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# BEHAVIOR OF QUASIREGULAR SEMIGROUPS NEAR ATTRACTING FIXED POINTS

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**Abstract.** We study the dynamical behavior of quasiconformal groups and of quasiregular semigroups near attracting fixed points. In the homeomorphic case we show that there is a 1-parameter quasiconformal loxodromic group of $\overline{\mathbf{R}}^n$, $n \geq 3$, that cannot be quasiconformally linearized. This implies that Martin's result, which says that the cyclic quasiconformal loxodromic groups are all quasiconformal conjugates of $\langle x \mapsto 2x \rangle$, cannot be quantitative. In the non-injective situation we exhibit an example of a quasiregular semigroup whose elements have a common attracting fixed point and which cannot be quasiconformally linearized at this point. For the dynamics of uniformly quasiregular mappings at superattracting fixed points, we show that there are several distinct quasiconformal conjugacy classes by classifying the power mappings of [My2].

## 1. Introduction

A semigroup $\mathscr{F}$ of uniformly quasiregular mappings is called a *quasiregular semigroup*. For invertible mappings, we consider *quasiconformal groups*, i.e. groups of uniformly quasiconformal mappings. In each case we consider semigroups whose elements are defined on $\overline{\mathbf{R}}^n$, $n \geq 3$, and have a common attracting fixed point. We are interested in the classification of the dynamics of $\mathscr{F}$ at such a point by means of quasiconformal conjugacy.

The simplest quasiconformal groups whose elements have a common attracting fixed point are the cyclic loxodromic ones, that is $G = \langle g \rangle$ is loxodromic if there are distinct points $p, q \in \overline{\mathbf{R}}^n$ so that either $g^k$ or $g^{-k}$ converges uniformly on compact sets of $\overline{\mathbf{R}}^n \setminus \{q\}$ to $p$ as $k \to \infty$.

**Theorem** (Martin). *A cyclic quasiconformal group generated by a loxodromic element is quasiconformally conjugate to the Möbius group generated by* $x \mapsto 2x$.

In other words, up to quasiconformal conjugacy, all the cyclic quasiconformal groups are identical. For general uniformly quasiregular mappings, there is the following corresponding local result.

**Theorem** (Hinkkanen and Martin). *Suppose that $f$ is a uniformly quasiregular mapping of $\overline{\mathbf{R}}^n$ having an attracting fixed point $p$. Then there is a quasiconformal mapping $h$ of $\overline{\mathbf{R}}^n$ with $h(p) = 0$ and such that $h \circ f \circ h^{-1}(x) = \frac{1}{2}x$ in a neighborhood of the origin* $0$.

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Our first aim is to establish sharpness of these results. Their proofs are based on deep topological facts that do not allow one to get quantitative results. More precisely, it was not known if the conjugating mapping can be chosen to have a dilatation depending only on the dilatation of the semigroup and possibly the dimension. We prove that this is not true in general and that these results cannot be extended to the non-discrete case: there are examples of non-discrete uniformly quasiregular semigroups which are not locally quasiconformally conjugate at a repelling fixed point to the corresponding conformal model.

We also consider the case of superattracting fixed points. These are fixed points which are at the same time branch points. Note that in this case there is no conformal model since 1-quasiregular mappings have no branch points. In [My2] we gave examples of uniformly quasiregular mappings having superattracting fixed points. Here we classify them and, in particular, we show that it is possible to choose two of them having the same degree and which are not, locally at the superattracting fixed points, in the same quasiconformal conjugacy class.

## 2. Definition of quasiregular mappings and notations

Let $D \subset \mathbf{R}^n$ be a domain and $f\colon D \to \mathbf{R}^n$ a mapping of Sobolev class $W^{1,n}_{\mathrm{loc}}(D)$. We consider only orientation preserving mappings, thus the Jacobian determinant $J_f(x) \geq 0$ for a.e. $x \in D$. Such a mapping is said to be $K$*-quasiregular*, where $1 \leq K < \infty$, if

$$\max_{|h|=1} |f'(x)h| \leq K \min_{|h|=1} |f'(x)h| \quad \text{for a.e. } x \in D.$$

The smallest number $K$ for which the above inequality holds is called the linear dilatation of $f$. A non-constant quasiregular mapping can be redefined on a set of measure zero so as to be continuous, open and discrete, and we shall always assume that this has been done. If $D$ is a domain of the compactification $\overline{\mathbf{R}}^n$ (equipped with the spherical metric; thus $\overline{\mathbf{R}}^n$ is isometric via stereographic projection with the $n$-sphere $\mathbf{S}^n$), then we use the chart at infinity $x \mapsto x/|x|^2$ to extend, in the obvious manner, the notion of quasiregularity to mappings $f\colon D \to \overline{\mathbf{R}}^n$. Such mappings are said to be *quasimeromorphic*. The *branch set* $B_f$ is the set of points $x \in D$ for which $f$ is not locally homeomorphic at $x$. The notation $i(x,f)$ stands for the *local index* of $f$ at $x$, as defined in [R, p. 18].

A mapping $f$ of a domain $D$ into itself is called *uniformly quasiregular* if there is some $1 \leq K < \infty$ such that all the iterates $f^k$ are $K$-quasiregular. We abbreviate this by $f \in \mathrm{UQR}(D)$. *Quasiconformal groups* and *quasiregular semigroups* are (semi-) groups whose elements are all $K$-quasiconformal, $K$-quasiregular respectively, for some fixed $K \geq 1$.

## 3. The quasiconformal group case

We consider here cyclic and 1-parameter quasiconformal loxodromic groups. Using an example of Tukia [T] we find:

**Theorem 1.** *For any $K > 1$, there exists a 1-parameter quasiconformal loxodromic group that cannot be conjugate by a quasiconformal mapping to a Möbius group.*

Before proving this, let us first deduce the announced distortion result in the discrete case. Given $G$ a cyclic quasiconformal loxodromic group, we denote by $K(G)$ the smallest possible dilatation $K \geq 1$ for which there exists a $K$-quasiconformal mapping $h$ with $h \circ G \circ h^{-1}$ a Möbius group. By Martin's result, $K(G) < \infty$.

**Corollary 3.1.** *For any $K > 1$, there exists a sequence of cyclic $K$-quasiconformal loxodromic groups $G_k$ with $\lim_{k\to\infty} K(G_k) = \infty$.*

*Proof.* Let $K > 1$ and $G = \{g_t : t \in \mathbf{R}\}$ be a 1-parameter $K$-quasiconformal loxodromic group which cannot be quasiconformally conjugate to a Möbius group. Such a group exists by Theorem 1. Denote by $G_k$ the cyclic subgroup of $G$ generated by $g_{2^{-k}}$. This is an increasing sequence of subgroups exhausting $G$. Suppose now that there is $H > 1$ and, for any $k \geq 1$, a $H$-quasiconformal mapping $h_k$, normalized so that it fixes $0, 1, \infty$, with $h_k \circ G_k \circ h_k^{-1} \subset \mathrm{M\ddot{o}b}(\overline{\mathbf{R}}^n)$. The normalization implies that there is a uniformly convergent subsequence $h_{k_j}$ with limit $h$ a quasiconformal mapping. We have now a contradiction since this limit mapping conjugates $G$ to a subgroup of $\mathrm{M\ddot{o}b}(\overline{\mathbf{R}}^n)$: let $g \in G$ and choose $g_{t(k_j)} \in G_{k_j}$ such that $g_{t(k_j)}$ converges uniformly to $g$. Then we see that

$$\varphi = h \circ g \circ h^{-1} = \lim_{j\to\infty} h_{k_j} \circ g_{t(k_j)} \circ h_{k_j}^{-1}$$

is a Möbius transformation. □

We give now a version of Tukia's group, the main ingredient for the proof of Theorem 1. Let $f$ be a quasiconformal mapping of $\mathbf{R}^2$ such that $f_{|\mathbf{R}}$ is bihölder $\alpha$ with $\alpha \in ]\frac{1}{2}, 1[$: there is $c \geq 1$ with

$$\frac{1}{c}|x-y|^\alpha \leq |f(x) - f(y)| \leq c|x-y|^\alpha \qquad \text{for all } x, y \in \mathbf{R}. \tag{1}$$

So $\Gamma = f(\mathbf{R})$ is a $1/\alpha$-dimensional fractal curve. In Tukia's example it is the Von Koch Snowflake curve and, more generally, one can use other fractal bilipschitz homogeneous curves [My1]. The mapping $f$ can be chosen to have a dilatation $K$ tending to 1 when $\alpha \to 1$ [Mc]. The 1-parameter quasiconformal loxodromic group we look for is the conjugate of the group

$$H = \{x = (u, v) \in \mathbf{R}^2 \times \mathbf{R}^{n-2} \mapsto h_t(x) = (e^t u, e^{\alpha t} v) : t \in \mathbf{R}\}$$

by the mapping $F = f \times \mathrm{Id}\colon \mathbf{R}^2 \times \mathbf{R}^{n-2} \to \mathbf{R}^2 \times \mathbf{R}^{n-2}$:

$$G = F \circ H \circ F^{-1} = \{g_t = F \circ h_t \circ F^{-1} : t \in \mathbf{R}\}. \tag{2}$$

*Proof of Theorem* 1. Let $u \in \Gamma$, $v_0 \in \mathbf{R}^{n-2}$ and $p_0 = (u, v_0)$. Consider the segment $\sigma = [(u,0), p_0]$. For $p = (u,v)$ we have $g_t(p) = \bigl(f\bigl(e^t f^{-1}(u)\bigr), e^{\alpha t} v\bigr)$. From this and from the bihölder $\alpha$ inequality (1) follows that there exists $t_0 > 0$ and $C \ge 1$ such that

$$\frac{|t|^\alpha}{C} \le |g_t(p) - p| \le C|t|^\alpha \quad \text{for every } p \in \sigma \text{ and } 0 \le |t| \le t_0. \tag{3}$$

Suppose now that there is a quasiconformal mapping $h$ conjugating $G$ to a subgroup of $\mathrm{Möb}(\overline{\mathbf{R}}^n)$. After normalization we have that $h$ is a mapping of $\overline{\mathbf{R}}^n$ fixing $0$ and that the conformal group $\Phi = h \circ G \circ h^{-1}$ has the form $\Phi = \{\varphi(x) = e^t U_t x : t \in \mathbf{R}\}$ with $U_t \in O(n)$. Evidently, the orthogonal matrices $U_t$ depend continuously on $t$ and $U_0 = \mathrm{Id}$. Since the conjugation induces an isomorphism of groups, there is $r \ne 0$ so that $h \circ g_t = \varphi_{rt} \circ h$.

Take $p \in \sigma$, $t \in [-t_0, 0]$ and let $a \in \mathbf{R}^n$ be any point with $|a| = |g_t(p) - p|$. Since $h$ is quasiconformal, we have

$$|h(p+a) - h(p)| \lesssim |h\bigl(g_t(p)\bigr) - h(p)| = |\varphi_{rt} \circ h(p) - h(p)|,$$

where $\lesssim$ stands for "$\le$ up to a constant depending only on $n$ and the dilatation constant of $h$". If $t_0$ is sufficiently small, this can be majorized by

$$|h(p+a) - h(p)| \lesssim |e^{rt} - 1|\, |h(p)| \lesssim |t h(p)|.$$

On the other hand, $|t| \le (C|a|)^{1/\alpha}$ in view of inequality (3). Therefore

$$\lim_{a \to 0} |h(p+a) - h(p)|/|a| = 0$$

for all $p \in \sigma$. In other words, $h$ restricted to $\sigma$ is constant which is impossible. □

## 4. Attracting fixed points

For the notion of an *attracting fixed point* we use the definition of [HM]: a fixed point $p$ of $f$ is called attracting if there is a topological ball $U$ with $\overline{f(U)} \subset U$ and that $f_{|U}$ is injective.

The proof of Theorem 1 is entirely local and thus these groups cannot be made conformal by a quasiconformal change of coordinates in any neighborhood of the fixed point $0$. Therefore, if the restriction of a quasiregular semigroup to a neighborhood of an attracting fixed point (which in our cases will always be common to all the elements of the semigroup) behaves like the quasiconformal wild groups of Theorem 1 around $0$, then this semigroup cannot be quasiconformally linearized at this point. It turns out that we can construct such a semigroup as follows.

**Theorem 2.** *There exists a quasiregular semigroup $\mathscr{F}$ generated by two mappings, with $0$ a common attracting fixed point for all the mappings $f \in \mathscr{F}$ and for which there is no quasiconformal mapping defined in a neighborhood $W$ of $0$ conjugating the restriction to $W$ of the elements of $\mathscr{F}$ to conformal mappings.*

Note that the restriction of $\mathscr{F}$ to a neighborhood of $0$ is an abelian semigroup with two generators, so in some sense our example is as close as possible to the cyclic case.

We now explain how to get such a semigroup. Let $f$ be a uniformly quasiregular mapping of $\overline{\mathbf{R}}^n$ with attracting fixed point $p$. The existence of non-homeomorphic such mappings can be found in [IM]. Hinkkanen and Martin's result asserts that there is $h \in \mathrm{QC}(\overline{\mathbf{R}}^n)$ with

$$h \circ f \circ h^{-1}(x) = \tfrac{1}{2}x \quad \text{in some neighborhood } W \text{ of } 0.$$

We set $\tilde{f} = h \circ f \circ h^{-1}$. Let $G$ be the quasiconformal group of (2) and $g = g_{-2} \in G$. This time we use Martin's linearization result to get $H \in \mathrm{QC}(\overline{\mathbf{R}}^n)$ conjugating $g$:

$$H^{-1} \circ g \circ H(x) = \tfrac{1}{2}x \qquad \text{for all } x \in \overline{\mathbf{R}}^n.$$

The first generator of our semigroup will be

$$f_2 = (H \circ h) \circ f \circ (H \circ h)^{-1} = H \circ \tilde{f} \circ H^{-1}.$$

The second generator will behave like the mapping $g_{-3}$ of $G$ near $0$ and be identically $f_2$ outside some ball. To obtain it we modify the construction of $g_{-2}$. For this we suppose that the neighborhood $W$ satisfies $H(W) \supset F\bigl(B(0,1)\bigr)$ where $F$ is the mapping of (2). Let $r\colon \mathbf{R}^n \to [-3,-2]$ be smooth with $r \equiv -2$ in $|x| > 1$, $r \equiv -3$ in

$$h_{-2}\bigl(B(0,1)\bigr) = \{(e^{-2}u, e^{-2\alpha}v) : x = (u,v) \in B(0,1)\}$$

and so that the restriction of $r$ to any ray $\tau_y = \{sy : s > 0\}$, $y \in \mathbf{S}^{n-1}$, is increasing. The conjugate of the mapping $x = (u,v) \mapsto \varphi(x) = (e^{r(x)}u, e^{\alpha r(x)}v)$ by the mapping $F$ of (2), so the mapping $\tilde{g} = F \circ \varphi \circ F^{-1}$, is uniformly quasiconformal and loxodromic. It also admits a quasiconformal conjugacy $\widetilde{H}$, that we need to construct explicitly starting from the conjugacy $H$ of $g$: since $g(x) = \tilde{g}(x)$ for $x \in \Omega' = F(\{|x| > 1\})$ we can define $\widetilde{H} = H$ in $H^{-1}(\Omega')$. For $x \notin H^{-1}(\Omega')$ choose $k \in \mathbf{N}$ so that $2^k x \in H^{-1}(\Omega')$. Then define $\widetilde{H}(x) = \tilde{g}^k \circ H \circ (2^k x)$. Now it is clear that $\widetilde{H}$ is a quasiconformal mapping with

$$\widetilde{H}^{-1} \circ \tilde{g} \circ \widetilde{H}(x) = \tfrac{1}{2}x \qquad \text{for all } x \in \mathbf{R}^n.$$

The mapping we are looking for is

$$f_3 = (\widetilde{H} \circ h) \circ f \circ (\widetilde{H} \circ h)^{-1} = \widetilde{H} \circ \tilde{f} \circ \widetilde{H}^{-1}.$$

Consider now the semigroup $\mathscr{F} = \langle f_2, f_3 \rangle$ generated by these mappings. By construction, $f_j \equiv g_{-j}$ in $\Omega = F \circ h_{-2}\bigl(B(0,1)\bigr)$, $j = 2,3$, where the $g_{-j}$ are the elements of the group $G$. This gives the following elementary properties of $\mathscr{F}$:

**Lemma 4.1.** *The semigroup $\mathscr{F}$ is a quasiregular semigroup and its restriction to $\Omega$ is an abelian semigroup generated by $g_{-2}$ and $g_{-3}$.*

*Proof.* We show that any mapping $f \in \mathscr{F}$ is $K^3$-quasiregular, where $K$ is supposed to be the dilatation constant common to $f_2$, $f_3$ and all its iterates. Let $x \in \overline{\mathbf{R}}^n$, $y = f(x)$ and consider the following points which correspond to a (not necessarily unique) decomposition of $f$,

$$(4) \qquad x_0 = x, x_1, \dots, x_k = y \qquad \text{where } x_l = f_2(x_{l-1}) \text{ or } x_l = f_3(x_{l-1}).$$

Then we have two cases: either all the $x_l \in \Omega'$, $l = 0, \dots, k$, and in that case we have a dilatation $K$ of $f$ at $x$ due to the fact that $H \equiv \widetilde{H}$ in $H^{-1}(\Omega')$. In the other case there is a first point $x_j$ outside $\Omega'$. If $j = k$, then the dilatation of $f$ at $x$ is still $K$ since we can choose in (4) always the same mapping $f_2$ or $f_3$ for every $l = 0, \dots, k-1$. If $j < k-1$, then we proceed as follows. The condition $H(W) \supset F\big(B(0,1)\big)$ implies $f_2 = g_{-2}$ and $f_3 = \tilde{g}$ in ${}^c\Omega'$. Since $g({}^c\Omega') = \tilde{g}({}^c\Omega') = \Omega$, necessarily $x_l \in \Omega$ for all $l > j$. In $\Omega$ the mappings coincide with the mappings $g_{-2}, g_{-3}$ of the (uniformly) quasiconformal group $G$. Therefore we have in this last case a dilatation $K$ for the composition of the mappings that map $x_0$ to $x_j$, a factor $K$ when $x_j$ is mapped to $x_{j+1}$ and one more dilatation factor $K$ for the remaining step when $x_{j+1}$ is mapped to $x_k$. □

*Proof of Theorem* 2. We prove by contradiction that the semigroup $\mathscr{F} = \langle f_2, f_3 \rangle$ described above cannot be quasiconformally linearized at 0. Suppose that there is $h$ a quasiconformal mapping defined on a neighborhood $W$ of 0 such that $h \circ f_{j|W} \circ h^{-1}$ is the restriction of a Möbius transformation $\varphi_j$, $j = 2, 3$. Choose $\varrho \in ]0, e^{-2}[$ so that $U = F\big(B(0,\varrho)\big) \subset W$. Then $f_{j|U} = g_{-j|U}$ for $j = 2, 3$ and $\overline{g_t(U)} \subset U$ for all $t < 0$. Since the $f_j$ are commuting in $U$, $\varphi_2$ and $\varphi_3$ must have the same fixed points. The group $\Phi = \overline{\langle \varphi_2, \varphi_3 \rangle}$ is a 1-parameter loxodromic group since it cannot be discrete. It follows that

$$h \circ \{g_{t|U} : t < 0\} \circ h^{-1} \subset \Phi_{|h(U)}$$

which is impossible in view of the proof of Theorem 1. □

## 5. Superattracting fixed points

A fixed point of a uniformly quasiregular mapping $f$ which is also a branch point is called a superattracting fixed point. Recall that $f$ is attracting at such a point, an immediate consequence of the local behavior of quasiregular mappings; see [R], [My2].

**5.1. Counterparts of power mappings.** In [My2] we gave examples of uniformly quasiregular mappings of $\overline{\mathbf{R}}^3$ having superattracting fixed points.

These mappings are the natural counterparts of the power mappings and they are obtained in a similar manner: Let $h\colon \mathbf{R}^3 \to \mathbf{R}^3 \setminus \{0\}$ be Zorich's mapping (see [R] for a construction). This is the three-dimensional counterpart of the exponential mapping and is characterized by the fact that it is automorphic with respect to the group of isometries

$$\Gamma = \langle (z,t) \mapsto (-z,t) : (z,t) \mapsto (z+2,t) : (z,t) \mapsto (z+2i,t) \rangle.$$

The analogue of the power mappings are the solutions $f_A$ of Schröder's equation

$$f_A \circ h = h \circ A \tag{5}$$

where $A(z,t) = (\lambda z, |\lambda| t)$ with $\lambda \in \mathbf{C}$, $|\lambda| > 1$ and where we identified $x \in \mathbf{R}^3$ with $(z,t) \in \mathbf{C} \times \mathbf{R}$. This conformal mapping $A$ has to satisfy $A \circ \Gamma \circ A^{-1} \subset \Gamma$ in order to get a well defined solution $f_A$ of Schröder's equation. The superattracting fixed points of these mappings are $0$ and $\infty$.

Note that to each such mapping $f_A$ there corresponds a Lattès rational mapping $\varphi_A$ which is again a solution of a Schröder equation:

$$\varphi_A \circ \wp(z) = \wp(\lambda z), \qquad z \in \mathbf{C},$$

where $\wp$ is the Weierstrass $P$-function. Moreover, after a quasiconformal change of coordinates, we may suppose that the restriction of $f_A$ to the two-sphere $\mathbf{S}^2$ is precisely $\varphi_A$ [My2].

**5.2. Classification.** As in the attracting case we are interested in the quasiconformal classification of the dynamics of uniformly quasiregular mappings near superattracting fixed points. In contrast to the attracting case there is no conformal model, since Liouville's theorem asserts that in the conformal case there can be no branching. We classify the above power mappings and deduce in particular that there cannot be a unique model.

Since we are dealing with quasiconformal conjugacy we have to respect a topological invariant, namely the local index. Two mappings $f_1$, $f_2$ can only be locally conjugate near their superattracting fixed points $p_1$, $p_2$ respectively, if they have the same local index there: $i(p_1, f_1) = i(p_2, f_2)$. Hence conjugate power mappings must have the same degree given by $|\lambda|^2$.

**Theorem 3.** *If $f_A$ and $f_B$ are two solutions of Schröder's equation* (5) *having the same degree, then there is a local quasiconformal conjugacy between these mappings near their superattracting fixed point $0$ if and only if $A = B$.*

This result shows in particular that there are mappings $f_A$ and $f_B$ having the same degree but for which there is no local quasiconformal conjugacy relating them near their superattracting fixed point $0$.

It suffices to choose $A(z,t)=(dz,dt)$ and $B(z,t)=(idz,dt)$, $d\in\{2,3,\dots\}$.

Since there is no conformal model and motivated by the particular construction of the above power mappings—which gives in dimension $n=2$ precisely the mappings $z\mapsto z^d$—we are led to think that these power mappings may in fact be the standard models for the local behavior of uniformly quasiregular mappings at their superattracting fixed points. We do not know any examples of a different kind.

*Proof.* When we express Zorich's mapping by $h(z,t)=(r,\xi)$, where $(r,\xi)\in\mathbf{R}^+\times\mathbf{S}^2$ are spherical coordinates, then $r=e^t$ and $\xi=\xi(z)$ is independent of $t$. Due to this special form we see that $f_A(r,\xi)=\bigl(r^{|\lambda|},g_A(\xi)\bigr)$, with $g_A\in\mathrm{UQR}(\mathbf{S}^2)$ and, moreover, up to a quasiconformal change of coordinates, $g_A$ is the Lattès mapping $\varphi_A$ corresponding to $f_A$.

Suppose now that there is a quasiconformal mapping $\Phi$ defined near 0 with $f_A\circ\Phi=\Phi\circ f_B$ still near the origin. We show that this implies that $\varphi_A$ and $\varphi_B$ are equivalent, which means that there are $\theta_1$, $\theta_2$ homeomorphisms that are isotopic rel $P_{\varphi_B}$, the post critical set of $\varphi_B$, with $\theta_1\circ\varphi_B=\varphi_A\circ\theta_2$ (cf. [DH]).

First we show that

$$|\Phi(r,\xi)|=s_r,\qquad\text{with } s_r \text{ independent of } \xi\in\mathbf{S}^2. \tag{6}$$

If this is not true, then there is $r>0$, $\xi_1,\xi_2\in\mathbf{S}^2$ and $s_1<s_2$ with $|\Phi(r,\xi_1)|=s_1<s_2=|\Phi(r,\xi_2)|$. Then it follows that

$$f_A^k\bigl(\Phi(r,\xi_i)\bigr)=\Phi\bigl(f_B^k(r,\xi_i)\bigr)=\Phi\bigl(r^{|\lambda|^k},g_B^k(\xi_i)\bigr)$$

and, when we set $(s_i,\eta_i)=\Phi(r,\xi_i)$,

$$f_A^k\bigl(\Phi(r,\xi_i)\bigr)=f_A^k(s_i,\eta_i)=\bigl(s_i^{|\lambda|^k},g_A^k(\eta_i)\bigr).$$

Therefore, for every $k\in\mathbf{N}$ there is $a_{i,k}=g_B^k(\xi_i)$ for which

$$\frac{|\Phi(r^{|\lambda|^k},a_{1,k})|}{|\Phi(r^{|\lambda|^k},a_{2,k})|}=\Bigl(\frac{s_1}{s_2}\Bigr)^{|\lambda|^k}\longrightarrow 0\qquad(k\to\infty).$$

But this conflicts with our assumption of the quasiconformality of $\Phi$.

Equation (6) means that $\Phi(r\mathbf{S}^2)=s_r\mathbf{S}^2$. Set $\Phi_r(\xi)=\Phi(r,\xi)$. This is a quasiconformal mapping of $\mathbf{S}^2$ with

$$g_A\circ\Phi_r=\Phi_{r^{|\lambda|}}\circ g_B.$$

It is clear that $\Phi_r(C_{g_B})=C_{g_A}$ and the post critical set is also fixed by these mappings: $\Phi_r(P_{g_B})=P_{g_A}$. This implies that $g_A$ and $g_B$ are equivalent and therefore the same is true for the associated Lattès mappings $\varphi_A$, $\varphi_B$. The assertion $A=B$ follows. □

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