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GROWTH PROPERTIES OF SPHERICAL MEANS FOR MONOTONE BLD FUNCTIONS IN THE UNIT BALL

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Abstract. Let u be a monotone (in the sense of Lebesgue) function on the unit ball $\mathbf{B}$ of $\mathbf{R}^n$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\alpha \, dx < \infty,$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \alpha < p-1$. Then u has a boundary limit $f(\xi)$ for almost every $\xi \in \partial\mathbf{B}$, and u may be considered as a Dirichlet solution for the boundary function f. Our aim in this paper is to deal with growth properties of the spherical means

$$S_q(u_r - f) \equiv \left(\int_{\partial\mathbf{B}} |u(r\xi) - f(\xi)|^q \, dS(\xi) \right)^{1/q}.$$

In fact, we prove that

$$\lim_{r\to 1-0} (1-r)^{-\omega} S_q(u_r - f) = 0$$

when $p \leqq q \leqq \infty$ and $\omega = (n-1)/q - (n-p+\alpha)/p > 0$.

1. Introduction

If f is a pth summable function on the boundary $\mathbf{S}$ of the unit ball $\mathbf{B}$ in $\mathbf{R}^n$, then we can find a (weak) Dirichlet solution u which is harmonic in $\mathbf{B}$ and

$$\lim_{r\to 1-0} S_p(u_r - f) = 0, \tag{1}$$

where $1 \leqq p < \infty$, $u_r(z) = u(rz)$ for $z \in \mathbf{S}$ and

$$S_p(v) = \left(\frac{1}{\sigma_n} \int_{\mathbf{S}} |v(z)|^p \, dS(z) \right)^{1/p}$$

with σ_n denoting the surface area of $\mathbf{S}$. Furthermore, for functions f in some Lipschitz spaces $\Lambda_\beta^{p,p}(\mathbf{S})$ (see Stein [16]), we can find Dirichlet solutions u in BLD

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(Beppo Levi and Deny) spaces for which (1) is satisfied. Conversely, if u is a harmonic function on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\alpha \, dx < \infty, \tag{2}$$

where $1 < p < \infty$ and $-1 < \alpha < p-1$, then it is well known that u has nontangential limits $f(\xi)$ at many boundary points $\xi \in \mathbf{S}$ and f belongs to a certain Lipschitz space (see Stein [16] and the first author [9], [10]). Of course, u is a Dirichlet solution for f.

In this paper, we are concerned with weighted limits of $S_q(u_r - f)$ for monotone BLD functions u on $\mathbf{B}$ with boundary values f. We say that a continuous function u is monotone in an open set G, in the sense of Lebesgue, if both

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x) \qquad \text{and} \qquad \min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold for every relatively compact open set D with the closure $\overline{D} \subset G$ (see [5]). Clearly, harmonic functions are monotone, and more generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone. For these facts, see Gilbarg–Trudinger [2], Heinonen–Kilpeläinen–Martio [3], Reshetnyak [13], Serrin [14] and Vuorinen [17] and [18].

Our starting point is a result of Gardiner [1, Theorem 2] which states that if u is a Green potential in the unit ball $\mathbf{B}$, then

$$\liminf_{r \to 1-0} (1-r)^{(n-1)(1-1/q)} S_q(u_r) = 0$$

when $(n-3)/(n-1) < 1/q \leqq (n-2)/(n-1)$ and $q > 0$.

Our first aim in this paper is to show the following result.

Theorem 1. *Let u be a monotone function on $\mathbf{B}$ satisfying (2) with $n-1 < p \leqq n+\alpha$. If $p \leqq q < \infty$ and*

$$\omega = \frac{n-1}{q} - \frac{n-p+\alpha}{p} > 0,$$

then

$$\lim_{r \to 1-0} (1-r)^{-\omega} S_q(U_r) = 0,$$

where $U_r(\xi) = u(r\xi) - u(\xi)$ for $\xi \in \mathbf{S}$.

The sharpness of the exponent will be discussed in the final section. We also find a BLD function u satisfying (2) and

$$\limsup_{r \to 1-0} (1-r)^{-\omega} S_q(U_r) = \infty,$$

when $\alpha < 0$.

Corollary 1. *Let u be a coordinate function of a quasiregular mapping on* $\mathbf{B}$ *satisfying* (2). *If $n-1 < p \leqq n+\alpha$, $p \leqq q < \infty$ and $\omega = (n-1)/q-(n-p+\alpha)/p > 0$, then*

$$\lim_{r\to 1-0}(1-r)^{-\omega} S_q(U_r) = 0.$$

For the definition and basic properties of quasiregular mappings, we refer to [3], [13], and [17]. In particular, a coordinate function $u = f_i$ of a quasiregular mapping $f = (f_1, \dots, f_n)\colon \mathbf{B} \to \mathbf{R}^n$ is $\mathscr{A}$-harmonic (see [3, Theorem 14.39] and [13]) and monotone in $\mathbf{B}$, so that Theorem 1 gives the present corollary.

It is well known that the coordinate functions of a bounded quasiconformal mapping on $\mathbf{B}$ have finite n-Dirichlet integral (see Vuorinen [18]), so that Corollary 1 gives the following result.

Corollary 2. *Let u be a coordinate function of a bounded quasiconformal mapping on* $\mathbf{B}$. *If $n \leqq q < \infty$, then*

$$\lim_{r\to 1-0}(1-r)^{-(n-1)/q} S_q(U_r) = 0.$$

Next we are concerned with the case $q = \infty$. In order to give a general result, we consider a nondecreasing positive function φ on the interval $[0, \infty)$ such that φ is of log-type, that is, there exists a positive constant M satisfying

$$\varphi(r^2) \leqq M\varphi(r) \qquad \text{for all } r \geqq 0.$$

Set $\Phi_p(r) = r^p\varphi(r)$ for $p > 1$. Our next aim is to study the boundary behavior of monotone BLD functions u on $\mathbf{B}$, which satisfy the weighted condition

$$\int_{\mathbf{B}} \Phi_p(|\nabla u(x)|)(1-|x|)^\alpha \, dx < \infty. \tag{4}$$

Consider the function

$$\kappa(r) = \left[\int_0^r \left(t^{n-p+\alpha}\varphi(t^{-1})\right)^{-1/(p-1)} \frac{dt}{t}\right]^{1-1/p}$$

for $r > 0$. We see (cf. [15, Lemma 2.4]) that if $n-p+\alpha < 0$, then

$$\kappa(r) \sim \left[r^{n-p+\alpha}\varphi(r^{-1})\right]^{-1/p} \qquad \text{as } r \to 0$$

and if $n-p+\alpha = 0$ and $\varphi(r) = \left(\log(e+r)\right)^\sigma$ with $\sigma > p-1$, then

$$\kappa(r) \sim \left[\log(1/r)\right]^{(p-1-\sigma)/p} \qquad \text{as } r \to 0.$$

Theorem 2. *Let u be a monotone function on* $\mathbf{B}$ *satisfying* (4) *with $-1 < \alpha \leqq p-n$. If $\kappa(1) < \infty$, then*

$$\lim_{r\to 1-0}\left[\kappa(1-r)\right]^{-1} S_\infty(U_r) = 0,$$

where $U_r(\xi) = u(r\xi) - u(\xi)$ for $\xi \in \mathbf{S}$.

Corollary 3. *Let u be a coordinate function of a quasiregular mapping on* $\mathbf{B}$ *satisfying* (4) *with* $-1 < \alpha \leqq p - n$. *If* $\kappa(1) < \infty$, *then*

$$\lim_{r \to 1-0} \left[\kappa(1-r)\right]^{-1} S_\infty(U_r) = 0.$$

For related results, we also refer to Herron–Koskela [4], the first author [8], [11] and the authors [12].

Finally we wish to express our deepest appreciation to the referee for his useful suggestions.

2. Proof of Theorem 1

Throughout this paper, let M denote various constants independent of the variables in question.

For a proof of Theorem 1, we need the following result, which gives an essential tool in treating monotone functions.

Lemma 1 (cf. [4], [6], [10]). *Let* $p > n-1$. *If* u *is a monotone BLD function on* $B(x_0, 2r)$, *then*

$$(5) \quad |u(x) - u(y)|^p \leqq M r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \qquad \text{whenever } x, y \in B(x_0, r).$$

Lemma 1 is a consequence of Sobolev's theorem, so that the restriction $p > n-1$ is needed; for a proof of Lemma 1, see for example [4, Lemma 7.1] or [10, Theorem 5.2, Chapter 8].

Now we are ready to prove Theorem 1, along the same lines as in the proof of [12, Theorem 2].

Proof of Theorem 1. Let u be a monotone function on $\mathbf{B}$ satisfying (2) with $n - 1 < p \leqq n + \alpha$. If $|s - t| \leqq r < \frac{1}{2}(1-t)$, then Lemma 1 gives

$$\begin{aligned} |S_q(u_s - u_t)| &= \left(\frac{1}{\sigma_n} \int_{\mathbf{S}} |u(s\xi) - u(t\xi)|^q \, dS(\xi) \right)^{1/q} \\ &\leqq M r^{(p-n)/p} \left(\int_{\mathbf{S}} \left(\int_{B(t\xi,2r)} |\nabla u(z)|^p \, dz \right)^{q/p} dS(\xi) \right)^{1/q}, \end{aligned}$$

so that Minkowski's inequality for integrals yields

$$\begin{aligned} |S_q(u_s - u_t)| \leqq M r^{(p-n)/p} & \left(\int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p \right. \\ & \left. \times \left(\int_{\{\xi \in \mathbf{S} : |\xi - z/t| < 2r/t\}} dS(\xi) \right)^{p/q} dz \right)^{1/p} \\ \leqq M r^{(p-n)/p} (2r/t)^{(n-1)/q} & \left(\int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p \, dz \right)^{1/p}. \end{aligned}$$

Let $r_j = 2^{-j-1}$, $t_j = 1-r_{j-1}$ and $A_j = B(0,1-r_j)-B(0,1-3r_j)$ for $j = 1,2,\dots$. For simplicity, set

$$\omega = -(n-p+\alpha)/p + (n-1)/q > 0.$$

Then we find

$$|S_q(u_{t_j} - u_r)| \leqq M r_{j+1}^{\omega} \left(\int_{A_j} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p}$$

for $t_j \leqq r < t_j + r_{j+1}$,

$$|S_q(u_r - u_s)| \leqq M r_{j+2}^{\omega} \left(\int_{A_j} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p}$$

for $t_j + r_{j+1} \leqq r < s < t_j + r_{j+1} + r_{j+2}$, and

$$|S_q(u_s - u_{t_{j+1}})| \leqq M r_{j+2}^{\omega} \left(\int_{A_{j+1}} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p}$$

for $t_j + r_{j+1} + r_{j+2} \leqq s < t_{j+1}$. Collecting these results, we have

$$\begin{aligned} |S_q(u_{t_j} - u_r)| \leqq{}& M r_j^{\omega} \left(\int_{A_j} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p} \\ &+ M r_{j+1}^{\omega} \left(\int_{A_{j+1}} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p} \end{aligned}$$

for $t_j \leqq r < t_{j+1}$. Hence it follows that

$$|S_q(u_r - u_{t_{j+m}})| \leqq M \sum_{l=j}^{j+m} r_l^{\omega} \left(\int_{A_l} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p}$$

for $t_j \leqq r < t_{j+m}$. Since $A_l \cap A_k = \emptyset$ for $l \geqq k+2$, Hölder's inequality gives

$$\begin{aligned} |S_q(u_r - u_{t_{j+m}})| &\leqq M \left(\sum_{l=j}^{j+m} r_l^{p'\omega} \right)^{1/p'} \left(\sum_{l=j}^{j+m} \int_{A_l} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p} \\ &\leqq M r_j^{\omega} \left(\int_{B(0,1-r_{j+m})-B(0,1-3r_j)} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p} \end{aligned}$$

for $t_j \leqq r < t_{j+m}$, where $1/p + 1/p' = 1$. Now, letting $m \to \infty$, we establish

$$|S_q(U_r)| \leqq M(1-r)^{\omega} \left(\int_{\mathbf{B}-B(0,1-3r_j)} |\nabla u(z)|^p (1-|z|)^{\alpha}\, dz \right)^{1/p}$$

for $t_j \leqq r < t_{j+1}$, which implies that

$$\lim_{r\to 1} (1-r)^{-\omega} S_q(U_r) = 0,$$

as required.

3. Proof of Theorem 2

If $B(x,2r) \subset \mathbf{B}$, then, applying Lemma 1 and dividing the domain of integration into two parts

$$E_1 = \{z \in B(x,2r) : |\nabla u(z)| > r^{-\delta}\},$$
$$E_2 = B(x,2r) - E_1,$$

we have

$$(6) \qquad |u(x)-u(y)|^p \leqq Mr^{(1-\delta)p} + Mr^{p-n}\left[\varphi(r^{-1})\right]^{-1}\int_{B(x,2r)} \Phi_p(|\nabla u(z)|)\,dz$$

for $y \in B(x,r)$, where M may depend on δ, $0<\delta<1$.

Let $r_j = 2^{-j-1}$, $j = 0,1,\dots$. For $x \in \mathbf{B}$, let $x_j = (1-2r_j)\xi$ with $\xi = x/|x|$. Then we find

$$|u(x_j)-u(x)| \leqq Mr_j^{1-\delta} + Mr_j^{(p-n)/p}\left[\varphi(r_j^{-1})\right]^{-1/p}\left(\int_{B(x_j,r_j)} \Phi_p(|\nabla u(z)|)\,dz\right)^{1/p}$$

when $\frac{3}{2}r_j < \varrho(x) < \frac{5}{2}r_j$, where $\varrho(x) = \big|1-|x|\big|$. Hölder's inequality gives

$$\begin{aligned}
|u(x_{j+m})-u(x_j)| &\leqq M\sum_{l=j}^{j+m} r_l^{1-\delta} + \sum_{l=j}^{j+m} r_l^{\omega}\left[\varphi(r_l^{-1})\right]^{-1/p} \\
&\qquad \times \left(\int_{B(x_l,r_l)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p} \\
&\leqq Mr_j^{1-\delta} + M\left(\sum_{l=j}^{j+m} r_l^{p'\omega}\left[\varphi(r_l^{-1})\right]^{-p'/p}\right)^{1/p'} \\
&\qquad \times \left(\sum_{l=j}^{j+m}\int_{B(x_l,r_l)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p} \\
&\leqq Mr_j^{1-\delta} + M\kappa(r_j)\left(\int_{\mathbf{B}-B(0,1-3r_j)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p},
\end{aligned}$$

where $\omega = (p-n-\alpha)/p$. Since $\lim_{r\to 0}\kappa(r) = 0$, we see that $\{u(x_j)\}$ is a Cauchy sequence. Thus (6) implies that $\lim_{r\to 1} u(r\xi)$ exists and is finite for every $\xi \in \mathbf{S}$. Now, letting $m \to \infty$, we have

$$\begin{aligned}
|U(x_j)| &= |u(x_j)-u(\xi)| \\
&\leqq Mr_j^{1-\delta} + M\kappa(r_j)\left(\int_{\mathbf{B}-B(0,1-3r_j)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p}.
\end{aligned}$$

Hence we establish

$$|U(x)| \leqq M\varrho(x)^{1-\delta} + M\kappa\big(\varrho(x)\big)\left(\int_{\mathbf{B}-B(0,1-3\varrho(x))} \Phi_p(|\nabla u(z)|)\varrho(z)^\alpha \, dz\right)^{1/p}.$$

Note that

$$\kappa(r) \geqq \left[\int_{r/2}^{r} \big(t^{n-p+\alpha}\varphi(t^{-1})\big)^{-1/(p-1)} \frac{dt}{t}\right]^{1-1/p} \geqq \big[r^{n-p+\alpha}\varphi(r^{-1})\big]^{-1/p}.$$

If we take δ so that $(n-p+\alpha)/p+1 > \delta > 0$, then

$$\lim_{r\to 0} \big[\kappa(r)\big]^{-1} r^{1-\delta} = 0.$$

Now it follows that

$$\limsup_{|x|\to 1} \big[\kappa(\varrho(x))\big]^{-1} |U(x)| = 0,$$

as required.

The above proof also shows the following (see [7] and [9]).

Proposition 1. *Let u be a monotone function on $\mathbf{B}$ satisfying* (4) *with $-1 < \alpha \leqq p-n$. If $\kappa(1) < \infty$, then u has a finite nontangential limit at every $\xi \in \mathbf{S}$.*

Proposition 2. *Let u be a monotone function on $\mathbf{B}$ satisfying* (4) *with $n-1 < p \leqq n+\alpha$. If $\kappa(1) = \infty$, then*

$$\lim_{r\to 1} (1-r)^{-\omega} \big[\varphi\big((1-r)^{-1}\big)\big]^{1/p} S_q(U_r) = 0$$

whenever $p \leqq q < \infty$ and $\omega = (n-1)/q - (n-p+\alpha)/p > 0$.

4. Sharpness

1. *The sharpness of the exponent $-\omega$.* Let $-1 < \alpha < p-1$. For $\delta > 0$, consider the function u on $\mathbf{B}$ defined by

$$u(x) = (1-|x|)^{1+a}|x-e|^{-b},$$

where $a = \delta - (\alpha+1)/p$, $b = (n-1)/p$ and $e = (1,0,\dots,0)$. Then the function u is monotone on $\mathbf{B}$. To show this, let D be a relatively compact open set with the closure $\overline{D} \subset \mathbf{B}$, and suppose u attains a maximum on $\overline{D}$ at an interior point $c \in D$, and set

$$E = \{x \in \overline{\mathbf{B}} : (1-|x|)^{1+a} = u(c)|x-e|^b\}.$$

Since $e \in E$,

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x)$$

holds. We also see that u attains a minimum on $\overline{D}$ at a boundary point. Hence u is monotone on $\mathbf{B}$. Further we have

$$\int_{\mathbf{B}} |\nabla u(x)|^p \varrho(x)^\alpha \, dx < \infty.$$

If $k(x) = |x-e|^{-b}$, then

$$S_q(u_r) = M(1-r)^{1+a} S_q(k_r) \geqq M(1-r)^{1+a+(1/q-1/p)(n-1)} = M(1-r)^{\omega+\delta}.$$

This implies that the exponent $-\omega$ is sharp in Theorem 1.

2. *The limits for BLD functions.* Theorem 1 fails to hold for BLD functions, when $\alpha < 0$ and $(n-p+\alpha)/p < (n-1)/q < (n-p)/p$.

For $0 < r_j < 1$ and $\gamma > 1$, set

$$C_j = \{x : |x - r_j e| \leqq (1-r_j)^\gamma\},$$

where $|e| = 1$. We take $\{r_j\}$ such that $\{C_j\}$ are mutually disjoint. For $\varphi \in C_0^\infty\bigl(B(0,1)\bigr)$ such that $\varphi = 1$ on $B\bigl(0, \frac{1}{2}\bigr)$, define

$$u_j(x) = \varphi(|x - r_j e|/(1-r_j)^\gamma)|x - r_j e|^{-a},$$

where

$$(n-p+\alpha)/p < a < (n-p+\alpha/\gamma)/p. \tag{7}$$

Then

$$\int_{\mathbf{B}} |\nabla u_j(x)|^p (1-|x|)^\alpha \, dx \leqq M(1-r_j)^{\alpha+\gamma(n-(a+1)p)}.$$

Note further that

$$S_q\bigl((u_j)_{r_j}\bigr) \geqq c(1-r_j)^{\gamma(-a+(n-1)/q)}$$

with a positive constant $c > 0$, when $a < (n-1)/q$. If we set $u = \sum_j u_j$, then

$$\lim_{j\to\infty} (1-r_j)^{-\omega} S_q(u_{r_j}) = \infty,$$

when a and γ are chosen so that

$$(n-p+\alpha/\gamma)/p + (1-\gamma^{-1})\{(n-1)/q - (n-p)/p\} < a < (n-1)/q. \tag{8}$$

Now we suppose that a satisfies (7) and (8). Then, since $\alpha+\gamma\bigl(n-(a+1)p\bigr)>0$, we can take $\{r_j\}$ so that

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\alpha \, dx < \infty.$$

Thus u satisfies (2) and

$$\limsup_{r\to 1} (1-r)^{-\omega} S_q(U_r) = \infty.$$

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