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EXISTENCE OF QUASI-ISOMETRIC MAPPINGS AND ROYDEN COMPACTIFICATIONS

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Abstract. It is shown that Royden p-compactifications for $1 < p < d$ of two orientable and countable Riemannian manifolds of the same dimension $d \geq 2$ are homeomorphic if and only if there exists a homeomorphism between these two Riemannian manifolds which is quasi-isometric outside some of their compact subsets.

1. Introduction

Given a d-dimensional ($d \geq 2$) Riemannian manifold D of class C^∞ which is orientable and countable but not necessarily connected and an exponent $1 < p < \infty$, the *Royden p-algebra* $M_p(D)$ of D is defined by $M_p(D) := L^{1,p}(D) \cap L^\infty(D) \cap C(D)$. It is a commutative Banach algebra, i.e. the so-called normed ring, under pointwise addition and multiplication with $\|u; M_p(D)\| := \|u; L^\infty(D)\| + \|\nabla u; L^p(D)\|$ as norm, where $L^{1,p}(D)$ is the Dirichlet space, i.e. the space of locally integrable real valued functions u on D whose distributional gradients ∇u of u belong to $L^p(D)$ considered with respect to the metric structure on D. The maximal ideal space D_p^* (cf. e.g. p. 298 in [20]) of $M_p(D)$ is referred to as the *Royden p-compactification* of D, which is also characterized as the compact Hausdorff space containing D as its open and dense subspace such that every function in $M_p(D)$ is continuously extended to D_p^* and $M_p(D)$ is uniformly dense in $C(D_p^*)$ (cf. e.g. [17], [18], [11] and also p. 154 in [14]).

Suppose that D and D' are d-dimensional ($d \geq 2$) Riemannian manifolds of class C^∞ which are orientable and countable but not necessarily connected. Moreover, in this paper we assume that none of the components of D and D' is compact, which is, however, not an essential restriction, and postulated only for the sake of simplicity. In 1982, the present author and H. Tanaka [13] (see also [10]) jointly showed that two conformal Royden compactifications D_d^* and $(D')_d^*$ are homeomorphic if and only if there exists an almost quasiconformal mapping of D onto D'. Here we say that a homeomorphism f of D onto D' is an *almost quasiconformal mapping* of D onto D' if there exists a compact subset $E \subset D$

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such that $f = f \mid D\backslash E$ is a quasiconformal mapping of $D\backslash E$ onto $D'\backslash f(E)$. There are many ways of defining quasiconformality but the following metric definition is convenient for applying to Riemannian manifolds (cf. e.g. p. 113 in [19]): the homeomorphism f of $D \setminus E$ onto $D' \setminus f(E)$ is *quasiconformal*, by definition, if

$$\sup_{x\in D\backslash E}\left(\limsup_{r\downarrow 0}\frac{\max_{\varrho(x,y)=r}\varrho'\big(f(x),f(y)\big)}{\min_{\varrho(x,y)=r}\varrho'\big(f(x),f(y)\big)}\right)<\infty, \tag{2}$$

where ϱ and ϱ' are geodesic distances on $D \setminus E$ and $D' \setminus f(E)$. It has been an open question for a long period, since the above result was obtained, as for what can be said about the counterpart of the above result for nonconformal case, i.e. if the exponent d in the result is replaced by $1 < p < d$. The *purpose* of this paper is to settle this question (the main theorem).

To state our result we need to introduce a class of a special kind of almost quasiconformal mappings. A homeomorphism f of D onto D' is said to be an *almost quasi-isometric mapping* of D onto D' if there exists a compact set $E \subset D$ such that $f = f \mid D\backslash E$ is a quasi-isometric mapping of $D\backslash E$ onto $D'\backslash f(E)$. Here the homeomorphism f of $D \setminus E$ onto $D' \setminus f(E)$ is *quasi-isometric*, by definition, if there exists a constant $K \in [1,\infty)$ such that

$$\frac{1}{K}\varrho(x,y)\le\varrho'\big(f(x),f(y)\big)\le K\varrho(x,y) \tag{3}$$

for every pair of points x and y in $D \setminus E$, where we always set $\varrho(x,y) = \varrho'\big(f(x),f(y)\big) = \infty$ if the component of $D \setminus E$ containing x and that containing y are different. From (3) it follows that

$$\frac{1}{K}r\le\min_{\varrho(x,y)=r}\varrho'\big(f(x),f(y)\big)\le\max_{\varrho(x,y)=r}\varrho'\big(f(x),f(y)\big)\le Kr$$

for any fixed $x \in D$ and for any sufficiently small positive number $r > 0$, which implies that the left-hand side term of (2) is dominated by K^2. Thus a quasi-isometric mapping is automatically a quasiconformal mapping but obviously there exists a quasiconformal mapping which is not a quasi-isometric mapping. Then the main result of this paper is stated as follows.

4. Main theorem. *When $1 < p < d$, Royden compactifications D_p^* and $(D')_p^*$ are homeomorphic if and only if there exists an almost quasi-isometric mapping of D onto D'. More precisely, any almost quasi-isometric mapping of D onto D' is uniquely extended to a homeomorphism of D_p^* onto $(D')_p^*$; conversely, the restriction to D of any homeomorphism of D_p^* onto $(D')_p^*$ is an almost quasi-isometric mapping of D onto D'.*

It may be interesting to compare the above topological result with the former relevant algebraic results obtained by the present author [8] and [9], Lewis [6], and Lelong-Ferrand [5] (cf. also Soderborg [15]): Royden algebas $M_d(D)$ and $M_d(D')$ are algebraically isomorphic if and only if there exists a quasiconformal mapping of D onto D'; when $1 < p < d$, $M_p(D)$ and $M_p(D')$ are algebraically isomorphic if and only if there exists a quasi-isometric mapping of D onto D'. All these results, including our present main theorem, are shown to be invalid when $d < p < \infty$ by giving a counterexample, which will be discussed elsewhere. Another important problem related to the main result is the following: does the existence of an almost quasi-isometric (almost quasiconformal, respectively) mapping of D onto D' imply that of a quasi-isometric (quasiconformal, respectively) mapping of D onto D'? It is affirmative for the quasiconformal case if D is the unit ball in the d-dimensional Euclidean space $\mathbf{R}^d$ (Gehring [2], see also Soderborg [16]); it is also affirmative for the quasiconformal case if the dimensions of D and D' are 2. Except for these partial results the problem is widely open.

This paper consists of six sections including this first section titled *Introduction.* In this paper, sections, formulas, theorems, lemmas, etc. are altogether consequtively numbered, which may not yet be one of the standard customs but anyone would admit it to be extremely convenient to refer to, beyond his stylistic preference, especially in short papers such as this one. In the second section, number 5 titled *Royden compactifications of Riemannian manifolds*, basic notion and notation on objects in the title are fixed. A useful separation lemma on the topology of Royden compactifications in terms of capacities of rings is given in the third section (8) titled *Capacities of rings.* Quasi-isometric mappings are considered in the fourth section (19) titled *Analytic properties of quasi-isometric mappings* from the view point of differentiability. In the fifth section (33) *Distortion of rings and their capacities*, an important characterization of quasi-isometries in terms of distortions of capacities of rings originally due to Gehring [3] is discussed, which plays one of the key roles in this paper. The main theorem of this paper will be proven in the final section (43) titled *Proof of the main theorem.*

5. Royden compactifications of Riemannian manifolds. By a *Riemannian manifold* D of dimension $d \geq 2$ we mean an orientable and countable but not necessarily connected C^∞ manifold D of dimension d with a metric tensor (g_{ij}) of class C^∞. For the sake of simplicity we also assume that any component of D is not compact.

We say that U or more precisely (U, x) is a *parametric domain* on D if the following two conditions are satisfied: first, U is a domain, i.e. a connected open set, in D; secondly x is a C^∞ diffeomorphism of U onto a domain $x(U)$ in the Euclidean space $\mathbf{R}^d$ of dimension $d \geq 2$. The map $x = (x^1, \ldots, x^d)$ is referred to as a *parameter* on U. We often identify a generic point P of U with its parameter $x(P)$ and denote it by the same letter x, for example. In other words, we view

U to be embedded in $\mathbf{R}^d$ by identifying U with $x(U)$ so that U itself may be considered as a Riemannian manifold (U, g_{ij}) with a metric tensor (g_{ij}) restricted on U and at the same time as an Euclidean subdomain (U, δ_{ij}) with the natural metric tensor (δ_{ij}), δ_{ij} being the Kronecker delta.

Take a parametric domain (U, x) on D. The metric tensor (g_{ij}) on D gives rise to a $d \times d$ matrix $(g_{ij}(x))$ of functions $g_{ij}(x)$ on U. We say that (U, x) is a λ-*domain* with $\lambda \in [1, \infty)$ if the following matrix inequalities hold:

$$\frac{1}{\lambda}(\delta_{i,j}) \leq (g_{ij}(x)) \leq \lambda(\delta_{ij}) \tag{6}$$

for every $x \in U$. It is important that any point of D has a λ-domain as its neighborhood for any $\lambda \in (1, \infty)$. This comes from the fact that there exists a parametric ball (U, x) at any point $P \in D$ (i.e. a parametric domain (U, x) such that $x(P) = 0$ and $x(U)$ is a ball in $\mathbf{R}^d$ centered at the origin) such that $\big(g_{ij}(x)\big)$ with respect to (U, x) satisfies $g_{ij}(0) = \delta_{ij}$.

The metric tensor (g_{ij}) on D defines the line element ds on D by $ds^2 = g_{ij}(x)\, dx^i\, dx^j$ in each parametric domain $(U, x = (x^1, \ldots, x^d))$. Here and hereafter we follow the Einstein convention: whenever an index i appears both in the upper and lower positions, it is understood that summation for $i = 1, \ldots, d$ is carried out. The length of a rectifiable curve γ on D is given by $\int_\gamma ds$. The *geodesic distance* $\varrho(x, y)$ between two points x and y in D is given by

$$\varrho(x, y) = \varrho_D(x, y) = \inf_\gamma \int_\gamma ds,$$

where the infimum is taken with respect to rectifiable curves γ connecting x and y. Needless to say, if there is no such curve γ, i.e. if x and y are in different components of D, then, as the infimum of the empty set, we understand that $\varrho(x, y) = \infty$. When (U, x) is a parametric domain and considered as the Riemannian manifold (U, δ_{ij}), then $\varrho_U(x, y)$ can also be given by

$$\varrho(x, y) = \varrho_U(x, y) = \inf \sum_{i=1}^{n} |x_i - x_{i-1}|,$$

where the infimum is taken with respect to every polygonal line

$$x = x_0, x_1, \ldots, x_{n-1}, x_n = y$$

such that every line segment $[x_{i-1}, x_i] = \{(1-t)x_{i-1} + tx_i : 0 \leq t \leq 1\} \subset U$ for each $i = 1, \ldots, n$.

We write $(g^{ij}) := (g_{ij})^{-1}$ and $g := \det(g_{ij})$. We denote by dV the volume element on D so that

$$dV(x) = \sqrt{g(x)}\, dx^1 \wedge \cdots \wedge dx^d$$

in each parametric domain $(U, x = (x^1, \dots, x^d))$. On (U, δ_{ij}) we also have the volume element (Lebesgue measure) $dx = dx^1 \cdots dx^d$. Sometimes we use dx to mean $(dx^1, \dots, dx^d)$ but there will be no confusion by context. The Riemannian volume element $dV(x)$ and the Euclidean (Lebesgue) volume element dx are mutually absolutely continuous and the Radon–Nikodym densities $dV(x)/dx = \sqrt{g(x)}$ and $dx/dV(x) = 1/\sqrt{g(x)}$ are locally bounded on U. Thus a.e. dV and a.e. dx are identical and we can loosely use a.e. without referring to dV or dx.

For each $x \in D$, the tangent space to D at x will be denoted by T_xD. We denote by $\langle h, k\rangle$ the inner product of two tangent vectors h and k in T_xD and by $|h|$ the length of $h \in T_xD$ so that if $(h_1, \dots, h_d)$ and $(k_1, \dots, k_d)$ are covariant components of h and k, then

$$\langle h, k\rangle = g^{ij}h_ik_j \qquad \text{and} \qquad |h| = \langle h, h\rangle^{1/2} = (g^{ij}h_ih_j)^{1/2}.$$

Since we may consider two metric tensors (g_{ij}) and $(\delta_{i,j})$ on a parametric domain (U, x), we occasionally write $\langle h, k\rangle_{g_{ij}}$ or $\langle h, k\rangle_{\delta_{ij}}$ and similarly $|h|_{g_{ij}}$ or $|h|_{\delta_{ij}}$ to make clear whether they are considered on (U, g_{ij}) or on (U, δ_{ij}).

Let G be an open subset of D. In this paper we use the notation $L^p(G)$ $(1 \le p \le \infty)$ in two ways. The first is the standard use: $L^p(G) = L^p(G; g_{ij})$ is the Banach space of measurable functions u on G with the finite norm $\|u; L^p(G)\|$ given by

$$\|u; L^p(G)\| := \left(\int_G |u|^p \, dV\right)^{1/p} \qquad (1 \le p < \infty)$$

and $\|u; L^\infty(G)\|$ is the essential supremum of $|u|$ on G. The second use: for a measurable vector field X on G we write $X \in L^p(G) = L^p(G; g_{ij})$ if $|X| = |X|_{g_{ij}} \in L^p(G)$ in the first sense and we set

$$\|X; L^p(G)\| := \| \, |X|; L^p(G)\|.$$

The *Dirichlet space* $L^{1,p}(G) = L^{1,p}(G; g_{ij})$ $(1 \le p \le \infty)$ is the class of functions $u \in L^1_{\mathrm{loc}}(G)$ with the distributional gradients $\nabla u \in L^p(G)$, where ∇u is determined by the relation

$$\int_G \langle \nabla u, \Psi\rangle \, dV = -\int_G u \operatorname{div}\Psi \, dV$$

for every C^∞ vector field Ψ on G with compact support in G. In the parametric domain (U, x) in G we have $\nabla u = (\partial u/\partial x^1, \dots, \partial u/\partial x^d)$. If $\Psi = (\psi_1, \dots, \psi_d)$ in U, then

$$\operatorname{div}\Psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\big(\sqrt{g}\, g^{ij}\psi_j\big).$$

The *Sobolev space* $W^{1,p}(G) = W^{1,p}(G, g_{ij})$ $(1 \le p \le \infty)$ is the Banach space $L^{1,p}(G) \cap L^p(G)$ equipped with the norm

$$\|u; W^{1,p}(G)\| := \|u; L^p(G)\| + \|\nabla u; L^p(G)\|.$$

Given a Riemannian manifold D of dimension $d \ge 2$ and given an exponent $1 < p < \infty$, the *Royden p-algebra* $M_p(D)$ is the Banach algebra $L^{1,p}(D) \cap L^\infty(D) \cap C(D)$ equipped with the norm

$$\|u; M_p(D)\| := \|u; L^\infty(D)\| + \|\nabla u; L^p(D)\|. \tag{7}$$

By the standard mollifier method we can show that the subalgebra $M_p(D) \cap C^\infty(D)$ is dense in $M_p(D)$ with respect to the norm in (7). Hence $M_p(D)$ may also be defined as the completion of $\{u \in C^\infty(D) : \|u; M_p(D)\| < \infty\}$ without appealing to the Dirichlet space. It is important that $M_p(D)$ is closed under lattice operations $\cup$ and $\cap$ given by $(u \cup v)(x) = \max\big(u(x), v(x)\big)$ and $(u \cap v)(x) = \min\big(u(x), v(x)\big)$ (cf. e.g. p. 21 in [4]). The maximal ideal space D_p^* of $M_p(D)$ is referred to as the *Royden p-compactification*, which can also be characterized as the compact Hausdorff space containing D as its open and dense subspace such that every function $u \in M_p(D)$ is continuously extended to D_p^* and $M_p(D)$, viewed as a subspace of $C(D_p^*)$ by this continuous extension, is dense in $C(D_p^*)$ with respect to its supremum norm.

8. Capacities of rings. A *ring* R in a Riemannian manifold D is a subset R of D with the following properties: R is a subdomain of D so that R is contained in a unique component D_R of D; $D_R \setminus R$ consists of exactly two components one of which, denoted by C_1, is compact and the other, denoted by C_0, is noncompact. The set C_1 will be referred to as the *inner part* of $R^c := D \setminus R$ and the set $D \setminus (R \cup C_1)$ as the *outer part* of R^c. We denote by $W(R)$ the class of functions $u \in W^{1,1}_{\text{loc}}(R) \cap C(D)$ such that $u = 1$ on the inner part of R^c and $u = 0$ on the outer part of R^c which includes C_0. The *p-capacity* $\text{cap}_p R$ $(1 \le p \le \infty)$ of the ring $R \subset D$ is given by

$$\text{cap}_p R := \inf_{u \in W(R)} \|\nabla u; L^p(R)\|^p \tag{9}$$

for $1 \le p < \infty$ and $\text{cap}_\infty R := \inf_{u \in W(R)} \|\nabla u; L^\infty(R)\|$. Note that $\text{cap}_p R$ does not depend upon which Riemannian manifold D the ring R is embedded as long as the metric structure on R is unaltered. The following inequality will be made use of (cf. e.g. p. 32 in [4]): if $1 < p < \infty$, if R is a ring in D, and if R_k $(1 \le k \le n)$ are disjoint rings contained in R each of which separates the boundary components of R, then

$$(\text{cap}_p R)^{1/(1-p)} \ge \sum_{k=1}^n (\text{cap}_p R_k)^{1/(1-p)}. \tag{10}$$

Suppose that a ring R is contained in a parametric domain (U,x) on D for which two metric structures (g_{ij}) and (δ_{ij}) can be considered. If the need occurs to indicate that $\mathrm{cap}_p R$ is considered on (U,δ_{ij}), then we write

$$\mathrm{cap}_p R = \mathrm{cap}_p(R,\delta_{ij}) = \inf_{u\in W(R)} \int_R |\nabla u(x)|^p_{\delta_{ij}}\,dx;$$

if $\mathrm{cap}_p R$ is considered on (U,g_{ij}), then we write

$$\mathrm{cap}_p R = \mathrm{cap}_p(R,g_{ij}) = \inf_{u\in W(R)} \int_R |\nabla u|^p_{g_{ij}}\,dV$$

for $1\le p<\infty$. Similar remarks are applied to $\mathrm{cap}_\infty(R,g_{ij})$ and $\mathrm{cap}_\infty(R,\delta_{ij})$. If, moreover, U is a λ-domain for any $\lambda\in[1,\infty)$, then (6) implies that

$$\frac{1}{\lambda^{(d+p)/2}}\mathrm{cap}_p(R,\delta_{ij}) \le \mathrm{cap}_p(R,g_{ij}) \le \lambda^{(d+p)/2}\mathrm{cap}_p(R,\delta_{ij}). \tag{11}$$

In the case $p=\infty$, the inequality corresponding to (11) takes the following form: $\lambda^{-1/2}\mathrm{cap}_\infty(R,\delta_{ij}) \le \mathrm{cap}_\infty(R,g_{ij}) \le \lambda^{1/2}\mathrm{cap}_\infty(R,\delta_{ij})$, which, however, will not be used in this paper.

We fix a parametric domain (U,x) in D. It is possible that the parametric domain is the d-dimensional Euclidean space $\mathbf{R}^d$ itself. A ring R contained in U is said to be a *spherical ring* in (U,x) if

$$R=\{x\in U : a<|x-P|<b\}, \tag{12}$$

where $P\in U$ and a and b are positive numbers with $0<a<b<\inf_{\partial U}|x-P|$. At this point we must be careful: in the case where R happens to be included in another parametric domain (V,y) of D, R may not be a spherical ring in (V,y) even if R is a spherical ring in (U,x). Namely, the notion of spherical rings cannot be introduced to the general Riemannian manifold D and is strictly attached to the parametric domain in question. Let R be a spherical ring in a parametric domain (U,x) with the expression (12). Then we have (cf. e.g. p. 35 in [4])

$$\mathrm{cap}_p R = \mathrm{cap}_p(R,\delta_{ij}) = \begin{cases} \omega_d\left(\dfrac{b^q-a^q}{q}\right)^{1-p} & (1<p<\infty,\ p\ne d),\\ \omega_d\left(\log\dfrac{b}{a}\right)^{1-d} & (p=d), \end{cases} \tag{13}$$

where we have set $q=(p-d)/(p-1)$ and ω_d is the surface area of the Euclidean unit sphere S^{d-1}. In passing we state that $\mathrm{cap}_1(R,\delta_{ij})=\omega_d a^{d-1}$ and $\mathrm{cap}_\infty(R,\delta_{ij})=1/(b-a)$, again results not used in this paper.

Another important ring in $\mathbf{R}^d$ which we use later is a *Teichmüller ring* R_T defined by $R_T = \mathbf{R}^d \setminus \{te_1 : t \in [-1,0] \cup [1,\infty)\}$, where e_1 is the unit vector $(1,0,\dots,0)$ in $\mathbf{R}^d$. We set

$$t_d := \operatorname{cap}_d(R_T, \delta_{ij}). \tag{14}$$

Finally we state a separation lemma on the topology of the Royden compactification. Let $(R_n)_{n\ge1}$ be a sequence of rings R_n in D $(n = 1,2,\dots)$ with the following properties: $(R_n \cup C_{n1}) \cap (R_m \cup C_{m1}) = \emptyset$ for $n \ne m$, where C_{n1} is the inner part of $(R_n)^c = D \setminus R_n$; $(R_n)_{n\ge1}$ does not accumulate in D, i.e. $\{n : E \cap (\overline{R_n} \cup C_{n1}) \ne \emptyset\}$ is a finite set for any compact set E in D. Such a sequence $(R_n)_{n\ge1}$ will be called an *admissible sequence*, which defines two disjoint closed sets X and Y in D as follows:

$$X := \bigcup_{n=1}^{\infty} C_{n1} \qquad \text{and} \qquad Y := \bigcap_{n=1}^{\infty} (D \setminus (R_n \cup C_{n1})).$$

We denote by $\operatorname{cl}(X; D_p^*)$ the closure of X in D_p^*. Although $X \cap Y = \emptyset$ in D, $\operatorname{cl}(X; D_p^*)$ and $\operatorname{cl}(Y; D_p^*)$ may intersect on the *Royden p-boundary*

$$\Gamma_p(D) := D_p^* \setminus D.$$

Concerning this we have the following result.

15. Lemma. *The set* $\operatorname{cl}\big(\bigcup_{n=1}^{\infty} R_n; D_p^*\big)$ *for an admissible sequence* $(R_n)_{n\ge1}$ *in D separates* $\operatorname{cl}(X; D_P^*)$ *and* $\operatorname{cl}(Y; D_p^*)$ *in* D_p^* *in the sense that*

$$\big(\operatorname{cl}(X; D_P^*)\big) \cap \big(\operatorname{cl}(Y; D_p^*)\big) = \emptyset \tag{16}$$

if and only if

$$\sum_{n=1}^{\infty} \operatorname{cap}_p R_n < \infty. \tag{17}$$

Proof. First we show that (16) implies (17). By (16) the Urysohn theorem assures the existence of a function $u \in C(D_p^*)$ such that $u = 3$ on $\operatorname{cl}(X; D_p^*)$ and $u = -2$ on $\operatorname{cl}(Y; D_p^*)$. Since $M_p(D)$ is dense in $C(D_p^*)$, there is a function $v \in M_p(D)$ such that $v > 2$ on X and $v < -1$ on Y. Finally, let $w = \big((v \cap 1) \cup 0\big) \in M_p(D)$, which satisfies $w \mid X = 1$, $w \mid Y = 0$ and $0 \le w \le 1$ on D. Set $w_n = w$ on $R_n \cup C_{n1}$ and $w_n = 0$ on $D \setminus (R_n \cup C_{n1})$ for $n = 1,2,\dots$. Clearly $w_n \in W(R_n)$ so that $\operatorname{cap}_p R_n \le \|\nabla w_n; L^p(R_n)\|^p$ $(n = 1,2,\dots)$ and $w = \sum_{n=1}^{\infty} w_n$. Since the supports of w_n in D $(n = 1,2,\dots)$ are mutually disjoint, we see that

$$\sum_{n=1}^{\infty} \operatorname{cap}_p R_n \le \sum_{n=1}^{\infty} \|\nabla w_n; L^p(R_n)\|^p = \|\nabla w; L^p(D)\|^p \le \|w; M_p(D)\|^p < \infty,$$

i.e. (17) has been deduced.

Conversely, suppose that (17) is the case. We wish to derive (16) from (17). Choose a function $w_n \in W(R_n)$ such that $\|\nabla w_n; L^p(R_n)\|^p < 2\mathrm{cap}_p R_n$ for each $n = 1, 2, \ldots$. We may suppose that $0 \le w_n \le 1$ on D by replacing w_n with $(w_n \cap 1) \cup 0$ if necessary (see e.g. p. 20 in [4]). Clearly $w := \sum_{n=1}^{\infty} w_n \in M_p(D)$ since $\|w; L^\infty(D)\| = 1$ and

$$\|\nabla w; L^p(D)\|^p = \sum_{n=1}^{\infty} \|\nabla w_n; L^p(D_n)\|^p \le 2\sum_{n=1}^{\infty} \mathrm{cap}_p R_n < \infty.$$

Observe that $w = 1$ on X and $w = 0$ on Y. Hence, by the continuity of w on D_p^*, we see that $w = 1$ on $\mathrm{cl}(X; D_p^*)$ and $w = 0$ on $\mathrm{cl}(Y; D_p^*)$, which yields (16). ◻

As a consequence of Lemma 15 we can characterize points in the Royden p-boundary $\Gamma_p(D) = D_p^* \setminus D$ among points in D_p^* in terms of their being not G_δ for $1 \le p \le d$. This is no longer true for $d < p \le \infty$. Recall that a point $\zeta \in D_p^*$ is said to be G_δ if there exists a countable sequence $(\Omega_i)_{i\ge 1}$ of open neighborhoods Ω_i of ζ such that $\bigcap_{i\ge 1} \Omega_i = \{\zeta\}$.

18. Corollary to Lemma 15. *A point ζ in D_p^* $(1 \le p \le d)$ belongs to D if and only if ζ is G_δ.*

Proof. We only have to show that $\zeta \in \Gamma_p(D) = D_p^* \setminus D$ is not G_δ. On the contrary, suppose ζ is G_δ so that there exists a sequence $(\Omega_i)_{i\ge 1}$ of open neighborhoods of ζ such that $\Omega_i \supset \mathrm{cl}(\Omega_{i+1}; D_p^*)$ $(i = 1, 2, \ldots)$ and $\bigcap_{i\ge 1} \Omega_i = \{\zeta\}$. Since D is dense in D_p^*, $H_i := D \cap \big(\Omega_i \setminus \mathrm{cl}(\Omega_{i+1}; D_p^*)\big)$ is a nonempty open subset of D for each i. Hence we can find a sequence $(P_n)_{n\ge 1}$ of points $P_n \in H_n$ $(n = 1, 2, \ldots)$ and a sequence $\big((U_n, x_n)\big)_{n\ge 1}$ of 2-domains (U_n, x_n) contained in H_n $(n = 1, 2, \ldots)$ such that $U_n = \{x_n : |x_n - P_n| < r_n\}$ $(r_n > 0,\ n = 1, 2, \ldots)$. Let $R_n := \{x_n : a_n < |x_n - P_n| < b_n\}$ $(0 < a_n < b_n := \frac{1}{2} r_n)$ be a spherical ring in (U_n, x_n). Clearly $(R_n)_{n\ge 1}$ is an admissible sequence. Since $\mathrm{cap}_p(R_n, \delta_{ij}) = \omega_d\big(|q|/\big(1 - (a_n/b_n)^{|q|}\big)\big)^{p-1} a_n^{|d-p|}$ by (13) for $1 < p < d$, $\mathrm{cap}_d(R_n, \delta_{ij}) = \omega_d/\big(\log(b_n/a_n)\big)^{d-1}$, and $\mathrm{cap}_1(R_n, \delta_{ij}) = \omega_d a_n^{d-1}$, we can see that $\mathrm{cap}_p(R_n, \delta_{ij}) < 2^{-n}$ by choosing $a_n \in (0, \frac{1}{2} r_n)$ small enough so that $\mathrm{cap}_p R = \mathrm{cap}_p(R, g_{ij}) \le 2^{(d+p)/2} \mathrm{cap}_p(R, \delta_{ij}) < 2^{(d+p)/2} 2^{-n}$ $(n = 1, 2, \ldots)$ by (11). Hence (17) is satisfied but (16) is invalid because the intersection on the left-hand side of (16) contains ζ due to the fact that $R_n \subset H_n$ $(n = 1, 2, \ldots)$. This is clearly a contradiction to Lemma 15. ◻

19. Analytic properties of quasi-isometric mappings. A *quasi-isometric* (or *quasiconformal*) mapping f of a Riemannian manifold D onto another D' is, as defined in the introduction, a homeomorphism f of D onto D' such that

$$K^{-1} \varrho(x, y) \le \varrho\big(f(x), f(y)\big) \le K \varrho(x, y)$$

for every pair of points x and y in D for some fixed $K \in [1,\infty)$, (or

$$\sup_{x\in D}\Bigl(\limsup_{r\downarrow 0}\Bigl(\Bigl(\max_{\varrho(x,y)=r}\varrho'\bigl(f(x),f(y)\bigr)\Bigr)\Big/\Bigl(\min_{\varrho(x,y)=r}\varrho'\bigl(f(x),f(y)\bigr)\Bigr)\Bigr)\Bigr)<\infty,$$

respectively), where ϱ and ϱ' are geodesic distances on D or D'. In this case we also say that f is a K-quasi-isometric referring to K. For simplicity, quasi-isometric (or quasiconformal) mappings will occasionally be abbreviated as qi (or qc). Consider a K-qi f of a d-dimensional Riemannian manifold D equipped with the metric tensor (g_{ij}) onto another d-dimensional Riemannian manifold D' equipped with the metric tensor (g'_{ij}). Fix an arbitrary $\lambda \in (0,\infty)$ and choose any λ-domain (U,x) in D and any λ-domain (U',x') in D' such that $f(U)=U'$. The mapping $f\colon (U,\delta_{ij}) \to (U',\delta_{ij})$ has the representation

$$x' = f(x) = \bigl(f^1(x),\dots,f^d(x)\bigr) \tag{20}$$

on U in terms of the parameters x and x'. As the composite mapping of id: $(U,\delta_{ij}) \to (U,g_{ij})$, $f\colon (U,g_{ij}) \to (U',g'_{ij})$, and id: $(U',g'_{ij}) \to (U',\delta_{ij})$, we see that the mapping $f\colon (U,\delta_{ij}) \to (U',\delta_{ij})$ is λK-qi since id: $(U,\delta_{ij}) \to (U,g_{ij})$ and id: $(U',g'_{ij}) \to (U',\delta_{ij})$ are $\sqrt{\lambda}$-qi as a consequence of $\lambda^{-1}\,|dx|^2 \le ds^2 \le \lambda\,|dx|^2$, where $dx = (dx^1,\dots,dx^d)$, $|dx|^2 = \delta_{ij}\,dx^i\,dx^j$, and $ds^2 = g_{ij}(x)\,dx^i\,dx^j$, which is deduced from $\lambda^{-1}(\delta_{ij}) \le (g_{ij}) \le \lambda(\delta_{ij})$. Hence we see that

$$\frac{1}{\lambda K}|x-y| \le |f(x)-f(y)| \le \lambda K|x-y| \tag{21}$$

whenever the line segment $[x,y] := \{(1-t)x+ty : t\in[0,1]\} \subset U$ and $[f(x),f(y)] \subset U'$. In particular (21) implies that

$$\limsup_{h\to 0}\frac{|f(x+h)-f(x)|}{|h|} \le \lambda K < \infty \tag{22}$$

for every $x \in U$ and

$$\liminf_{h\to 0}\frac{|f(x+h)-f(x)|}{|h|} \ge \frac{1}{\lambda K} > 0. \tag{23}$$

As an important consequence of (22), the Rademacher–Stepanoff theorem (cf. e.g. p. 218 in [1]) assures that $f(x)$ is differentiable at a.e. $x \in U$, i.e.

$$f(x+h)-f(x) = f'(x)h + \varepsilon(x,h)|h| \qquad (\lim_{h\to 0}\varepsilon(x,h)=0) \tag{24}$$

for a.e. $x \in U$, where $f'(x)$ is the $d\times d$ matrix $(\partial f^i/\partial x^j)$. Fix an arbitrary vector h with $|h|=1$. Then for any positive number $t>0$ we have, by replacing h in (24) with th,

$$|f'(x)h| - |\varepsilon(x,th)| \le \frac{|f(x+th)-f(x)|}{|th|}$$

and on letting $t \downarrow 0$ we obtain by (22) that $|f'(x)h| \le \lambda K$. Therefore

$$|f'(x)| := \sup_{|h|=1} |f'(x)h| \le \lambda K \tag{25}$$

for a.e. $x \in U$. Similarly we have

$$|f'(x)h| + |\varepsilon(x, th)| \ge \frac{|f(x+th) - f(x)|}{|th|}$$

and hence by (23) we deduce $|f'(x)h| \ge 1/\lambda K$. Thus

$$l\bigl(f'(x)\bigr) := \inf_{|h|=1} |f'(x)h| \ge \frac{1}{\lambda K}. \tag{26}$$

From (25) it follows that $|\partial f^i(x)/\partial x^j| \le |f'(x)| \le \lambda K$ for a.e. $x \in U$ $(i, j = 1, \dots, d)$ and thus $|\nabla f| = \bigl(\sum_{i=1}^d |\nabla f_i|^2\bigr)^{1/2} \in L^\infty(U)$. By (21), $f(x)$ is ACL (absolutely continuous on almost all straight lines which are parallel to coordinate axes). That $f(x)$ is ACL and $\nabla f \in L^\infty(U)$ is necessary and sufficient for f to belong to $L^{1,\infty}(U)$ (cf. e.g. pp. 8–9 in [7]) so that, by the continuity of f, we have

$$f \in W^{1,\infty}_{\mathrm{loc}}(D). \tag{27}$$

By (25) and (26) we have the matrix inequality

$$l\bigl(f'(x)\bigr)^2(\delta_{ij}) \le f'(x)^* f'(x) \le |f'(x)|^2(\delta_{ij})$$

for a.e. $x \in U$, where $f'(x)^*$ is the transposed matrix of $f'(x)$. Let $\lambda_1(x) \ge \dots \ge \lambda_d(x)$ be the square roots of the proper values of the symmetric positive matrix $f'(x)^* f'(x)$. Then

$$\frac{1}{\lambda K} \le l\bigl(f'(x)\bigr) = \lambda_d(x) \le \dots \le \lambda_1(x) = |f'(x)| \le \lambda K.$$

Observe that $\prod_{i=1}^d \lambda_i(x)^2 = \det\bigl(f'(x)^* f'(x)\bigr) = \bigl(\det f'(x)\bigr)^2$ is the square of the Jacobian $J_f(x)$ of f at x. Hence, by $\lambda K \lambda_i \ge 1$ $(i = 2, 3, \dots, d)$, we see that

$$\begin{aligned} |f'(x)|^p = \lambda_1(x)^p \le \lambda_1(x)(\lambda K)^{p-1} &\le \lambda_1(x)(\lambda K)^{p-1} \prod_{i=2}^d \bigl(\lambda K \lambda_i(x)\bigr) \\ = (\lambda K)^{d+p-2} \prod_{i=1}^d \lambda_i(x) &= (\lambda K)^{d+p-2} |J_f(x)|, \end{aligned}$$

i.e. we have deduced that

$$|f'(x)|^p \le (\lambda K)^{d+p-2} |J_f(x)| \tag{28}$$

for a.e. $x \in U$. This is used to prove the following result.

29. Proposition. *The pull-back $v = u \circ f$ of any u in $M_p(D')$ by a K-qi f of D onto D' belongs to $M_p(D)$ and satisfies the inequality*

$$\int_D |\nabla v(x)|^p_{g_{ij}} \sqrt{g(x)}\, dx \le K^{d+p-2} \int_{D'} |\nabla u(x')|^p_{g'_{ij}} \sqrt{g'(x')}\, dx' \tag{30}$$

and, in particular,

$$\|v; M_p(D)\| \le K^{(d+p-2)/p} \|u; M_p(D')\|. \tag{31}$$

Proof. The inequality (30) is nothing but

$$\|\nabla v; L^p(D)\| \le K^{(d+p-2)/p} \|\nabla u; L^p(D')\|.$$

This with $\|v; L^\infty(D)\| = \|u; L^\infty(D')\|$ implies (31). Suppose that Proposition 29 is true if $u \in M_p(D') \cap C^\infty(D')$. Since $M_p(D') \cap C^\infty(D')$ is dense in $M_p(D')$, for an arbitrary $u \in M_p(D')$, there exists a sequence $(u_k)_{k\ge1}$ in $M_p(D') \cap C^\infty(D')$ such that $\|u - u_k; M_p(D')\| \to 0$ $(k \to \infty)$. In particular $\|u_k - u_{k'}; M_p(D')\| \to 0$ $(k, k' \to \infty)$. By our assumption, $v_k := u_k \circ f \in M_p(D)$ $(k = 1, 2, \ldots)$. By (31), the inequalities $\|v_k - v_{k'}; M_p(D)\| \le K^{(d+p-2)/p} \|u_k - u_{k'}; M_p(D')\|$ assure that $\|v_k - v_{k'}; M_p(D)\| \to 0$ $(k, k' \to \infty)$. By the completeness of $M_p(D)$, since $\|v - v_k; L^\infty(D)\| \to 0$ $(k \to \infty)$, we see that $v \in M_p(D)$. By the validity of (30) (and hence of (31)) for v_k, we see that (30) is valid for v. For this reason we can assume $u \in M_p(D') \cap C^\infty(D')$ to prove Proposition 29.

It is clear by (25) that $v = u \circ f \in W^{1,\infty}_{\mathrm{loc}} \cap L^\infty(D) \cap C(D)$ if $u \in M_p(D') \cap C^\infty(D')$. Hence we only have to prove (30) to deduce $v \in M_p(D)$. Fix an arbitrary $\lambda \in (1, \infty)$. Let $D = \bigcup_{k=1}^\infty E_k$ be a union of disjoint Borel sets E_k in D such that each E_k is contained in a λ-domain U_k in D and $E'_k = f(E_k)$ in a λ-domain $U'_k = f(U_k)$ in D' for $k = 1, 2, \ldots$. Fix a k and consider the λK-qi f of (U_k, δ_{ij}) onto (U'_k, δ_{ij}) with the representation (20) on U_k in terms of the parameter x in U_k and x' in U'_k. By the chain rule we have

$$\nabla v(x) = f'(x)^* \nabla u\bigl(f(x)\bigr) \tag{32}$$

for a.e. $x \in U_k$. Since $|f'(x)^*| = |f'(x)|$, (28) and (32) yield

$$|\nabla v(x)|^p \le (\lambda K)^{d+p-2} \bigl|\nabla u\bigl(f(x)\bigr)\bigr|^p |J_f(x)|$$

for a.e. $x \in U_k$. In view of (22), the formula of the change of variables in integrations is valid for $x' = f(x)$:

$$\int_{E_k} \bigl|\nabla u\bigl(f(x)\bigr)\bigr|^p |J_f(x)|\, dx = \int_{E'_k} |\nabla u(x')|^p\, dx'.$$

From the above two displayed relations we deduce

$$\int_{E_k} |\nabla v(x)|^p \, dx \le (\lambda K)^{d+p-2} \int_{E'_k} |\nabla u(x')|^p \, dx'.$$

Observe that $|\nabla v|^p_{g_{ij}} \le \lambda^{p/2} |\nabla v|^p$ and $\sqrt{g} \le \lambda^{d/2}$, and, similarly, that $|\nabla u|^p \le \lambda^{p/2} |\nabla u|^p_{g'_{ij}}$ and $1 \le \lambda^{d/2}\sqrt{g'}$. The above displayed inequality then implies that

$$\int_{E_k} |\nabla v(x)|^p_{g_{ij}} \sqrt{g(x)} \, dx \le \lambda^{2(d+p-1)} K^{d+p-2} \int_{E'_k} |\nabla u(x')|^p \sqrt{g'(x')} \, dx'.$$

On adding these inequalities for $k = 1, 2, \ldots$ we obtain (30) with K^{d+p-2} replaced by $\lambda^{2(p+d-1)} K^{d+p-2}$. Since $\lambda \in (1, \infty)$ is arbitrary, we deduce (30) itself by letting $\lambda \downarrow 1$. □

33. Distortion of rings and their capacities. Throughout this section we fix two nonempty open sets V and V' in $\mathbf{R}^d$ (or, what amounts to the same, two parametric domains (V, x) and (V', x') in certain Riemannian manifolds D and D', respectively, considered as (V, δ_{ij}) and (V', δ_{ij})) and consider homeomorphisms f of V onto V'. We introduce two classes of such homeomorphisms f. The first class $\mathrm{Lip}(K) = \mathrm{Lip}(K; V, V')$ for a positive constant $K \in (0, \infty)$ is the family of homeomorphisms f of V onto V' such that

$$\limsup_{r \downarrow 0} \frac{\max_{|x-P|=r} |f(x) - f(P)|}{r} \le K \tag{34}$$

at every point $P \in V$. If the inverse f^{-1} of a homeomorphism f of V onto V' satisfies a similar property as (34), we should write $f^{-1} \in \mathrm{Lip}(K; V', V)$ but we often loosely write $f^{-1} \in \mathrm{Lip}(K)$. This class was first introduced by Gehring [3]. Note that $f(R)$ may be viewed as a ring in V' in the natural fashion along with a ring R in V: the inner part and the outer part of $f(R)^c = V' \setminus f(R)$ are the images of those of $R^c = V \setminus R$ under f, respectively. For each $p \in (1, \infty)$ the second class $Q_p(K, \delta) = Q_p(K, \delta; V, V')$ for two constants $K \in (0, \infty)$ and $\delta \in (0, \infty]$ is defined to be the family of homeomorphisms f of V onto V' satisfying the following condition:

$$\mathrm{cap}_p f(R) \le K \mathrm{cap}_p R \tag{35}$$

for every *spherical* ring R in V such that $\overline{R} \subset V$ and

$$\mathrm{cap}_p R < \delta. \tag{36}$$

In the case $\delta = \infty$ the condition (36) is redundant and thus the condition is given only by (35). The same remark as for the use of the notation $f^{-1} \in \mathrm{Lip}(K)$ also applies to the use of $f^{-1} \in Q_p(K, \delta)$. Clearly we see that $Q_p(K, \infty) \subset Q_p(K, \delta) \subset Q_p(K', \delta')$ for $0 < K \le K' < \infty$ and $0 < \delta' \le \delta \le \infty$. The class $Q_p(K, \infty)$ was introduced by Gehring [3] under the notation $Q_p(K)$. The following result plays a key role in the proof of Theorem 4 in this paper.

37. Lemma. *Suppose that* $1 \le p < d$, $0 < K < \infty$, *and* $0 < \delta \le \infty$ *are arbitrarily given. Then* $f, f^{-1} \in Q_p(K, \delta)$ *implies that* $f, f^{-1} \in \mathrm{Lip}(K_1)$, *where* K_1 *depends only upon* d, p, *and* K *and does not depend on* δ. *Explicitly,* K_1 *can be chosen as*

$$K_1 = K_1(K) := K^{1/(d-p)} \exp\bigl(\bigl(2^{d+1}\omega_d^{1+1/d} K^{2(d-1)/(d-p)} t_d^{-1/d}\bigr)^{d/(d-1)}\bigr). \tag{38}$$

Recall that t_d was given in (14). Lemma 37 is partly a generalization of Gehring's theorem ([3]): $f, f^{-1} \in Q_p(K, \infty)$ for $1 \le p < \infty$ with $p \ne d$ and $0 < K < \infty$ implies $f, f^{-1} \in \mathrm{Lip}(K')$, where K' depends only upon d, p, and K. Namely, Lemma 37 contains Gehring's theorem for $1 \le p < d$. However, Lemma 37 is no longer true for small finite positive numbers $\delta > 0$ if $1 \le p < d$ is replaced by $d < p \le \infty$. Nevertheless, Lemma 37 can be proven by suitably modifying Gehring's original proof ([3]) of the theorem. A complete proof of Lemma 37 can be found in [12].

If we assume that f is K_1-qi, then $f, f^{-1} \in \mathrm{Lip}(K_1)$, which is the conclusion of Lemma 37, follows immediately. We now prove the converse of this so that $f, f^{-1} \in \mathrm{Lip}(K)$ can be used for the definition of K-qi in the case of mappings between space open sets.

39. Lemma. *If* $f, f^{-1} \in \mathrm{Lip}(K)$, *then* f *is a* K*-qi of* V *onto* V'.

Proof. We define positive numbers $s(r) > 0$ for sufficiently small positive numbers $r > 0$ by $\min_{|x-P|=r} |f(x) - f(P)| =: s(r)$ for an arbitrarily fixed $P \in V$. On setting $P' := f(P)$ we see that $\max_{|x'-P'|=s(r)} |f^{-1}(x') - f^{-1}(P')| = r$. Observe that $s(r) \downarrow 0$ as $r \downarrow 0$. Hence, by $f^{-1} \in \mathrm{Lip}(K) = \mathrm{Lip}(K; V', V)$, we see that

$$\begin{aligned}\limsup_{r\downarrow 0} \frac{r}{s(r)} &= \limsup_{r\downarrow 0} \frac{\max_{|x'-P'|=s(r)} |f^{-1}(x') - f^{-1}(P')|}{s(r)} \\ &\le \limsup_{s\downarrow 0} \frac{\max_{|x'-P'|=s} |f^{-1}(x') - f^{-1}(P')|}{s} \le K.\end{aligned}$$

Therefore we infer that

$$\begin{aligned}\limsup_{r\downarrow 0} \frac{\max_{|x-P|=r} |f(x) - f(P)|}{\min_{|x-P|=r} |f(x) - f(P)|} &= \limsup_{r\downarrow 0} \left(\frac{\max_{|x-P|=r} |f(x) - f(P)|}{r} \cdot \frac{r}{s(r)} \right) \\ &\le \left(\limsup_{r\downarrow 0} \frac{\max_{|x-P|=r} |f(x) - f(P)|}{r} \right) \cdot \left(\limsup_{r\downarrow 0} \frac{r}{s(r)} \right) \le K^2,\end{aligned}$$

concluding that f is a qc of V onto V' by the metric definition (2) of quasiconformality. This assures that f is differentiable a.e. on V and $f \in W^{1,d}_{\mathrm{loc}}(V)$ (cf. e.g. pp. 109–111 in [19]). The latter in particular implies that f is ACL in an arbitrarily given direction l: f is absolutely continuous on almost all straight lines which are parallel to l. Suppose that f is differentiable at $x \in V$, i.e.

$$f(x+h) - f(x) = f'(x)h + \varepsilon(x,h)|h| \qquad (\lim_{h\to 0} \varepsilon(x,h) = 0).$$

For any $|h| = 1$ and any small $t > 0$, we have

$$|f'(x)h| \le \frac{|f(x+th) - f(x)|}{|th|} + |\varepsilon(x,th)| \le \frac{\max_{|y-x|=t} |f(y) - f(x)|}{t} + |\varepsilon(x,th)|.$$

On letting $t \downarrow 0$ we deduce $|f'(x)h| \le K$ since $f \in \mathrm{Lip}(K)$. We can thus conclude that

$$|f'(x)| = \sup_{|h|=1} |f'(x)h| \le K \tag{40}$$

for a.e. $x \in U$. We now maintain that

$$|f(x) - f(y)| \le K|x - y| \tag{41}$$

for any line segment $[x,y] = \{(1-t)x + ty : t \in [0,1]\} \subset V$. Since f is ACL in the direction of $[x,y]$, we see that f is absolutely continuous in V on almost all straight lines L parallel to $[x,y]$. As a consequence of (40), $|f'(x)| \le K$ in V on almost all straight lines L parallel to $[x,y]$ a.e. with respect to the linear measure on L. Hence we can find a sequence of line segments $[x_n, y_n] \subset V$ with the following properties: $x_n \to x$ and $y_n \to y$ as $n \to \infty$; f is absolutely continuous on $[x_n, y_n]$; $|f'(x)| \le K$ a.e. on $[x_n, y_n]$ with respect to the linear measure. Then

$$\begin{aligned} |f(x_n) - f(y_n)| &\le \int_{[x_n,y_n]} |df(z)| = \int_{[x_n,y_n]} |f'(z)\,dz| \\ &\le \int_{[x_n,y_n]} |f'(z)|\,|dz| \le K \int_{[x_n,y_n]} |dz| = K|x_n - y_n|, \end{aligned}$$

i.e. $|f(x_n) - f(y_n)| \le K|x_n - y_n|$ $(n = 1, 2, \dots)$, from which (41) follows by the continuity of f. By the symmetry of the situation for f and f^{-1}, we deduce the same inequality for f^{-1} so that

$$\frac{1}{K}|x - y| \le |f(x) - f(y)| \le K|x - y|$$

for every x and y in V with $[x,y] \subset V$ and $[f(x), f(y)] \subset V'$. Thus we can show the validity of (3) with respect to δ_{ij}-geodesic distances ϱ on V and ϱ' on V' so that f: $V \to V'$ is a K-qi. $\square$

Combining Lemmas 37 and 39, we obtain the following result, which will be used in the final part of the proof of the main theorem.

42. Theorem. *Suppose that* $1 \le p < d$, $0 < K < \infty$, *and* $0 < \delta \le \infty$ *are arbitrarily given. Then* $f, f^{-1} \in Q_p(K,\delta)$ *implies that* f *is a* K_1*-qi of* V *onto* V', *where* $K_1 = K_1(K)$ *is given by* (38) *so that it is independent of* δ.

43. Proof of the main theorem. In this section we assume that the exponent p is fixed in $(1,d)$ and we choose two Riemannian manifolds D and D' of the same dimension $d \ge 2$ which are orientable and countable and no component of D and D' is compact. The proof of Theorem 4 consists of two parts.

First part. Assume that there exists an almost quasi-isometric mapping f of D onto D', i.e. f is a homeomorphism of D onto D' and there exists a compact subset $E \subset D$ such that $f = f \mid D \setminus E$ is a K-quasi-isometric mapping of $D \setminus E$ onto $D' \setminus E'$, where $E' = f(E)$ is a compact subset of D' and K a constant in $[1,\infty)$. We are to show that f can be extended to a homeomorphism f^* of the Royden compactification D_p^* of D onto the Royden compactification $(D')_p^*$ of D'. Choose an arbitrary point ξ in the Royden p-boundary $\Gamma_p(D) = D_p^* \setminus D$. Since D is dense in D_p^*, the point ξ is an accumulation point of D.

We first show that the net $\big(f(x_\lambda)\big)$ in D' converges to a point $\xi' \in \Gamma_p(D')$ for any net (x_λ) in D convergent to ξ. Clearly $\big(f(x_\lambda)\big)$ does not accumulate at any point in D' along with (x_λ) so that the cluster points of $\big(f(x_\lambda)\big)$ are contained in $\Gamma_p(D')$. On the contrary we assume the existence of two subnets $(x_{\lambda'})$ and $(x_{\lambda''})$ of (x_λ) such that $\big(f(x_{\lambda'})\big)$ and $\big(f(x_{\lambda''})\big)$ are convergent to η' and η'' in $\Gamma_p(D')$, respectively, with $\eta' \ne \eta''$. Since $M_p(D')$ is dense in $C\big((D')_p^*\big)$ and forms a lattice, we can find a function $u \in M_p(D')$ such that $u \equiv 0$ in a neighborhood G' of E', $u(\eta') = 0$, and $u(\eta'') = 1$. Viewing $u \in M_p(D' \setminus E')$, we see by Proposition 29 that $v := u \circ f \in M_p(D \setminus E)$. Since $v \equiv 0$ in the neighborhood $G = f^{-1}(G')$ of $E = f^{-1}(E')$, we can conclude that $v \in M_p(D)$. From $v(x_{\lambda'}) = u\big(f(x_{\lambda'})\big)$ and $v(x_{\lambda''}) = u\big(f(x_{\lambda''})\big)$ it follows that $v(\xi) = u(\eta') = 0$ and $v(\xi) = u(\eta'') = 1$, which is a contradiction.

We next show that the nets $\big(f(x_{\lambda'})\big)$ and $\big(f(y_{\lambda''})\big)$ in D' converge to a point in $\Gamma_p(D')$ for any two nets $(x_{\lambda'})$ and $(y_{\lambda''})$ convergent to $\xi \in \Gamma_p(D)$. In fact, let (z_λ) be a net convergent to ξ such that (z_λ) contains $(x_{\lambda'})$ and $(y_{\lambda''})$ as its subnets. Then we see that $\lim_\lambda f(x_{\lambda'}) = \lim_{\lambda''} f(y_{\lambda''}) = \lim_\lambda f(z_\lambda)$. Hence we have shown that $f^*(\xi) := \lim_{x \in D, x \to \xi} f(x) \in \Gamma_p(D')$ for any $\xi \in \Gamma_p(D)$. On setting $f^* = f$ on D, we see that f^* is a continuous mapping of D_p^* onto $(D')_p^*$. The uniqueness of f^* on D_p^* is a consequence of the denseness of D in D_p^*. Similarly we can show that f^{-1} can also be uniquely extended to a continuous mapping $(f^{-1})^*$ of $(D')_p^*$ onto D_p^*. Since $(f^{-1})^* \circ f^*$ and $f^* \circ (f^{-1})^*$ are identities on D_p^* and $(D')_p^*$, respectively, as the unique extensions of id: $D \to D$ and id: $D' \to D'$, respectively, we see that f^* is a homeomorphism of D_p^* onto $(D')_p^*$ which is the unique extension of f: $D \to D'$. □

Second part. Suppose the existence of a homeomorphism f^* of D_p^* onto $(D')_p^*$. We are to show that $f := f^* \mid D$ is an almost quasi-isometric mapping of D onto D', which is the essential part of this paper.

Choose an arbitrary point $x \in D$. Since x is G_δ, $f^*(x) \in (D')_p^*$ is also G_δ so that $f^*(x) \in D'$ by Corollary 18. Thus we have shown that $f^*(D) \subset D'$. Similarly we can conclude that $(f^*)^{-1}(D') \subset D$. These show that $f^*(D) = D'$ and therefore $f := f^* \mid D$ is a homeomorphism of D onto D'. We are to show that f is an almost quasi-isometric mapping of D onto D'.

We fix a family $\mathscr{V} = \mathscr{V}_D = \{V\}$ of open sets V in D with the following properties: V is contained in a 2-domain U_V in D and $V' := f(V)$ is contained in the 2-domain $U'_{V'} = f(U_V)$ in D'; $\bigcup_{V\in\mathscr{V}} V = D$. This is possible since the family of 2-domains forms a base of open sets on any Riemannian manifold and $f\colon D \to D'$ is a homeomorphism. We set $\mathscr{V}' := \{V' : V' = f(V),\ V \in \mathscr{V}\}$, which enjoys the same properties as $\mathscr{V}$ does. We also fix an exhaustion $(\Omega_n)_{n\ge1}$ of D, i.e. Ω_n is a relatively compact open subset of D $(n = 1,2,\dots)$, $\overline{\Omega_n} \subset \Omega_{n+1}$ $(n = 1,2,\dots)$, and $\bigcup_{n\ge1}\Omega_n = D$. Then $(\Omega'_n)_{n\ge1}$ with $\Omega'_n := f(\Omega_n)$ $(n = 1,2,\dots)$ also forms an exhaustion of D'. We set $D_n := D\setminus\overline{\Omega_n}$ and $D'_n := f(D_n) = D'\setminus\overline{\Omega'_n}$ $(n = 1,2,\dots)$. Then $(D_n)_{n\ge1}$ ($(D'_n)_{n\ge1}$, respectively) is a decreasing sequence of open sets D_n (D'_n, respectively) with compact complements $D\setminus D_n$ ($D'\setminus D'_n$, respectively) such that $\bigcap_{n\ge1} D_n = \emptyset$ ($\bigcap_{n\ge1} D'_n = \emptyset$, respectively). If we set $\mathscr{V}_{D_n} := \{V\cap D_n : V \in \mathscr{V}_D \text{ and } V\cap D_n \ne \emptyset\}$ $(n = 1,2,\dots)$, then $\mathscr{V}_{D_n}$ plays the same role for D_n as $\mathscr{V}$ does for D. Take an arbitrary $n \in \{1,2,\dots\}$. If $f \in Q_p(2^{n+p-1}, 2^{-n}; V\cap D_n, V'\cap D'_n)$ ($f^{-1} \in Q_p(2^{n+p-1}, 2^{-n}; V'\cap D'_n, V\cap D_n)$, respectively) for every $V \in \mathscr{V}$ with $V\cap D_n \ne \emptyset$ (so that $V'\cap D'_n \ne \emptyset$), where $V' = f(V)$ and $V'\cap D'_n = f(V)\cap f(D_n) = f(V\cap D_n)$, then we write

$$f \in (n) \qquad (f^{-1} \in (n),\ \text{respectively}).$$

Hence, for example, $f \notin (n)$ means that there exists a $V \in \mathscr{V}$ with $V\cap D_n \ne \emptyset$ such that $f \notin Q_p(2^{n+p-1}, 2^{-n}; V\cap D_n, V'\cap D'_n)$. We maintain

44. Assertion. *If $f \in (n)$ ($f^{-1} \in (n)$, respectively) for some n, then $f \in (m)$ ($f^{-1} \in (m)$, respectively) for every $m \ge n$.*

In fact, $f \in (n)$ assures that $f \in Q_p(2^{n+p-1}, 2^{-n}; V\cap D_n, V'\cap D'_n)$ for every $V \in \mathscr{V}$ with $V\cap D_n \ne \emptyset$. Choose any $V \in \mathscr{V}$ with $V\cap D_m \ne \emptyset$. Since $D_m \subset D_n$, $V\cap D_n \ne \emptyset$ along with $V\cap D_m \ne \emptyset$ and therefore $f \in Q_p(2^{n+p-1}, 2^{-n}; V\cap D_n, V'\cap D'_n)$. In view of the fact that $2^{n+p-1} \le 2^{m+p-1}$ and $2^{-n} \ge 2^{-m}$, we have the inclusion relation $Q_p(2^{m+p-1}, 2^{-m}; V\cap D_m, V'\cap D'_m) \supset Q_p(2^{n+p-1}, 2^{-n}; V\cap D_n, V'\cap D'_n)$ so that $f \in Q_p(2^{m+p-1}, 2^{-m}; V\cap D_m, V'\cap D'_m)$, i.e. $f \in (m)$, which completes the proof of Assertion 44. Next we assert

45. Assertion. *If $f \in (n)$ and $f^{-1} \in (n)$ for some n, then $f = f \mid D_n$ is a qi of D_n onto D'_n.*

Indeed, by Theorem 42, we see that $f\colon (V\cap D_n,\delta_{ij})\to(V'\cap D'_n,\delta_{ij})$ is a K_1-qi with $K_1=K_1(2^{n+p-1})$ (cf. (38)). Clearly id: $(V\cap D_n,g_{ij})\to(V\cap D_n,\delta_{ij})$ and id: $(V'\cap D'_n,\delta_{ij})\to(V'\cap D'_n,g'_{ij})$ are $\sqrt{2}$-qi, where (g'_{ij}) is the metric tensor on D'. Therefore, as the suitable composition of these maps above, we see that $f\colon (V\cap D_n,g_{ij})\to(V'\cap D'_n,g'_{ij})$ is a $2K_1$-qi. Since this is true for every $V\in\mathscr{V}$ with $V\cap D_n\neq\emptyset$ and $\bigcup_{V\in\mathscr{V}}V=D\supset D_n$, we can conclude that $f\colon D_n\to D'_n$ is a $2K_1$-qi. The proof of Assertion 45 is thus complete.

To complete the proof of the second part it is sufficient to show that $f\colon D_n\to D'_n$ is a qi for some n. We prove it by contradiction. To get a contradiction, suppose that $f\colon D_n\to D'_n$ is not qi for every $n=1,2,\ldots$. Then we maintain that either $f\notin(n)$ for every n or $f^{-1}\notin(n)$ for every n. In fact, if $f\notin(n)$ for every n, then we are done. Otherwise, there is a k with $f\in(k)$. Then by Assertion 44 we have $f\in(n)$ for every $n\geq k$. In this case we must have $f^{-1}\notin(n)$ for every n and the assertion is assured. To see this assume that $f^{-1}\in(l)$ for some l. Then $f^{-1}\in(n)$ for every $n\geq l$ again by Assertion 44. Then $f\in(k\cup l)$ and $f^{-1}\in(k\cup l)$. By Assertion 45 we see that f is a qi of $D_{k\cup l}$ onto $D'_{k\cup l}$, contradicting our assumption. On interchanging the roles of f and f^{-1} (and thus those of D and D') if necessary, we can assume that

$$f\notin(n)\qquad(n=1,2,\ldots),$$

from which we will derive a contradiction.

The fact that $f\notin(1)$ implies the existence of a 2-domain $V\in\mathscr{V}_{D_1}$ such that $f\notin Q_p\big(2^{1+p-1},2^{-1};V,f(V)\big)$. We can then find a spherical ring $S_1\subset V(\subset D_1)$ such that

$$\mathrm{cap}_pS_1<2^{-1},\qquad \mathrm{cap}_pf(S_1)>2^{1+p-1}\mathrm{cap}_pS_1.$$

Here cap_pS_1 means $\mathrm{cap}_p(S_1,\delta_{ij})$. We set $n_1:=1$. Let n_2 be the least integer such that $n_2\geq n_1+1$ (and hence $D_{n_1+1}\supset D_{n_2}$) and $\overline{D_{n_2}}\cap\overline{S_{n_1}}=\emptyset$. Since $f\notin(n_2)$, there exists a $V\in\mathscr{V}_{D_{n_2}}$ with $f\notin Q_p\big(2^{n_2+p-1},2^{-n_2};V,f(V)\big)$. Hence we can find a spherical ring $S_{n_2}\subset V(\subset D_{n_2})$ such that

$$\mathrm{cap}_pS_{n_2}<2^{-n_2},\qquad \mathrm{cap}_pf(S_{n_2})>2^{n_2+p-1}\mathrm{cap}_pS_{n_2},$$

where $\mathrm{cap}_pS_{n_2}$ means $\mathrm{cap}_p(S_{n_2},\delta_{ij})$. Repeating this process we can construct a sequence $(S_{n_k})_{k\geq1}$ of spherical rings S_{n_k} with the following properties: $n_k+1\leq n_{k+1}$; $S_{n_k}\subset D_{n_k}$; $\overline{S_{n_k}}\cap\overline{S_{n_l}}=\emptyset$ $(k\neq l)$;

$$\mathrm{cap}_pS_{n_k}<2^{-n_k},\qquad \mathrm{cap}_pf(S_{n_k})>2^{n_k+p-1}\mathrm{cap}_pS_{n_k}\qquad(k=1,2,\ldots).\tag{46}$$

Fix a k and set $T=S_{n_k}$. Since it is a spherical ring in a 2-domain $(U_{V_{n_k}},x)$ and contained in V_{n_k}, T has a representation $T=\{x:a<|x-P|<b\}$, where

$P \in V_{n_k}$ and $0 < a < b < \infty$. Let $l = [(2^{-n_k}/\mathrm{cap}_p T)^{1/(p-1)}] > 0$, where $[\ ]$ is the Gaussian symbol, which means that

$$l^{p-1} \le \frac{2^{-n_k}}{\mathrm{cap}_p T} < (l+1)^{p-1} \le 2^{p-1} l^{p-1}. \tag{47}$$

Using the notation $q = (p-d)/(p-1)$ (cf. (13)) we set

$$t_j := \left(\frac{(l-j)a^q + jb^q}{l} \right)^{1/q} \qquad (j = 0, 1, \dots, l).$$

We divide the ring T into l small sphherical rings T_j given by

$$T_j := \{ x : t_{j-1} < |x - P| < t_j \} \qquad (j = 1, \dots, l).$$

By (13) we have $\mathrm{cap}_p T = \mathrm{cap}_p(T, \delta_{ij}) = \omega_d \big((b^q - a^q)/q \big)^{1-p}$. Similarly

$$\begin{aligned} \mathrm{cap}_p T_j &= \omega_d \left(\frac{t_j^q - t_{j-1}^q}{q} \right)^{1-p} \\ &= \omega_d \left(\frac{1}{q} \left(\frac{1}{l} \big((l-j)a^q + jb^q \big) - \frac{1}{l} \big((l-j+1)a^q + (j-1)b^q \big) \right) \right)^{1-p} \\ &= \omega_d \left(\frac{b^q - a^q}{q} \right)^{1-p} l^{p-1} = l^{p-1} \mathrm{cap}_p T, \end{aligned}$$

i.e., we have shown that $\mathrm{cap}_p T_j = l^{p-1} \mathrm{cap}_p T$. Therefore we have the following identity for the subdivision $\{T_j\}_{1 \le j \le l}$ of T:

$$\sum_{j=1}^{l} (\mathrm{cap}_p T_j)^{1/(1-p)} = (\mathrm{cap}_p T)^{1/(1-p)}. \tag{48}$$

Concerning the induced subdivision $\{f(T_j)\}$ of $f(T)$, the general inequality (10) implies the inequality

$$\sum_{j=1}^{l} \big(\mathrm{cap}_p f(T_j) \big)^{1/(1-p)} \le \big(\mathrm{cap}_p f(T) \big)^{1/(1-p)}. \tag{49}$$

Now suppose that $\mathrm{cap}_p f(T_j) \le 2^{n_k + p - 1} \mathrm{cap}_p T_j$ for every $1 \le j \le l$. Then $\big(\mathrm{cap}_p f(T_j) \big)^{1/(1-p)} \ge 2^{(n_k + p - 1)/(1-p)} (\mathrm{cap}_p T_j)^{1/(1-p)}$ for every $1 \le j \le l$. Using

(49) and (48) we deduce

$$\begin{aligned}\big(\mathrm{cap}_p f(T)\big)^{1/(1-p)} &\ge \sum_{j=1}^{l}\big(\mathrm{cap}_p f(T_j)\big)^{1/(1-p)}\\ &\ge 2^{(n_k+p-1)/(1-p)}\sum_{j=1}^{l}(\mathrm{cap}_p T_j)^{1/(1-p)}\\ &= 2^{(n_k+p-1)/(1-p)}(\mathrm{cap}_p T)^{1/(1-p)},\end{aligned}$$

which means that $\mathrm{cap}_p f(T) \le 2^{n_k+p-1}\mathrm{cap}_p T$. This contradicts (46) since $T = S_{n_k}$. Therefore there must exist a number $j_0 \in \{1,\dots,l\}$ such that

$$\mathrm{cap}_p f(T_{j_0}) > 2^{n_k+p-1}\mathrm{cap}_p T_{j_0}. \tag{50}$$

We now set $R_k := T_{j_0}$. By (47) we have $l^{p-1}\mathrm{cap}_p T \le 2^{-n_k} \le 2^{p-1}l^{p-1}\mathrm{cap}_p T$. Since $l^{p-1}\mathrm{cap}_p T = \mathrm{cap}_p T_{j_0} = \mathrm{cap}_p R_k$, we see that

$$\mathrm{cap}_p R_k \le 2^{-n_k} \le 2^{p-1}\mathrm{cap}_p R_k.$$

This is equivalent to $\mathrm{cap}_p R_k \le 2^{-n_k}$ ($< 2^{-k}$ since $n_k > k$) and $\mathrm{cap}_p R_k \ge 2^{-n_k-p+1}$. The latter inequality with (50) implies that

$$\mathrm{cap}_p f(R_k) > 2^{n_k+p-1}\mathrm{cap}_p R_k \ge 2^{n_k+p-1}\cdot 2^{-n_k-p+1} = 1.$$

By (46), $\mathrm{cap}_p(R_k, g_{ij}) < 2^{(d+p)/2}\cdot 2^{-k}$ and $\mathrm{cap}_p\big(f(R_k), g_{ij}\big) > 2^{(d+p)/2}$.

We have thus constructed an admissible sequence $(R_k)_{k\ge 1}$ of rings R_k in D in the sense of Section 8 (cf. Lemma 15) such that $\mathrm{cap}_p R_k = \mathrm{cap}_p(R_k, g_{ij})$ and $\mathrm{cap}_p f(R_k) = \mathrm{cap}_p\big(f(R_k), g'_{ij}\big)$ satisfy

$$\mathrm{cap}_p R_k < 2^{(d+p)/2}\cdot 2^{-k} \qquad \text{and} \qquad \mathrm{cap}_p f(R_k) > 2^{(d+p)/2} \tag{51}$$

for every $k = 1, 2, \dots$. Let C_{k1} be the inner part of $R_k^c = D \setminus R_k$ and we set

$$X := \bigcup_{k=1}^{\infty} C_{k1} \qquad \text{and} \qquad Y := \bigcap_{k=1}^{\infty}\big(D \setminus (R_k \cup C_{k1})\big)$$

as in Section 8 (cf. Lemma 15). The first inequality in (51) implies that

$$\sum_{k=1}^{\infty}\mathrm{cap}_p R_k < \sum_{k=1}^{\infty} 2^{(d+p)/2}\cdot 2^{-k} = 2^{(d+p)/2} < \infty$$

and therefore Lemma 15 assures that

$$\big(\mathrm{cl}(X;D_p^*)\big)\cap\big(\mathrm{cl}(Y;D_p^*)\big)=\emptyset.$$

Due to the fact that f^* is a homeomorphism of D_p^* onto $(D')_p^*$, we see that

$$\begin{aligned}\big(\mathrm{cl}\big(f(X);(D')_p^*\big)\big)\cap\big(\mathrm{cl}\big(f(Y);(D')_p^*\big)\big)&=f^*\big(\mathrm{cl}(X,D_p^*)\big)\cap f^*\big(\mathrm{cl}(Y;D_p^*)\big)\\&=f^*\big(\big(\mathrm{cl}(X;D_p^*)\big)\cap\big(\mathrm{cl}(Y;D_p^*)\big)\big)\\&=f^*(\emptyset)=\emptyset.\end{aligned}$$

Since again $\big(f(R_k)\big)_{k\ge1}$ is an admissible sequence of rings $f(R_k)$ on D', the above relation must imply by Lemma 15 that $\sum_{k=1}^\infty \mathrm{cap}_p f(R_k)<\infty$. However, the second inequality in (51) implies that

$$\sum_{k=1}^\infty \mathrm{cap}_p f(R_k)\ge\sum_{k=1}^\infty 2^{(d+p)/2}=\infty,$$

which is a contradiction. This comes from the erroneous assumption that $f\colon D_n\to D'_n$ is not a qi for every $n=1,2,\dots$, and thus we have established the existence of an n such that $f=f\mid D_n$ is a qi of D_n onto D'_n. The second part of the proof of the main theorem is herewith complete. □

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