromorphic in $|z| < R_1 \le \infty$, then $T_1(r, I-F)$ is well defined, nonnegative, nondecreasing in $r < R_1$ such that it is convex as a function of $\log r$. It satisfies*

$$T_\infty(r, I-F) \le T_1(r, I-F).$$

*If $G$ is another function in $\mathscr{F}_1(R_1)$, then*

$$T_1\big(r, (I-F)(I-G)\big) \le T_1(r, I-F) + T_1(r, I-G). \tag{2.26}$$

*Proof.* Positivity of $T_1$ is clear from the definition. It is increasing and convex in the variable $\log r$ by Lemma 2.11 as this is a general fact of mean values of subharmonic functions, see [5, p. 127]. The inequality (2.26) follows from the next lemma.

**Lemma 2.12.** *If $A, B \in \mathscr{S}_1$, then*

$$s\big((I-A)(I-B)\big) \le s(I-A) + s(I-B). \tag{2.27}$$

*Proof.* It is clear from the previous proofs that this inequality again follows by approximation arguments if we have for all $d \times d$-matrices $A, B$

$$s(AB) \le s(A) + s(B).$$

However, this follows easily from the following inequality of A. Horn, see [7, Theorem 3.3.4]: for all $k \le d$

$$\prod_1^k \sigma_j(AB) \le \prod_1^k \sigma_j(A)\sigma_j(B).$$

The analogue of Theorem 2.2 holds also for $T_1$, by the same argument, since $s(I-F)$ is subharmonic.

**Theorem 2.5.** *If $F$ is an analytic $\mathscr{S}_1$-function for $|z| < R_1$ and $0 < r < \theta r < R_1$, then*

$$T_1(r, I-F) \le \log^+ M_1(r, I-F) \le \frac{\theta+1}{\theta-1} T_1(\theta r, I-F). \tag{2.28}$$

### 3. An identity for inversion

We shall establish the identity starting from the corresponding identity for the determinant function.

**Lemma 3.1.** *Let $f$ be a meromorphic scalar function in $|z| < R$. Let $\{a_j\}$ denote the zeros and $\{b_j\}$ the poles of $f$ (only nonzero ones and repeated if multiple). Assume that at the origin*

$$f(z) = \sum_{j=-\nu}^{\infty} c_j z^j, \tag{3.1}$$

*where $c_{-\nu}$ is the first nonvanishing coefficient. Then for $r < R$*

$$\log|c_{-\nu}| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\phi})|\,d\phi - \sum \log^+ \frac{r}{|a_j|} + \sum \log^+ \frac{r}{|b_j|} + \nu \log r. \tag{3.2}$$

*Proof.* This is the standard starting point in proving the first main theorem, see [15].

**Lemma 3.2.** *If $F \in \mathscr{F}_1(R)$, then $\det\big(I - F(z)\big)$ is meromorphic for $|z| < R$ and for $z$ not a pole of $F$, nor a zero of $\det\big(I - F(z)\big)$ we have*

$$\log\big|\det\big(I - F(z)\big)\big| = s\big(I - F(z)\big) - s\Big(\big(I - F(z)\big)^{-1}\Big). \tag{3.3}$$

*Proof.* Let $F$ be given and fix $\eta < R$. Then there are only a finite number of poles $b_j$ such that $|b_j| \le \eta$. We can write

$$F = \sum P_j + G$$

where $P_j$ is the principal part of $F$ at $b_j$, and $G$ is analytic for $r < \eta + \delta$ for some $\delta > 0$. By Approximation Lemma 2.5 there exist finite rank polynomials $G_n$ approximating $G$ in $\mathscr{S}_1$ uniformly in the disc $|z| \le \eta$. Set now $F_n := \sum P_j + G_n$. As $F_n$ is of finite rank, $\det\big(I - F_n(z)\big)$ is a rational function. Since the poles are not moving with $n$ take small neighborhoods of poles out and then the rational functions converge uniformly to $\det\big(I - F(z)\big)$ by definition of det for operators of the form $I - A$ with $A \in \mathscr{S}_1$, see e.g. [3]. The limit function is analytic outside these small neighborhoods. Now remove the poles by multiplying the rational functions by $(z - b_j)^{\nu_j}$ and conclude that the limit function $\det\big(I - F(z)\big)$ has a pole of the same multiplicity $\nu_j$ at $b_j$. Notice in particular that there may be poles of $F$ which are not poles of $\det(I - F)$, i.e. $\nu_j$ can be 0.

To prove the identity (3.3) fix $z$ which is not a pole of $F$ and such that $\det\big(I - F(z)\big)$ does not vanish. Denote $F(z) =: A$. We want to show that if 1 is not an eigenvalue of $A \in \mathscr{S}_1$, then

$$\log|\det(I - A)| = s(I - A) - s\big((I - A)^{-1}\big).$$

Choose a sequence $\{A_n\}$ converging to $A$ in $\mathscr{S}_1$ with $\operatorname{rank}(A_n) \le n$. Let $H_n$ be a subspace of dimension $d \le 2n$ such that $A_n$ is invariant in $H_n$ and vanishes in the orthogonal complement. We may also assume $n$ to be large enough so that for $j \ge n$

$$\|(I - A_j)^{-1}\| \le 2\|(I - A)^{-1}\|.$$

In particular, then

$$\|(I - A)^{-1} - (I - A_j)^{-1}\|_1 \le 2\|(I - A)^{-1}\|^2 \|A - A_j\|_1$$

shows that also the differences of the inverses converge in $\mathscr{S}_1$. Let $B_n$ denote the restriction of $I - A_n$ to $H_n$. Then also, $B_n^{-1}$ is the restriction of the inverse of $I - A_n$ to $H_n$ and the determinant satisfies $\det B_n = \det(I - A_n)$. However, the singular values of $B_n$ and those of $I - A_n$ agree only as long as they are larger than 1. But when $\sigma_k(B_n) < 1$ we can write

$$\sigma_k(B_n) = \frac{1}{\sigma_{d+1-k}(B_n^{-1})}$$

and here $\sigma_{d+1-k}(B_n^{-1})$ is again a singular value of $(I - A_n)^{-1}$. Since $|\det B_n| = \prod_1^d \sigma_k(B_n)$ we obtain

$$\log|\det(B_n)| = \sum \log^+ \sigma_k(B_n) - \sum \log^+ \sigma_k(B_n^{-1})$$

and, written differently,

$$\log|\det(I - A_n)| = s(I - A_n) - s\big((I - A_n)^{-1}\big).$$

Letting $n \to \infty$ gives now the identity as both $\det$ and $s$ are continuous in $\mathscr{S}_1$.

Lemma 3.2 gives immediately, with $f = \det(I - F)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\phi})|\, d\phi = m_1(r, I - F) - m_1\big(r, (I - F)^{-1}\big). \tag{3.4}$$

Also, (3.3) and Definition 2.9 imply that

$$N(r, f) - N\left(r, \frac{1}{f}\right) = N_1(r, I - F) - N_1\big(r, (I - F)^{-1}\big)) \tag{3.5}$$

where $N(r, f)$ denotes the usual averaged counting function of $f = \det(I - F)$:

$$N(r, f) = \sum \log^+ \frac{r}{|b_j|} + \max\{\nu, 0\}\log r. \tag{3.6}$$

**Theorem 3.1.** *Let $F$ be finitely $\mathscr{S}_1$-meromorphic in $|z| < R$ and if around origin*

$$\det\big(I - F(z)\big) = c_{-\nu} z^{-\nu} + c_{-\nu+1} z^{-\nu+1} + \cdots$$

*then for $r < R$*

$$T_1(r, I - F) = T_1\big(r, (I - F)^{-1}\big) + \log|c_{-\nu}|. \tag{3.7}$$

*Proof.* We only have to substitute (3.4) and (3.5) into (3.2).

**Remark 3.1.** Notice that Lemma 2.9 and (3.3) give

$$\log|c_{-\nu}| = a(I-F) - a\big((I-F)^{-1}\big) \tag{3.8}$$

where

$$a(I-F) := \limsup_{z\to 0}\left[s\big(I-F(z)\big) - \mu(0)\log\frac{1}{|z|}\right].$$

This allows one to write the identity (3.7) in a symmetric form with no reference to the determinant function:

$$T_1(r, I-F) - a(I-F) = T_1\big(r,(I-F)^{-1}\big) - a\big((I-F)^{-1}\big). \tag{3.9}$$

If $F$ is finitely meromorphic in a larger Schatten class, then Theorem 3.1 can still be used to give an upper bound for the inverse.

**Theorem 3.2.** *If $m$ is a positive integer such that $F^m \in \mathscr{F}_1(R)$, then for $r<R$*

$$T_\infty\big(r,(I-F)^{-1}\big) \le T_1(r, I-F^m) + (m-1)\big[T_\infty(r,F)+\log 2\big] - \log|c_{-\nu}| \tag{3.10}$$

*where $c_{-\nu}$ is as in (3.1) with $f=\det(I-F^m)$.*

*Proof.* For any bounded $A$ with 1 not in the spectrum we have

$$(I-A)^{-1} = (I-A^m)^{-1}\prod_{j=1}^{m-1}(I-e^{i\phi_j}A), \tag{3.11}$$

where $\phi_j := 2\pi j/m$. Thus, by (2.7) we have

$$T_\infty\big(r,(I-F)^{-1}\big) \le T_\infty\big(r,(I-F^m)^{-1}\big) + T_\infty\left(r, \prod_{j=1}^{m-1}(I-e^{i\phi_j}F)\right).$$

Here, by Theorem 3.1

$$T_\infty\big(r,(I-F^m)^{-1}\big) \le T_1\big(r,(I-F^m)^{-1}\big) = T_1(r, I-F^m) - \log|c_{-\nu}|$$

while Theorem 2.1 implies

$$T_\infty\left(r, \prod_{j=1}^{m-1}(I-e^{i\phi_j}F)\right) \le \sum_{j=1}^{m-1}\big[T_\infty(r,F)+\log 2\big].$$

The first main theorem in the scalar case is obtained from the inversion identity by noting that $f$ and $f-a$ are large at the same time, whatever constant $a$ we choose. Inverting $f-a$ results in a statement on how often $f$ is close to $a$. As

$$T(r,f) = T\left(r, \frac{1}{f-a}\right) + \mathscr{O}(1)$$

$f$ visits $a$ equally often, independently of the value $a$. In the operator valued case nothing like this can hold if the "value" is taken as an "arbitrary" operator. We formulate a version here where the function $F$ in $\mathscr{F}_1$ is perturbed by a small constant operator $A$. Our main use of Theorem 3.1 is in perturbation theory and that is discussed in the next chapter.

**Theorem 3.3.** *Let $F \in \mathscr{F}_1(R)$ and $A \in \mathscr{S}_1$ be given. Then*

$$T_1\big(r,(I-F-A)^{-1}\big) = T_1(r, I-F) + \log|c_{-\nu}| + \varepsilon(r,A) \tag{3.12}$$

*where $c_{-\nu}$ is the first nonzero coefficient in the expansion of $\det[(I-F-A)^{-1}]$ at origin and*

$$|\varepsilon(r,A)| \le \|A\|_1.$$

*Proof.* By the Continuity Lemma we have $|m_1(I-F-A)-m_1(I-F)| \le \|A\|_1$ and since $F$ and $F-A$ have the same poles

$$|T_1(r,I-F-A)-T_1(r,I-F)| = |m_1(I-F-A)-m_1(I-F)| \le \|A\|_1.$$

## 4. Perturbation results

Our first perturbation result concerns with perturbations of meromorphic operators by finite rank ones.

**Theorem 4.1.** *Assume that $F$, $F^{-1}$ and $G$ are meromorphic for $|z| < R$ and* rank $G \le k$. *Then $(F+G)^{-1}$ is meromorphic for $|z|<R$ and*

$$T_\infty\big(r,(F+G)^{-1}\big) \le (k+1)T_\infty(r,F^{-1}) + kT_\infty(r,G) + C, \tag{4.1}$$

*where*

$$C \le k\log 2 - \log|c_{-\nu}|$$

*and $c_{-\nu}$ is the first nonzero coefficient in the Laurent series of $\det(I+F^{-1}G)$ at the origin.*

*Proof.* Away from poles we can write $F+G = F(I+F^{-1}G)$ and thus

$$T_\infty\big(r,(F+G)^{-1}\big) \le T_\infty(r,F^{-1}) + T_\infty\big(r,(I+F^{-1}G)^{-1}\big). \tag{4.2}$$

Here $F^{-1}G \in \mathscr{F}_1(R)$ as $G$ is of finite rank. This allows us to use the inversion identity:

$$T_\infty\big(r,(I+F^{-1}G)^{-1}\big) \le T_1\big(r,(I+F^{-1}G)^{-1}\big) = T_1(r,I+F^{-1}G) - \log|c_{-\nu}|. \tag{4.3}$$

If $a,b \ge 0$ then always

$$\log(1+ab) \le \log^+ a + \log^+ b + \log 2.$$

We use this in estimating $\log^+\sigma_j(I+F^{-1}G)$. For $j>k$, $\sigma_j(I+F^{-1}G)=1$ while for $j \le k$ we have $\sigma_j(I+F^{-1}G) \le 1 + \|F^{-1}\|\|G\|$. Thus

$$s(I+F^{-1}G) \le k[\log^+\|F^{-1}\| + \log^+\|G\| + \log 2]$$

and

$$T_1(r,I+F^{-1}G) \le k[T_\infty(r,F^{-1}) + T_\infty(r,G) + \log 2]$$

from which the claim follows.

**Definition 4.1.** Let $F$ be a meromorphic operator valued function in the whole plane. Then the order $\omega_\infty$ is

$$\omega_\infty := \limsup_{r\to\infty} \frac{\log T_\infty(r,F)}{\log r} \tag{4.4}$$

and if $0<\omega_\infty<\infty$ then the type $\tau_\infty$ is

$$\tau_\infty := \limsup_{r\to\infty} \frac{T_\infty(r,F)}{r^{\omega_\infty}}. \tag{4.5}$$

If $F$ is in $\mathscr{F}_1(\infty)$, then the order $\omega_1$ and type $\tau_1$ are defined in the same way for $F$ and for $I-F$.

Notice that if $F$ is entire, then it also has order and type as an entire function based on $M_\infty(r)$. The order is then the same as a meromorphic function while the type can be somewhat different. In fact, an easy estimate between the types can be obtained from Theorem 2.2. This holds as such also for entire $\mathscr{S}_1$-valued functions by Theorem 2.5.

**Corollary 4.1.** *If in addition to the assumptions of Theorem* 4.1, $R=\infty$ *and* $F^{-1}$ *and* $G$ *are of finite orders,* $\omega_\infty(F^{-1})$ *and* $\omega_\infty(G)$ *respectively, then* $(F+G)^{-1}$ *is also of finite order and*

$$\omega_\infty\big((F+G)^{-1}\big) = \max\{\omega_\infty(F^{-1}), \omega_\infty(G)\}.$$

*If* $F^{-1}$ *and* $G$ *are also of finite type,* $\tau_\infty(F^{-1})$ *and* $\tau_\infty(G)$, *then the type of* $(F+G)^{-1}$ *satisfies*

$$\tau_\infty\big((F+G)^{-1}\big) \le (k+1)\tau_\infty(F^{-1}), \quad \text{if} \quad \omega_\infty(F^{-1})>\omega_\infty(G),$$
$$\tau_\infty((F+G)^{-1}) \le k\,\tau_\infty(G), \quad \text{if} \quad \omega_\infty(F^{-1})<\omega_\infty(G),$$

*and*

$$\tau_\infty((F+G)^{-1}) \le (k+1)\tau_\infty(F^{-1}) + k\,\tau_\infty(G), \quad \text{if} \quad \omega_\infty(F^{-1})=\omega_\infty(G).$$

**Example 4.1.** Let $F(z):=(1-z)I$ and $G(z):=-z\,\mathrm{diag}(\beta_1,\dots,\beta_k,0,0,\dots)$, where $\beta_1>\beta_2>\cdots>\beta_k>0$. We have poles at $1$ and at $1/(1+\beta_j)$ and so

$$N_\infty\big(r,(F+G)^{-1}\big) = \log^+ r + \sum_1^k \log^+\big((1+\beta_j)r\big) \ge (k+1)\log^+ r.$$

For $r\ge 2$ we have $T_\infty(r,F^{-1}) = \log^+ r$ and thus

$$T_\infty\big(r,(F+G)^{-1}\big) \ge (k+1)T_\infty(r,F^{-1}) \qquad \text{for } r\ge 2.$$

Thus, multiplying with $k+1$ is really needed.

**Example 4.2.** Now we look at the term $kT_\infty(r,G)$. Let

$$F := \operatorname{diag}(k, k-1, k-2, \dots, 2, 1, 1, 1, \dots)$$

so that $T_\infty(r, F^{-1}) = 0$ for all $r$, while $G(z) := e^z\, I_k$ where $I_k = \operatorname{diag}(1, \dots, 0, \dots)$ is the rank $k$ projection onto the $k$ first components. Then

$$T_\infty(r,G) = \frac{r}{\pi}.$$

On the other hand, $(F+G)^{-1}$ has poles at $z = \log j + 2\pi i n$ for all $n$ and for $j = 1, 2, \dots, k$. Therefore

$$T_\infty\big(r, (F+G)^{-1}\big) \ge k\frac{r}{\pi} + \mathscr{O}(\log^+ r).$$

Again, multiplying with $k$ is really needed.

**Example 4.3.** The previous examples have all been based on diagonal mappings. Here we point out that rank 1 updates can change the nature of the operator dramatically. In fact, we have two operators, $V^2$ and $K$ where $V^2$ is a quasinilpotent Volterra operator while $K$ is a self-adjoint Fredholm operator which are rank 1 perturbations of each other. Now the resolvent (see the next section) of $V^2$ is entire and thus $N_\infty\big(r, (I - zV^2)^{-1}\big) = 0$ while as $K$ is self-adjoint, $m_\infty\big(r, (I - zK)^{-1}\big) = \mathscr{O}(1)$ (see Theorem 5.5). So, $V^2$ corresponds to the Picard exceptional case while $K$ is as "regular" as possible. The space is $L_2[0,1]$, and

$$V^2 f(t) = \int_0^t (t-s) f(s)\, ds$$

and

$$Kf(t) = \int_0^1 k(t,s) f(s)\, ds$$

where $k(t,s) = k(s,t)$ and for $0 \le s \le t \le 1$

$$k(t,s) = s(t-1).$$

Here $V^2$ is the solution operator for the initial value problem $u'' = f$ with $u(0) = u'(0) = 0$ while $K$ solves the same problem with boundary conditions $u(0) = u(1) = 0$. Thus

$$Kf(t) = V^2 f(t) - V^2 f(1) t.$$

It can be shown that both resolvents are of order $1/2$ and of type $2/\pi$.

The next result relaxes the assumption on finite rank to $\mathscr{S}_1$ but assumes $F^{-1}$ to be analytic instead.

**Theorem 4.2.** *Assume that $F$ is meromorphic, $F^{-1}$ analytic, and $G$ finitely $\mathscr{S}_1$-meromorphic for $|z| < R$. Then $(F+G)^{-1}$ is meromorphic and for $r < R$*

$$T_\infty\big(r,(F+G)^{-1}\big) \le T_\infty(r,F^{-1}) + \max\{M_\infty(r,F^{-1}),1\}\big[\hat{m}_1(r,G)+N_1(r,G)\big]+C, \tag{4.6}$$

*where*

$$\hat{m}_1(r,G) := \sum_{j=1}^{\infty} m\big(r,1+\sigma_j(G)\big),$$

$$C \le -\log|c_{-\nu}|$$

*and $c_{-\nu}$ is the first nonzero coefficient in the Laurent series of $\det(I+F^{-1}G)$ at the origin.*

*Proof.* We start the proof in the same way as before but the estimation of the term $T_1(r, I+F^{-1}G)$ in (4.3) is different. Let, again, $a, b \ge 0$. Then we have

$$\log(1+ab) \le \max\{a,1\}\log(1+b).$$

Put for short, $M := \max\{M_\infty(r,F^{-1}),1\}$. Then this inequality gives

$$\log^+\sigma_j(I+F^{-1}G) \le M\log\big(1+\sigma_j(G)\big)$$

and so

$$s(I+F^{-1}G) \le M\sum \log\big(1+\sigma_j(G)\big).$$

Now the claim follows from this.

**Corollary 4.2.** *If, in addition to the assumptions in Theorem 4.2, $G$ is analytic for $|z| < R$, then for $r < R$*

$$T_\infty\big(r,(F+G)^{-1}\big) \le \max\{M_\infty(r,F^{-1}),1\}\sup_{|z|\le r}\|G(z)\|_1 + T_\infty(r,F^{-1}) + C, \tag{4.7}$$

*where*

$$C \le -\log|c_{-\nu}|.$$

*Proof.* We have $\log\big(1+\sigma_j(G)\big) \le \sigma_j(G)$ which gives the term $\|G\|_1$.

**Example 4.4.** $F(z) = e^z I$ and $G = \mathrm{diag}(j^{-1-\varepsilon})$. For $r$ large one has

$$N_\infty\big(r,(F+G)^{-1}\big) \ge e^{r/(1+2\varepsilon)}$$

while the right hand side in (4.7) is bounded from above by $\mathscr{O}(1/\varepsilon)e^r + (r/\pi) + C$. Thus, the order and type of $(F+G)^{-1}$ are the same as those of the bound.

The previous results are natural in a setting where one of the functions is treated as large and the other one as a perturbation. If both functions are finitely $\mathscr{S}_1$-meromorphic we can formulate a result directly for their sum without inverting as the inverse is equally large by Theorem 3.1.

**Theorem 4.3.** *If both $F$ and $G$ are in $\mathscr{F}_1(R)$, then for $r < R$*

$$T_1(r, I+F+G) \le 2T_1(r, I+2F) + 2T_1(r, I+2G). \tag{4.8}$$

The result follows immediately from the following lemma.

**Lemma 4.1.** *If $A, B \in \mathscr{S}_1$, then*

$$s(I+A+B) \le 2s(I+2A) + 2s(I+2B). \tag{4.9}$$

*Proof of the lemma.* We have for any $a, b$

$$\log^+ \tfrac{1}{2}(a+b) \le \log^+ a + \log^+ b. \tag{4.10}$$

On the other hand, writing $I+A+B = \frac{1}{2}[(I+2A)+(I+2B)]$ and using the fact that singular values are obtained as distances to finite rank operators, we get

$$\sigma_{2j-1}(I+A+B) \le \tfrac{1}{2}[\sigma_j(I+2A) + \sigma_j(I+2B)].$$

As $\sigma_{2j}(I+A+B) \le \sigma_{2j-1}(I+A+B)$ this and (4.10) imply (4.9).

**Example 4.5.** If $F = G$ then we have trivially

$$T_1(r, I+F+G) = \tfrac{1}{2}[T_1(r, I+2F) + T_1(r, I+2G)].$$

## 5. Growth of resolvents

In the following we specialize to resolvents. It is natural to write them here in the form $(I - zA)^{-1}$ instead of the usual $(\lambda - A)^{-1}$.

We start with a result which generalizes a similar statement for quasinilpotent trace class operators, see Theorem 2.2 in Chapter X, [3].

**Theorem 5.1.** *Assume $A \in \mathscr{S}_p$. Then $(I - zA)^{-1}$ is of order $\omega_\infty \le p$ and if $\omega_\infty = p$ then it is of zero type.*

*Proof.* We prove first the result in the case $p \le 1$. To that end it suffices to show that for any $\varepsilon > 0$ we have

$$T_\infty\big(r, (I - zA)^{-1}\big) \le \varepsilon r^p + \mathscr{O}(\log r).$$

Since $A \in \mathscr{S}_p$ there exists an $m$, large enough so that

$$\frac{1}{p} \sum_{m+1}^{\infty} \sigma_j(A)^p < \varepsilon.$$

Then, however, we can proceed as follows:

$$\begin{aligned}T_\infty\big(r,(I-zA)^{-1}\big) &\le T_1(r,I-zA) \le \log M_1(r,I-zA)\\ &\le \sum_1^m \log\big(1+r\sigma_j(A)\big) + \frac{1}{p}r^p\sum_{m+1}^\infty \sigma_j(A)^p\\ &\le \mathscr{O}(\log r) + \varepsilon r^p.\end{aligned}$$

Here we used the inequality $\log(1+x) \le x^p/p$, valid for $x>0$ and $0<p\le 1$.

In the general case, let $k$ be a positive integer such that $k<p\le k+1$. Then in particular $A^{k+1}\in\mathscr{S}_1$, and in fact

$$\sum \sigma_j(A^{k+1})^{p/(k+1)} \le \sum \sigma_j(A)^p,$$

see e.g. Corollary II.4.2 in [4]. We have, see Theorem 3.2, or Theorem 5.3 below,

$$T_\infty\big(r,(I-zA)^{-1}\big) \le T_1(r,I-z^{k+1}A^{k+1}) + k\log(1+r\|A\|).$$

Here we proceed as above and in particular use

$$\log\big(1+r^{k+1}\sigma_j(A^{k+1})\big) \le \frac{k+1}{p} r^p\sigma_j(A^{k+1})^{p/(k+1)}$$

to split the sum at a proper place in order to have the growth again bounded by $\varepsilon r^p + \mathscr{O}(\log r)$.

The proof above is based on

$$T_1(r,I-z^{k+1}A^{k+1}) = T_1\big(r,(I-z^{k+1}A^{k+1})^{-1}\big),$$

valid for $k+1\le p$. We shall next study the behavior of $k^{-1}T_1\big(r,(I-z^kA^k)^{-1}\big)$ as $k$ grows.

**Lemma 5.1.** *Let $A\in\mathscr{S}_1$ and $\{\lambda_j\}$ denote the spectrum, indexed so that $|\lambda_j|\ge|\lambda_{j+1}|$, each eigenvalue repeated according to the dimension of the corresponding eigenspace. Then*

$$N_1\big(r,(I-zA)^{-1}\big) = \sum_j \log^+|\lambda_j r|. \tag{5.1}$$

*Proof.* Choose $r$ and take the Riesz spectral projection of $A$ including all eigenvalues which are larger than, say, $1/(r+1)$ in modulus. This gives a finite rank operator, say, $A_r$. Then $A-A_r$ can be approximated arbitrarily well with another finite rank operator and this shows that $N_1\big(r,(I-zA)^{-1}\big)$ only depends on $A_r$, compare with the Continuity Lemma 2.4. But since this is of finite rank, it can be transformed unitarily into a finite dimensional upper triangular (that is, a sum of diagonal and nilpotent) operator and then with a similarity transformation into a form where the nilpotent part is made arbitrarily small and so $N_1$ only depends on the eigenvalues. In fact, if $S$ denotes the similarity transformation, and if $B = SCS^{-1}$ with $d = \dim B$, then

$$s(B) \le d \log[\|S\| \, \|S^{-1}\|] + s(C) \tag{5.2}$$

and therefore the multiplicity $\mu(1/\lambda_j)$ is not affected by the similarity transformation.

Observe that the right hand side of (5.1) makes sense for all compact operators as it is always a finite sum for any fixed $r$. We introduce the following notation for it:

$$N(r, \{\lambda_j\}) := \sum_j \log^+ |\lambda_j r|. \tag{5.3}$$

Now the following holds.

**Theorem 5.2.** *Assume $A \in \mathscr{S}_p$ with some $p$. Then*

$$\lim_{k\to\infty} \frac{1}{k} T_1\big(r, (I - z^k A^k)^{-1}\big) = N(r, \{\lambda_j\}), \tag{5.4}$$

*where $\{\lambda_j\}$ denotes the spectrum of $A$ with multiplicities counted according to the dimensions of the corresponding eigenspaces.*

*Proof.* Recall that if $A \in \mathscr{S}_p$ then $A^k \in \mathscr{S}_1$ for $k \ge p$. Then for such $k$ $T_1\big(r, (I - z^k A^k)^{-1}\big)$ and $T_1(r, I - z^k A^k)$ are both well defined and equal. The proof is given by several simple lemmas, some of which have some independent interest.

**Lemma 5.2.** *If $A$ is compact, then*

$$\lim_{k\to\infty} [\sigma_j(A^k)]^{1/k} = |\lambda_j|. \tag{5.5}$$

*Proof of Lemma* 5.2. This is Proposition 2.d.6 in [8].

The aim is to show that

$$\frac{1}{k} m_1(r, I - z^k A^k) \to N(r, \{\lambda_j\}). \tag{5.6}$$

This would imply (5.4) as

$$T_1\big(r, (I - z^k A^k)^{-1}\big) = T_1(r, I - z^k A^k)$$

and, trivially,

$$N_1(r, I - z^k A^k) = 0.$$

We shall first reduce the claim into a finite dimensional problem. Since our basic claim is about a limit with a fixed $r$ we can without lack of generality set $r = 1$ in the following. Choose a small $0 < \delta < 1$. Then take a spectral decomposition of $A = A_1 \oplus A_2$ as follows:

$$A_2 := \frac{1}{2\pi i} \int_{|\lambda|=1-\delta} \lambda(\lambda I - A)^{-1}\, d\lambda.$$

By the spectral radius formula we have for large enough $n$

$$\|A_2^n\|^{1/n} \le 1 - \tfrac{1}{2}\delta.$$

**Lemma 5.3.** *Assume that $A \in \mathscr{S}_p$ and $\rho(A) < 1$. Then we have*

$$m_1(1, I - z^k A^k) \to 0$$

*as $k \to \infty$.*

*Proof of Lemma* 5.3. If $\rho(A) < \rho < 1$ then for large enough $n$ we have $\|A^n\| \le \rho^n$. If also $n \ge k$ where $k$ such that $A^k \in \mathscr{S}_1$, then we can estimate as $|z| = 1$,

$$\sigma_j(I - z^n A^n) \le 1 + \rho^{n-k} \sigma_j(A^k)$$

which shows that

$$s(I - z^n A^n) \le \rho^{n-k} \|A^k\|_1.$$

The claim follows.

**Lemma 5.4.** *If $A \in \mathscr{S}_1$ and $B$ is of finite rank and they operate in invariant subspaces $H_A$, $H_B$ respectively with $H_A \cap H_B = \{0\}$, then*

$$s\big(I + (A \oplus B)\big) \le s(I + A) + s(I + B) + \operatorname{rank}(B)[\log(1 + \|A\|) + \log 2]. \tag{5.7}$$

*Proof of Lemma* 5.4. If $j \le \operatorname{rank}(B) = d$ then

$$\sigma_j\big(I + (A \oplus B)\big) \le 1 + \|A\| + \sigma_j(I + B)$$

while for $j > d$ we have

$$\sigma_j\big(I + (A \oplus B)\big) \le \sigma_{j-d}(I + A).$$

We obtain (5.7) by taking the logarithm and summing up.

If $A = A_1 \oplus A_2$ as above and $\operatorname{rank} A_1 = d$ then Lemma 5.4 gives

$$m_1(1, I - z^n A^n) \le m_1(1, I - z^n A_1^n) + m_1(1, I - z^n A_2^n) + d[\log(1 + \|A_2^n\|) + \log 2]. \tag{5.8}$$

This follows because $A^n = A_1^n \oplus A_2^n$ allows us to apply Lemma 5.4 with $-z^n A^n$ in place of $A$. By Lemma 5.3 we have $m_1(1, I - z^n A_2^n) \to 0$ and since $\|A_2^n\| \to 0$, then inequality (5.8) implies

$$\limsup \frac{1}{n} m_1(1, I - z^n A^n) \le \limsup \frac{1}{n} m_1(1, I - z^n A_1^n). \tag{5.9}$$

What we need still to prove is the reverse inequality

$$\liminf \frac{1}{n} m_1(1, I - z^n A_1^n) \le \liminf \frac{1}{n} m_1(1, I - z^n A^n) \tag{5.10}$$

and that the limit exists and satisfies

$$\lim \frac{1}{n} m_1(1, I - z^n A_1^n) = N(1, \{\lambda_j\}). \tag{5.11}$$

Consider first (5.10). Let $P$ denote the spectral projection $A_1 = PA$. Then for $j \le d$ we have

$$\sigma_j(I + A_1) \le \|P\| \sigma_j(I + A)$$

while for $j > d$ we have $\sigma_j(I + A_1) = 1$. Thus

$$s(I + A_1) \le s(I + A) + d \log \|P\|.$$

Applying this to $-z^n A^n$ in place of $A$ gives (5.10).

To prove (5.11) observe first that by construction $N_1\big(1, (I - zA_1)\big)^{-1} = N(1, \{\lambda_j\})$. And recall that we have set $r = 1$. For $|z| = 1$ we have

$$-1 + \sigma_j(A^n) \le \sigma_j(I - z^n A_1^n) \le 1 + \sigma_j(A^n)$$

which implies, as $A_1$ is of rank $d$,

$$\left| \frac{1}{n} m_1(1, I - z^n A^n) - \frac{1}{n} \sum \log^+ \sigma_j(A^n) \right| \le \frac{1}{n} d \log 2.$$

By Lemma 5.2 we know that

$$\frac{1}{n} \sum \log^+ \sigma_j(A^n) \to N(1, \{\lambda_j\})$$

which proves (5.11). The proof of Theorem 5.2 is now completed.

**Definition 5.1.** We denote by $\rho_\infty(A)$ the smallest radius such that the resolvent $(I-zA)^{-1}$ is meromorphic for $|z|<1/\rho_\infty(A)$.

**Remark 5.1.** Notice that the analogous concept, say $\rho_1(A)$ in $\mathscr{F}_1$ is not needed as by the inversion identity everything depends on whether $-zA\in\mathscr{F}_1$ or not, that is, whether $A\in\mathscr{S}_1$.

**Theorem 5.3.** *If $A$ is bounded, then $(I-zA)^{-1}$ and $(I-z^kA^k)^{-1}$ are meromorphic at the same discs: $\rho_\infty(A)=\rho_\infty(A^k)^{1/k}$ and*

$$T_\infty\bigl(r,(I-z^kA^k)^{-1}\bigr)\le kT_\infty\bigl(r,(I-zA)^{-1}\bigr) \tag{5.12}$$

*while*

$$T_\infty\bigl(r,(I-zA)^{-1}\bigr)\le T_\infty\bigl(r,(I-z^kA^k)^{-1}\bigr)+(k-1)\log(1+r\|A\|). \tag{5.13}$$

*Proof.* Write, with $\phi_j:=2\pi j/k$,

$$(I-z^kA^k)=(I-zA)(I-e^{i\phi_1}zA)\cdots(I-e^{i\phi_{k-1}}zA).$$

This gives (5.12). Writing it for the inverses as in (3.11) gives in the same way (5.13).

So in particular, the order is preserved while the type can change somewhat. If the operator is quasinilpotent, so that the resolvent is entire, then we can also look at the growth of the maximum and here the type is preserved as well.

**Theorem 5.4.** *If $A$ is bounded and $\rho(A)$ denotes the spectral radius, then for $r<1/\rho(A)$ we have*

$$M_\infty\bigl(r,(I-z^kA^k)^{-1}\bigr)\le M_\infty\bigl(r,(I-zA)^{-1}\bigr), \tag{5.14}$$

*and*

$$M_\infty\bigl(r,(I-zA)^{-1}\bigr)\le[1+r\|A\|]^{k-1}M_\infty\bigl(r,(I-z^kA^k)^{-1}\bigr). \tag{5.15}$$

*Proof.* Here (5.15) is analogous to (5.13) while (5.14) follows from writing

$$(I-z^kA^k)^{-1}=\frac{1}{k}\sum_1^k(I-e^{i\phi_j}zA)^{-1}.$$

For reference, it is useful to formulate the following simple fact.

**Theorem 5.5.** *Let $A$ be bounded and selfadjoint. Then for $r<1/\rho_\infty(A)$*

$$m_\infty\bigl(r,(I-zA)^{-1}\bigr)\le\log 2. \tag{5.16}$$

*Proof.* Since the operator is self-adjoint, we have $\|(I-zA)^{-1}\|=1/d(z)$ where

$$d(re^{i\phi}):=\inf_{\lambda\in\sigma(A)}|1-z\lambda|\ge\inf_{\lambda\in R}|1-z\lambda|=|\sin\phi|.$$

Thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+\frac{1}{d(re^{i\varphi})}\,d\varphi\le\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\frac{1}{|\sin\varphi|}\,d\varphi=\log 2.$$

**References**

[1] Aupetit, B.: A Primer on Spectral Theory. - Springer-Verlag, New York, 1991.

[2] Aupetit, B.: On Log-subharmonicity of singular values of matrices. - Studia Math. 122(2), 1997, 195–200.

[3] Gohberg, I., S. Goldberg, and M.A. Kaashoek: Classes of Linear Operators, Vol. I. - Birkhäuser, Basel, 1990.

[4] Gohberg, I., and M. Krein: Introduction to the Theory of Linear Nonselfadjoint Operators. - Amer. Math. Soc. Transl. 18, 1969.

[5] Hayman, W.K., and P.B. Kennedy: Subharmonic Functions, Vol. 1. - Academic Press, 1976.

[6] Horn, R.A., and C.R. Johnson: Matrix Analysis. - Cambridge Univ. Press, 1985.

[7] Horn, R.A., and C.R. Johnson: Topics in Matrix Analysis. - Cambridge Univ. Press, 1991.

[8] König, H.: Eigenvalue Distribution of Compact Operators. - Birkhäuser, 1986.

[9] Nevanlinna, O.: Convergence of Iterations for Linear Equations. - Birkhäuser, 1993.

[10] Nevanlinna, O.: Meromorphic resolvents and power bounded operators. - BIT 36:3, 1996, 531–541.

[11] Nevanlinna, O.: Convergence of Krylov methods for sums of two operators. - BIT 36:4, 1996, 775–785.

[12] Nevanlinna, O.: A characteristic function for matrix valued meromorphic functions. - XVIth Rolf Nevanlinna Colloquium, edited by I. Laine and O. Martio, Walter de Gruyter & Co, Berlin, 1996, 171–179.

[13] Nevanlinna, O.: On the growth of the resolvent operators for power bounded operators. - Banach Center Publ. 38, 1997, 247–264.

[14] Nevanlinna, R.: Zur Theorie der meromorphen Funktionen. - Acta Math. 46, 1925, 1–99.

[15] Nevanlinna, R.: Analytic Functions. - Springer-Verlag, 1970.

[16] Ribaric, M., and I. Vidav: Analytic properties of the inverse $A(z)^{-1}$ of an analytic linear operator valued function $A(z)$. - Arch. Rational Mech. Anal. 32, 1969, 298–310.

Received 19 November 1997