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SUBSTANTIAL BOUNDARY POINTS FOR PLANE DOMAINS AND GARDINER'S CONJECTURE

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Abstract. The local dilatation H_p at a boundary point of a quasiconformal mapping on a plane domain of arbitrary connectivity is defined and it is shown that there is always a substantial point p, such that $H_p = H$, where H is the boundary dilatation. Infinitesimal local boundary dilatation is also defined and it is shown that the sets of infinitesimally substantial and substantial boundary points coincide.

Introduction

Let Ω be a plane domain with two or more boundary points and let $M(\Omega)$ be the space of L_∞-Beltrami coefficients μ defined on Ω with $\|\mu\|_\infty < 1$. With respect to the global parameter z for Ω, elements μ of $M(\Omega)$ are just functions and we arbitrarily put $\mu(z)$ identically equal to zero outside of Ω. Corresponding to any such μ there is a global quasiconformal self-mapping f^μ of the plane which solves the Beltrami equation [3], [2],

$$f_{\bar z}(z) = \mu(z) f_z(z), \tag{1}$$

and f^μ is defined uniquely up to postcomposition by a complex affine map of the plane. We say such a solution f^μ to this equation is induced by μ. Conversely, any quasiconformal mapping f defined on Ω has a Beltrami coefficient $\mu(z) = f_{\bar z}(z)/f_z(z)$ in $M(\Omega)$. μ is called the complex dilatation of f and $K = (1+k)/(1-k)$, where $k = k(\mu) = \|\mu\|_\infty$, is called the dilatation of f. K bounds the ratio of extremal length problems corresponding under f taken in the domain and range of f.

The Teichmüller space $T(\Omega)$ of Ω is a space of equivalence classes of Beltrami coefficients in $M(\Omega)$. Any Beltrami coefficient μ in $M(\Omega)$ induces a mapping f^μ which is a solution to (1), with μ identically equal to zero in the complement of Ω. Two such Beltrami coefficients μ_0 and μ_1 are equivalent if they induce mappings f_0 and f_1 such that there is a conformal map c from $f_0(\Omega)$ to $f_1(\Omega)$ and an isotopy through quasiconformal mappings g_t, $0 \le t \le 1$, from Ω to Ω which extend continuously to the boundary of Ω such that

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1. $g_0(z)$ is identically equal to z on Ω,
2. g_1 is identically equal to $f_1^{-1} \circ c \circ f_0$ on Ω, and
3. $g_t(p) = f_1^{-1} \circ c \circ f_0(p) = p$ for every point p in the boundary of Ω.

Note that this equivalence relation is nonlinear. It is known [6] that the equivalence relation induced by this isotopy condition is the same as the equivalence relation induced by the same conditions except the isotopy in condition 3 is replaced by an isotopy fixing the ideal boundary points of Ω.

This equivalence relation partitions $M(\Omega)$ into equivalence classes and the space of equivalence classes is by definition the Teichmüller space $T(\Omega)$. The equivalence class $[\mu]$ of an element μ of $M(\Omega)$ always contains an extremal representative, that is, a representative μ_0 with the property that $\|\mu_0\|_\infty \leq \|\mu\|_\infty$ for all other representatives μ of the same class. The proof of this fact depends on the equicontinuity of a family of quasiconformal mappings with bounded dilatation. We define

$$k_0([\mu]) = \|\mu_0\|_\infty,$$

where μ_0 is an extremal representative of its class. The dilatation of the class is $K_0([\mu]) = \big(1 + k_0([\mu])\big)/\big(1 - k_0([\mu])\big)$ and the Teichmüller metric on $T(\Omega)$ is defined to be

$$d([\mu], [\nu]) = \tfrac{1}{2} \log K_0([\sigma]),$$

where σ is the Beltrami coefficient of $f^\nu \circ (f^\mu)^{-1}$. Teichmüller's metric d is the quotient metric with respect to the Kobayashi metric on $M(\Omega)$.

There is another natural constant associated with any equivalence class τ in $M(\Omega)$. For any μ, define $h^*(\mu)$ to be the infimum over all compact subsets F contained in Ω of the essential supremum norm of the Beltrami coefficient $\mu(z)$ as z varies over $\Omega \setminus F$. Define $h(\tau)$ to be the infimum of $h^*(\mu)$ taken over all representatives μ of the class τ. The numbers

$$H^*(\mu) = \frac{1 + h^*(\mu)}{1 - h^*(\mu)} \qquad \text{and} \qquad H(\tau) = \frac{1 + h(\tau)}{1 - h(\tau)}$$

are called the boundary dilatations of μ and of the class $\tau = [\mu]$, respectively. It is obvious that $h(\tau) \leq k_0(\tau)$. By definition, τ in $T(\Omega)$ is called a Strebel point if $h(\tau) < k_0(\tau)$ (see [19], [7], and [14]).

Let p be any point in the boundary of $\partial\Omega$ and let μ in $M(\Omega)$ represent a class τ in $T(\Omega)$. Define $h_p^*(\mu)$ to be the infimum over all open sets U in the plane containing p of ess $\sup_{z \in U} |\mu(z)|$. $h_p(\tau)$ is the infimum over all μ representing the class τ of $h_p^*(\mu)$. The numbers

$$H_p^*(\mu) = \frac{1 + h_p^*(\mu)}{1 - h_p^*(\mu)} \qquad \text{and} \qquad H_p(\tau) = \frac{1 + h_p(\tau)}{1 - h_p(\tau)}$$

are called the local boundary dilatations at p of μ in $M(\Omega)$ and τ in $T(\Omega)$, respectively.

The definitions so far given depend on the nonlinear equivalence relation which determines the Teichmüller classes in $M(\Omega)$. There is an infinitesimal version of this equivalence relation which is linear. Let $A(\Omega)$ be the space of integrable holomorphic quadratic differentials $\varphi(z)$ defined on Ω with norm given by

$$\|\varphi\| = \iint_\Omega |\varphi(z)|\, dx\, dy.$$

We say two Beltrami coefficients μ_1 and μ_2 are linearly equivalent if

$$\iint_\Omega \mu_1(z)\varphi(z)\, dx\, dy = \iint_\Omega \mu_2(z)\varphi(z)\, dx\, dy$$

for all φ in $A(\Omega)$.

We find it convenient to stipulate that elements of φ of $A(\Omega)$ are identically equal to zero in the complement of Ω. Since $A(\Omega)$ is a closed subspace of the Banach space $L_1(\Omega)$, by the Hahn–Banach and Riesz representation theorems, the dual space $Z(\Omega)$ to $A(\Omega)$ is isomorphic to $L_\infty(\Omega)/N$. N is the space of complex-valued, L_∞-Beltrami differentials μ defined on Ω such that

$$\iint_\Omega \mu(z)\varphi(z)\, dx\, dy = 0,$$

for all φ in $A(\Omega)$. If v is an element of $Z(\Omega)$ and μ is a Beltrami differential in $L_\infty(\Omega)$, we say μ represents v if

$$v(\varphi) = \iint_\Omega \mu\varphi$$

for all φ in $A(\Omega)$. Thus, the linear equivalence classes of Beltrami differentials are in one-to-one correspondence with the elements v of $Z(\Omega)$.

Corresponding to this linear equivalence relation there is a notion of infinitesimal boundary dilatation $b(v)$ of the equivalence class determined by v:

$$\begin{aligned} b(v) &= \inf\{b^*(\mu) : \mu \text{ represents } v\} \qquad \text{where} \\ b^*(\mu) &= \inf\{\|\mu\,|_{\Omega\setminus E}\,\|_\infty : E \subset \Omega,\ E \text{ compact}\}. \end{aligned} \tag{2}$$

b is a semi-norm on $Z(\Omega)$. A sequence $\{\varphi_n\}$ in $A(\Omega)$ is called degenerating if $\|\varphi_n\| = 1$ and if $\varphi_n(z)$ converges uniformly to 0 on compact subsets of Ω. Another semi-norm β on $Z(\Omega)$ is defined by

$$\beta(v) = \sup_{\{\varphi_n\}} \limsup_n |v(\varphi_n)|, \tag{3}$$

where the supremum is taken over all degenerating sequences $\{\varphi_n\}$. Obviously, $\beta(v) \leq b(v)$ and in [4] it is shown that $\beta(v) = b(v)$.

We can also define the infinitesimal local boundary dilatation at any point in $\partial\Omega$:

$$\begin{aligned} &b_p(v) = \inf\{b_p^*(\mu) : \mu \text{ represents } v\} \qquad \text{where} \\ &b_p^*(\mu) = \inf\{\|\mu\,|_{\Omega\cap U}\,\|_\infty : U \text{ is a neighborhood of } p \text{ in the plane}\}. \end{aligned} \tag{4}$$

b_p is the infinitesimal version of h_p. Of course, there is also an analogy β_p to β. A degenerating sequence φ_n, (with $\|\varphi_n\| = 1$) in $A(\Omega)$ is said to degenerate towards p if for every open set U containing p,

$$\lim_{n\to\infty} \iint_U |\varphi_n(z)|\, dx\, dy = 1.$$

We define

$$\beta_p(v) = \sup_{\{\varphi_n\}} \limsup_n |v(\varphi_n)|, \tag{5}$$

where the supremum is taken over all sequences $\{\varphi_n\}$ in $A(\Omega)$ which degenerate towards p. Obviously, $\beta_p(v) \leq b_p(v)$ and one of our preliminary results is that $\beta_p(v) = b_p(v)$ for all v.

Two important results of this paper are the (affirmative) solution of Gardiner's conjecture

$$H(\tau) = \max_{p\in\partial\Omega} H_p(\tau) \tag{6}$$

and its infinitesimal version

$$b(v) = \max_{p\in\partial\Omega} b_p(v). \tag{7}$$

We first prove the infinitesimal statement (7) and we also show that $b_p(v) = \beta_p(v)$ for every $v \in Z(\Omega)$. In [4] it is shown that $\beta(v) = b(v)$ for all $v \in Z(\Omega)$. Thus, $\beta(v) = \max_{p\in\partial\Omega}\beta_p(v)$ for every $v \in Z(\Omega)$, and both semi-norms b_p and β_p achieve their maxima at the same boundary points. Every such point is called an infinitesimally substantial boundary point of Ω for v. Formula (6) is a generalization of Fehlmann's result which says that $H([\mu]) = \max_{p\in\partial\Delta} H_p([\mu])$ for every Beltrami coefficient μ in the unit disc Δ, (see [8] and [9]). Frederick P. Gardiner conjectured that Fehlmann's theorem generalizes to all plane domains. In [16], Reich showed that a similar version of our infinitesimal result (7) holds in the case of the unit disc using a different method of proof. Reich's approach also provided another proof of Fehlmann's result for all non-Strebel points in $T(\Delta)$.

The proof of the result (6) breaks into two cases, according to whether the Teichmüller class τ is a Strebel or non-Strebel point. The Teichmüller class τ is called a non-Strebel class if it is represented by an extremal Beltrami coefficient μ for which there is degenerating Hamilton sequence. A sequence φ_n with $\|\varphi_n\| = 1$ in $A(\Omega)$ is called a degenerating Hamilton sequence if it is degenerating and if

$$\lim_{n\to\infty} \iint_\Omega \varphi_n(z)\mu(z)\,dx\,dy = \|\mu\|_\infty.$$

It turns out that this notion can be expressed either in terms of the linear or non-linear equivalence relation on Beltrami coefficients. Suppose μ is extremal and represents the Teichmüller class τ. Then μ also represents a linear functional v in $Z(\Omega)$. It is known that $\beta(v) < \|v\|$ if, and only if, $h(\tau) < k_0(\tau)$. Strebel points τ in $T(\Omega)$ are those for which either one of these inequalities is strict and non-Strebel points are those for which either inequality is an equality.

The third important result of this paper concerns the case when μ represents a non-Strebel class. Assume μ is extremal in its Teichmüller class, let v be the linear functional in $Z(\Omega)$ represented by μ and let τ be the equivalence class of μ in $T(\Omega)$. Then the set of points p in the boundary of Ω for which any of the following equalities hold is the same set:

(1) $H(\tau) = H_p(\tau)$,
(2) $b(v) = b_p(v)$,
(3) $\beta(v) = \beta_p(v)$,
(4) there exists a Hamilton sequence for μ degenerating towards p.

Points in the boundary of Ω for which $H_p(\tau) = H(\tau)$ are called substantial points for τ and points for which $b_p(v) = b(v)$ are called infinitesimally substantial points for v. In (2) and (3) the sets are determined by the linear equivalence class of the linear functional v, whereas in (1) the set is determined by the nonlinear Teichmüller equivalence class of τ. For a non-Strebel class τ represented by an extremal Beltrami coefficient μ, this result implies that the set of infinitesimally substantial points corresponding to the element v in $Z(\Omega)$ represented by μ coincides with the set of substantial boundary points for the Teichmüller class τ represented by μ. Since H_p and b_p are upper semi-continuous functions defined on the compact boundary of Ω, the sets of points p satisfying (1) or (2) are non-empty.

The result generalizes Fehlmann's theorem on the existence of substantial points in two ways. First of all, exactly as Gardiner conjectured, it applies to all plane domains, not just the unit disc. Secondly, both for the unit disc and for any plane domain, it says that for non-Strebel classes the sets of substantial and infinitesimally substantial boundary points coincide.

This paper is organized into seven sections. In the first section, we prove $b(v) = \max_p b_p(v)$ for v in $Z(\textit{unit disc})$. Proving this inequality can be viewed as a problem of sewing together vector fields V with dilatation bounded by M

near a large number of points p on the boundary of Ω to obtain a globally defined vector field with dilatation no more than $M+\varepsilon$. Instead of sewing, we go all the way to the boundary of the unit disk and apply the Beurling–Ahlfors extension process to a suitably chosen alteration of the vector field defined on the boundary of the disc.

In the second section we define another quantity $G(\tau)$ associated to a Teichmüller class τ which turns out to be the same as the boundary dilatation $H(\tau)$. By definition, $G(\tau)=\big(1+g(\tau)\big)/\big(1-g(\tau)\big)$, where $g(\tau)$ is

$$\lim_{k\to\infty}\sup_{\{\varphi_n\}}\limsup_{n\to\infty}\operatorname{Re}\int_\Omega \mu_k\varphi_n\,dx\,dy$$

and where the sequence μ_k is in the equivalence class τ and is approximating in the sense that $h^*(\mu_k)<h(\tau)+1/k$ and the supremum is over all degenerating sequences $\{\varphi_n\}$ of quadratic differentials of norm equal to one. It turns out that because of the principle of Teichmüller contraction in asymptotic Teichmüller space [4], $g(\tau)$ does not depend on the selection of the approximating sequence μ_k in the equivalence class τ. The main result of this section is that $G(\tau)=H(\tau)$. The freedom to work with G as a replacement for H is an essential element in our proof of the first main result. In this section, we give another way to define boundary dilatation. We call it $S(\tau)$ and it is the maximal distortion of degenerating sequences of quadratic differentials under the mapping by heights induced by the Teichmüller class of τ. In the case Ω is the unit disc we show that $S=G=H$. We believe this result should be true for any domain Ω, but verification awaits the proof of a preliminary theorem, namely, that there is a well-defined "mapping by heights" for quadratic differentials on plane domains corresponding to any Teichmüller class. This topic has been recently explored by Strebel, Gardiner and Lakic (see [21] and [15] for more details).

In the third section, we prove $b(v)=\max_{p\in\partial\Omega}b_p(v)$ for any plane domain. The sewing step is avoided once again by going all the way to the boundary and applying the Sullivan–Thurston [22] extension process used in [5] for vector fields vanishing at the boundary.

In the fourth section, we prove $\beta_p(v)=b_p(v)$ in all cases. We use the same method which was used to show $\beta(v)=b(v)$ in [4].

In the fifth section, we give an alternative definition of local boundary dilatation at a point p in the boundary of Ω of a Teichmüller class τ. By definition $g_p(\tau)$ is a limit over sequences $\{\varphi_n\}$ degenerating towards p of integrals $\int_\Omega\mu_k\varphi_n$ where μ_k is a sequence of Beltrami coefficients in the class of τ whose local dilatations at the point p give better and better approximations to $h_p(\tau)$. We prove the local result that $g_p(\tau)=h_p(\tau)$ which is analogous to the global result of the second section. The proof depends on applying the fundamental inequalities for boundary dilatation in [4] and on truncating suitable Beltrami coefficients representing the class of τ.

In the sixth section, we prove Gardiner's conjecture $\max_{p\in\partial\Omega} H_p(\tau) = H(\tau)$ for non-Strebel points by using the result of the third section and the two forms of the main inequality to show that points p which realize the maximum of β_p coincide with points p which realize the maximum of h_p.

In the seventh section, we show the equality $\max_{p\in\partial\Omega} H_p(\tau) = H(\tau)$ also holds for Strebel points τ. Here the method is to use the two main inequalities for boundary dilatation given in [4].

In the appendix, we give a brief discussion of some consequences of our theorems for Strebel's chimney domain.

1. Unit disc case

Theorem 1. *For all v in $Z(\Delta)$, $b(v) = \max_{p\in\partial\Delta} b_p(v)$.*

Proof. Suppose that $\max_{p\in\partial\Delta} b_p(v) < b(v)$ for some v in $Z(\Delta)$. Choose c so that $\max_{p\in\partial\Delta} b_p(v) < c < b(v)$. Let φ_n be a degenerating sequence in $A(\Delta)$ so that $v(\varphi_n) \to \beta(v)$. Let μ be an extremal representative of v. Then

$$\int_\Delta \mu\varphi_n \to b(v),$$

and

$$\|\mu\|_\infty = \|v\|.$$

Since $c > \max_{p\in\partial\Delta} b_p(v)$, there exists $l > 0$ such that for every arc I on the unit circle of length less than l there exists a neighborhood U of I and a Beltrami differential ν equivalent to μ so that

$$\|\nu\ |_{U\cap\Delta}\ \| < c.$$

Following Fehlmann's idea (see [8] and [9]), we divide the unit circle into $N > 4\pi/l$ disjoint arcs I_i of equal length. Let the end points of arc I_i be a_i and a_{i+1}, with $a_{N+1} = a_1$. Let $\varepsilon > 0$. Choose an arc V_i on the unit circle with length less than $l/4$ and the center at the point a_i. Let R_i be the sector in Δ bounded by V_i and the two radial lines terminating at the end-points of V_i. Dividing the sector R_i into more than $1/\varepsilon$ disjoint sectors and observing that each φ_n has norm one, we see that there exist points x_i on V_i, sectors S_i and a subsequence ψ_n of φ_n such that x_i is a mid-point of the boundary arc of S_i and

$$\limsup_{n\to\infty} \int_{S_i} |\psi_n| < \varepsilon \qquad \text{for all } i.$$

Since the length of the open arc A_i from x_i to x_{i+1} is less than l, there exists a neighborhood U_i of A_i and a Beltrami differential ν_i equivalent to μ so that $\partial U_i \cup \partial\Delta = \{x_i, x_{i+1}\}$ and

$$\|\nu_i\ |_{U_i\cap\Delta}\ \| < c.$$

Let $\eta_i^1 = (\nu_i - \mu)\chi_{\Delta\setminus U_i}$ and $v_i = [\eta_i^1]$. Here χ is the characteristic function of a set. Then we also have $v_i = [\eta_i^2]$ where $\eta_i^2 = (\mu - \nu_i)\chi_{\Delta\cap U_i}$.

Let $\mathbf{H}$ be the upper half plane and let ζ_i^1 and ζ_i^2 be the pull-backs of η_i^1 and η_i^2 by a Möbius transformation m that maps the unit disc onto the upper half plane $\mathbf{H}$ and satisfies $m(x_i) = \infty$. The space $Z(\mathbf{H})$ is isomorphic to the space of all Zygmund bounded functions on the real axis, and the isomorphism sends each $v = [\zeta] \in Z(\mathbf{H})$ into a Zygmund bounded function V defined by

$$V(z) = \frac{-1}{\pi}\int_{\mathbf{C}} \frac{z(z-1)\zeta(w)}{w(w-1)(w-z)}\,du\,dv,$$

where ζ is extended to the lower half plane using the reflection $j(z) = \bar{z}$, i.e. $\zeta(\bar{z}) = \overline{\zeta(z)}$ (see 13]). Let V_i be a Zygmund bounded function corresponding to the linear functional $m_* v_i$ in $Z(\mathbf{H})$. Formula (21) in [13] shows that

$$\frac{V_i(z+t) + V_i(z-t) - 2V_i(z)}{t} = \frac{-1}{\pi}\int_{\mathbf{C}} \frac{\zeta_i^j(tw+z)}{w(w-1)(w+1)}\,du\,dv,$$

for $j = 1, 2$. Therefore V_i satisfies the little Zygmund condition

$$V_i(z+t) + V_i(z-t) - 2V_i(z) = o(t)$$

locally uniformly for all z in $\mathbf{R}/\{m(x_{i+1})\}$. Therefore, by pulling-back the Beurling–Ahlfors extension

$$F_i(x+iy) = \frac{1}{2y}\int_{x-y}^{x+y} V_i(t)\,dt + \frac{i}{y}\left[\int_x^{x+y} V_i(t)\,dt - \int_{x-y}^{x} V_i(t)\,dt\right], \tag{8}$$

it follows from Lemma 8.1 in [13] that a Beltrami differential $\eta_i = m^*\bar{\partial}F_i(z)$ in the unit disc satisfies $v_i = [\eta_i]$, $\|\eta_i\|_\infty \le C\|v_i\|$ and $\eta_i(z) \to 0$ if z tends to a point on $\partial\Delta - \{x_i, x_{i+1}\}$. Observe that

$$\|v_i\| \le \|\eta_i^2\|_\infty \le \|v\| + c.$$

Thus, all differentials η_i are uniformly bounded in the L^∞ norm.

With no loss of generality we may assume that the neighborhoods U_i are disjoint and that the set

$$F = \Delta \setminus \bigcup_{i=1}^{N} U_i$$

is a union of a compact set and of measure zero. Then,

$$v = [\mu] = \left[\sum_{i=1}^{N} \mu\chi_{U_i\cap\Delta} + \mu\chi_F\right] = \left[\sum_{i=1}^{N}(\eta_i + \nu_i\chi_{U_i\cap\Delta}) + \mu\chi_F\right].$$

Let

$$\tilde{\mu} = \sum_{i=1}^{N} (\eta_i + \nu_i \chi_{U_i \cap \Delta}) + \mu \chi_F.$$

Then $v = [\tilde{\mu}]$ and

$$\tilde{\mu} = \eta + \sum_{i=1}^{N} \nu_i \chi_{U_i \cap \Delta} + \mu \chi_F,$$

where

$$\eta = \sum_{i=1}^{N} \eta_i.$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \left| \int_{\cup S_i} \tilde{\mu} \psi_n \right| &\le \bigl(NC(\|v\| + c) + Nc\bigr) \limsup_{n\to\infty} \int_{\cup S_i} |\psi_n| \\ &\le \bigl(NC(\|v\| + c) + Nc\bigr)\varepsilon N.\end{aligned}$$

Furthermore, if $W_i = (U_i \cap \Delta) \setminus (S_i \cup S_{i+1})$, then

$$\begin{aligned}\limsup_{n\to\infty} \left| \int_{\cup W_i} \tilde{\mu} \psi_n \right| &\le \limsup_{n\to\infty} \sum_{i=1}^{N} \left| \int_{W_i} \nu_i \psi_n \right| + \sum_{i=1}^{N} \sum_{j=1}^{N} \limsup_{n\to\infty} \left| \int_{W_i} \eta_j \psi_n \right| \\ &= \limsup_{n\to\infty} \sum_{i=1}^{N} \left| \int_{W_i} \nu_i \psi_n \right| \le c \limsup_{n\to\infty} \sum_{i=1}^{N} \int_{W_i} |\psi_n| \le c.\end{aligned}$$

Thus,

$$b(v) = \lim_{n\to\infty} v(\psi_n) = \limsup_{n\to\infty} \int_{\Delta} \tilde{\mu} \psi_n \le c + \bigl(NC(\|v\| + c) + Nc\bigr)\varepsilon N,$$

which is a contradiction provided that ε is sufficiently small. □

2. Boundary dilatation

The boundary dilatation $H(\tau)$ determines Teichmüller's metric on the asymptotic Teichmüller space $\mathrm{AT}(\Omega) = T(\Omega) \bmod T_0(\Omega)$, (see [13] and [4]). It also determines the Strebel points in the Teichmüller space $T(\Omega)$. In this section we find several ways to express the boundary dilatation. They are analogous to the several different representations of the dilatation $K_0(\tau)$, which determines the Teichmüller's metric in $T(\Omega)$.

The first point of view is looking at the mapping by heights introduced by Strebel in [21]. If f is a quasiconformal homeomorphism of the unit disk Δ and

φ is in $A(\Delta)$, then there is a unique integrable holomorphic quadratic differential ψ such that the vertical φ-distance between any two boundary points r and s is equal to the vertical ψ-distance between $f(r)$ and $f(s)$, (see [21]). We say that ψ is the image of φ under the mapping by heights induced by f, and we denote ψ by $\mathrm{MH}(f,\varphi)$. Notice that if $[f_1]$ and $[f_2]$ are the same points in the universal Teichmüller space $T(\Delta)$ then there exists a conformal homeomorphism α of Δ such that $f_1(t) = \alpha \circ f_2(t)$ for every $t \in \partial\Delta$. Therefore $\psi_2 = \mathrm{MH}(f_2,\varphi)$ is a pull-back of $\psi_1 = \mathrm{MH}(f_1,\varphi)$ by α:

$$\psi_2 = \psi_1(\alpha)\alpha'^2.$$

That yields $\|\psi_1\| = \|\psi_2\|$. We define a function from $T(\Delta)\times A(\Delta)$ onto $A(\Delta)$ by $(\tau,\varphi) \to \mathrm{MH}(f,\varphi)$, where f is normalized to fix 1, -1 and i, and $[f]=\tau$. This function describes the mapping by heights up to the pull-backs by Möbius transformations, so we will also call it the mapping by heights and denote it by MH. It is proved in [15] that $K_0(\tau)$ is equal to the supremum of $\|\mathrm{MH}(\tau,\varphi)\|$ over all unit vectors φ in $A(\Delta)$. In a parallel manner we define

$$S(\tau) = \sup_{(\varphi_n)} \limsup_{n\to\infty} \|\mathrm{MH}(\tau,\varphi_n)\|.$$

Here the supremum is over all degenerating sequences of unit vectors φ_n.

A second point of view comes from looking at the semi-norm b, the infinitesimal version of the boundary dilatation h. Since $b(v)=\beta(v)$ for all v, we would like to find the corresponding statement for the boundary dilatation. We use the estimates for the Teichmüller metric given by the following Reich–Strebel inequalities:

$$K_0(\tau) \le \sup_{\|\psi\|=1} \int_\Omega |\psi| \frac{\left|1+\mu(\psi/|\psi|)\right|^2}{1-|\mu|^2} \tag{9}$$

$$\frac{1}{K_0(\tau)} \le \int_\Omega |\varphi| \frac{\left|1-\mu(\varphi/|\varphi|)\right|^2}{1-|\mu|^2} \tag{10}$$

for all $\tau=[\mu]\in T(\Omega)$ and all unit vectors φ in $A(\Omega)$, (see [11] for the proofs). We say that the inequalities (10) and (9) are the first and the second fundamental inequalities of Reich–Strebel, respectively. Reich and Strebel proved the inequalities (10) and (9) by studying the trajectory structure of the quadratic differential φ. Using further analysis of this structure and previous results of Hamilton and Krushkal, Reich and Strebel came to the following criterion for the extremality of the Beltrami coefficient μ: μ is extremal in its Teichmüller class if, and only if, there exists a Hamilton sequence for μ. Thus,

$$k_0(\tau) = \sup_{\|\varphi\|=1} \mathrm{Re} \int_\Omega \mu\varphi.$$

We note that when the class τ contains more than one extremal representative, this supremum takes the same value independently of which extremal representative is chosen. To define the analogous quantity corresponding to boundary dilatation we use Beltrami coefficients μ_k which nearly realize the boundary dilatation in their asymptotic Teichmüller class. We call a sequence μ_k in a given class τ an approximating sequence if $h^*(\mu_k) - h(\tau)$ approaches 0 as k approaches infinity. Let

$$G(\tau) = \frac{1+g(\tau)}{1-g(\tau)} \qquad \text{and}$$

$$g(\tau) = \sup_{\mu_k} \limsup_{k\to\infty} \sup_{\{\varphi_n\}} \limsup_{n\to\infty} \operatorname{Re} \int_\Omega \mu_k \varphi_n.$$

Here, the first supremum is over any sequence μ_k of approximating Beltrami coefficients for the class τ and the second supremum is over all degenerating sequences $\{\varphi_n\}$ in $A(\Omega)$.

Note that $\beta(v=[\mu])$ is the infinitesimal version of $g(\tau=[\mu])$ and $b(v=[\mu])$ is the infinitesimal version of $h(\tau=[\mu])$. The part of the following theorem which equates $G(\tau)$ and $H(\tau)$ is analogous to the theorem from [4] which equates $\beta(v)$ and $b(v)$.

Theorem 2. *The distortions H, G and S coincide. More precisely,*

$$\begin{aligned} G(\tau) &= S(\tau) = H(\tau) && \text{for all } \tau \in T(\Delta), \\ G(\tau) &= H(\tau) && \text{for all } \tau \in T(\Omega). \end{aligned}$$

Proof. $H(\tau) = S(\tau)$ by Theorem 5 in [15]. Clearly $g(\tau) \le h(\tau)$. To prove the reverse inequality we may assume that $H(\tau) > 1$. By the definition of $H(\tau)$, there exists a sequence of representatives μ_n of τ such that $h^*(\mu_n) \to h(\tau)$. Then, by the theorem on inequalities for the boundary dilatation in [4] or, in the case of the unit disc, by the main theorem of [12], $\beta([\mu_n]) \to h(\tau)$. Thus, $g(\tau) \ge h(\tau)$. □

3. Infinitesimal substantial points

In the proof of Theorem 1 we used Beurling–Ahlfors extension (8) of Zygmund bounded functions on the unit circle. In [16], Reich showed a similar result by considering the extension induced by the kernel

$$S(z,w) = \frac{(1-|z|^2)^3}{2\pi i(1-\bar{z}w)^3(w-z)}.$$

These extensions apply to the unit disc case and cannot be easily extended to an arbitrary plane domain case. In this section we generalize Theorem 1 to the plane domain case by looking at the infinitesimal version of the Sullivan–Thurston [22] extension of holomorphic motions.

Let Ω be a plane domain and let Λ be the complement of Ω. We assume that Λ contains at least three points.

Theorem 3. *For all v in $Z(\Omega)$, $b(v) = \max_{p\in\partial\Omega} b_p(v)$.*

Proof. Suppose that $\max_{p\in\partial\Omega} b_p(v) < b(v)$ for some v in $Z(\Omega)$. Choose c so that $\max_{p\in\partial\Omega} b_p(v) < c < b(v)$. Let φ_n be a degenerating sequence in $A(\Omega)$ so that $v(\varphi_n) \to \beta(v)$. Let μ be an extremal representative of v. Then

$$\int_\Omega \mu\varphi_n \to b(v)$$

and

$$\|\mu\|_\infty = \|v\|.$$

We may assume $\infty \in \Omega$. Then Λ is compact in the Euclidean metric, and for some positive integer $M > 0$, Λ is contained in the square of side length $2M$ centered at the origin. Since $c > \max_{p\in\partial\Omega} b_p(v)$, there exists $l > 0$ such that for every subset Y of Λ of diameter less than l there exists a neighborhood U of Y and a Beltrami differential ν representing v so that $\|\nu\ |_{U\cap\Omega}\| < c$ (as usual, we assume that the L^∞ norm of the characteristic function of an empty set is equal to zero). Also, we may assume μ, ν and φ_n are identically equal to zero in the complement of Ω. We use a two-dimensional analogue of an idea of Fehlmann (see [8] and [9]). Divide the square $[-M, M]^2$ into N squares $A_1, A_2, \ldots, A_N$ of equal diameter d with d less than $\frac{1}{2}l$. Fix $\varepsilon > 0$. For every square $A_i = [a, b] \times [c, d]$ we consider the frames $F_{i_k} = P_{i_k} \setminus Q_{i_k}$ where

$$P_{i_k} = \left[a - \frac{l}{10}\frac{k+1}{L}, b + \frac{l}{10}\frac{k+1}{L}\right] \times \left[c - \frac{l}{10}\frac{k+1}{L}, d + \frac{l}{10}\frac{k+1}{L}\right]$$

and

$$Q_{i_k} = P_{i_{k-1}} = \left[a - \frac{l}{10}\frac{k}{L}, b + \frac{l}{10}\frac{k}{L}\right] \times \left[c - \frac{l}{10}\frac{k}{L}, d + \frac{l}{10}\frac{k}{L}\right],$$

where $L > 1/\varepsilon$ and $k = 0, 1, 2, \ldots, L$. Since $\|\varphi_n\| = 1$ for all n, there is a subsequence ψ_n of φ_n and a frame $S_i = F_{i_k}$ such that

$$\limsup_{n\to\infty} \int_{S_i} |\psi_n| \le \varepsilon \qquad \text{for all } i.$$

Let

$$R'_{i_k} = \left[a - \frac{l}{10}\frac{k+\frac{1}{2}}{L}, b + \frac{l}{10}\frac{k+\frac{1}{2}}{L}\right] \times \left[c - \frac{l}{10}\frac{k+\frac{1}{2}}{L}, d + \frac{l}{10}\frac{k+\frac{1}{2}}{L}\right]$$

be the rectangle bounded by the core curve of the frame S_i. Also let $R_1 = R'_{1_k}$, $R_2 = R'_{2_k} \setminus R_1, \ldots, R_N = R'_{N_k} \setminus (R_1 \cup R_2 \cup \cdots \cup R_{N-1})$. Note that $\Lambda \subset R_1 \cup$

$R_2 \cup \cdots R_N$. The diameter of each R_i is less than l, thus there exists a Beltrami coefficient ν_i representing v and a neighborhood U_i of R_i such that

$$\partial U_i \subset \bigcup_{i=1}^{N} S_i \qquad \text{and} \qquad \|\nu_i \,|_{U_i \cap \Omega}\,\|_\infty < c.$$

Let $\eta_i^1 = (\nu_i - \mu)\chi_{\Omega/U_i}$. Then $[\eta_i^1] = [\eta_i^2]$ where $\eta_i^2 = (\mu - \nu_i)\chi_{U_i \cap \Omega}$. Pick three points λ_1, λ_2, λ_3 in Λ, let φ_z a rational function holomorphic in $\overline{\mathbf{C}}$ defined by

$$\varphi_z(w) = -\frac{1}{\pi}\frac{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}{(w-\lambda_1)(w-\lambda_2)(w-\lambda_3)(w-z)}$$

and let

$$V_i(z) = \int_\Omega \varphi_z(w)\eta_i^1(w)\,du\,dv.$$

Since the function $w \mapsto \varphi_z(w)$ belongs to $A(\Omega)$, we also have

$$V_i(z) = \int_\Omega \varphi_z(w)\eta_i^2(w)\,du\,dv.$$

V_i is a vector field on Λ with bounded cross ratio norm (see [5]), and any extension of V_i to a vector field $\widetilde{V}$ on $\mathbf{C}$ with bounded $\bar\partial$-derivative satisfies $[\eta_i^1] = [\bar\partial\widetilde{V}]$. Instead of Beurling–Ahlfors extension used in Section 2 or the kernel $S(z,w)$ used in [16] for the unit disc case, we now apply the infinitesimal version of the extension procedure used by Sullivan and Thurston [22] to extend a holomorphic motion. This extension was used in [5] to show that the infinitesimal Teichmüller norm is equivalent to the cross ratio norm. The extension $\widetilde{V}$ is obtained as the limit of extensions $\widetilde{V}_n$ applied to the finite sets $\Lambda_n = \{\lambda_1, \lambda_2, \ldots, \lambda_{n+2}\}$ such that the set $\{\lambda_1, \lambda_2, \ldots\}$ is dense in Λ. The vector fields $\widetilde{V}_n$ are obtained by pasting together the local extensions by a suitable (ample and uniform) partition of unity. The local extensions are achieved by restricting Λ_n to a three-point set or a four-point set depending on the thick-thin decomposition of the domain Ω_n complementary to Λ_n. In the case of a three point set we use the best affine extension (i.e. the extension with the smallest L^∞ norm of its $\bar\partial-$derivative). In the case of a four point set we use the canonical extension obtained by looking at the one-dimensional Teichmüller space of the (extended complex) plane punctured at those four points (see [5] for more details). Since $\eta_i^1(z) \to 0$ if z converges to a point on $\Lambda \cap U_i$ and $\eta_i^2(z) \to 0$ if z converges to a point on $\Lambda \setminus \overline{U_i}$, the proof of the Equivalence Theorem in [5] shows (see in particular Section 7.5 in [5]) that the extension $\widetilde{V}$ of V_i has $\bar\partial$-derivative η_i which satisfies $\eta_i(z) \to 0$ if z converges to a point on $\Lambda \setminus \partial U_i$. Furthermore $\|\bar\partial\widetilde{V}\|_\infty \le C\|\eta_i^2\|_\infty$ for some universal constant C. The rest of the proof is the same as in the unit disc case. □

4. Local boundary semi-norms

Let Ω be a plane domain whose complement Λ contains at least three points. It is shown in [4] that $\beta(v) = b(v)$ for all v in $Z(\Omega)$. Now we prove the local version of this theorem.

Theorem 4. *If* $v \in Z(\Omega)$, *then*

$$b_p(v) = \beta_p(v)$$

for all $p \in \partial\Omega$.

Proof. Let $p \in \partial\Omega$. It is easy to see that $0 \le \beta_p(v) \le b_p(v)$. We now show that $\beta_p(v) \ge b_p(v)$. Clearly, we may assume $b_p(v) > 0$. Let μ be an extremal Beltrami differential representing v. By the definition of $b_p(v)$, there exists a sequence of Beltrami differentials μ_n and a sequence of neighborhoods U_n of p such that U_{n+1} is contained in U_n for all n, $\bigcap_n U_n = \{p\}$, each μ_n represents v, and $|\mu_n(z)| \le b_p(v) + 1/n$ for all z in $U_n \cap \Omega$. The result of the previous section implies that there exists a Beltrami differential ν_n representing $[\mu_n \chi_{\Omega \setminus U_n}]$ such that $\|\nu_n\|_\infty \le C\|[\mu_n \chi_{\Omega\setminus U_n}]\|$ and $\nu_n(z) \to 0$ when z converges to a point on $U_n \cap \Lambda$ (ν_n is the $\bar\partial$-derivative of the extension in [5] of the vector field on Λ corresponding to $[\mu_n\chi_{\Omega\setminus U_n}]$). Thus, there are neighborhoods V_n of p such that $V_n \subset U_n$ and $|\nu_n(z)| < 1/n$ for all $z \in V_n \cap \Omega$. Since ν_n is equivalent to $\mu_n - \mu_n\chi_{U_n\cap\Omega}$, it is also equivalent to $\mu - \mu_n\chi_{U_n\cap\Omega}$. Furthermore $\|\mu - \mu_n\chi_{U_n\cap\Omega}\|_\infty \le \|v\| + b(v) + 1$. Thus,

$$\|\nu_n\|_\infty \le C\big(\|v\| + b(v) + 1\big)$$

for some universal constant C. Note that μ is equivalent to the Beltrami differential $\eta_n = \nu_n + \mu_n\chi_{U_n\cap\Omega}$. Let $v_n = [\eta_n\chi_{V_n\cap\Omega}]$. Choose a unit vector φ_n in $A(\Omega)$ such that $v_n(\varphi_n) > \|v_n\| - 1/n$. Then

$$\begin{aligned} b_p(v) &= b_p(v_n) \le \|v_n\| < v_n(\varphi_n) + 1/n \\ &\le \|\eta_n\chi_{V_n\cap\Omega}\|_\infty \int_{V_n\cap\Omega} |\varphi_n| + 1/n \le \Big(b_p(v) + \frac{2}{n}\Big)\int_{V_n\cap\Omega}|\varphi_n| + 1/n. \end{aligned}$$

Hence,

$$\int_{V_n\cap\Omega} |\varphi_n| \ge \frac{b_p(v) - 1/n}{b_p(v) + 2/n} \to 1 \qquad \text{as } n \to \infty.$$

Therefore φ_n degenerates towards p. Furthermore,

$$\begin{aligned} |v(\varphi_n)| &= \left|\int_\Omega \eta_n\varphi_n\right| \ge |v_n(\varphi_n)| - \left|\int_{\Omega\setminus V_n} \eta_n\varphi_n\right| \\ &> \|v_n\| - \frac{1}{n} - (C+1)\big(\|v\| + b(v) + 1\big)\frac{3/n}{b_p(v) + 2/n} \\ &\ge b_p(v) - \frac{1}{n} - (C+1)\big(\|v\| + b(v) + 1\big)\frac{3/n}{b_p(v) + 2/n}. \end{aligned}$$

Therefore

$$\beta_p(v) \geq \limsup_{n\to\infty} |v(\varphi_n)| \geq b_p(v). \qquad \square$$

5. Local boundary dilatation

In Section 2 we introduced the formula for the boundary dilatation using degenerating sequences. In this section we study the corresponding local situation. Let p be a boundary point of the plane domain Ω and let τ be a point in $T(\Omega)$. In a parallel manner we define

$$g_p(\tau) = \lim_{k\to\infty} \sup_{\{\varphi_n\}} \limsup_{n\to\infty} \operatorname{Re} \int_\Omega \mu_k \varphi_n.$$

Here μ_k is a Beltrami coefficient in the class of τ such that $h_p^*(\mu_k) < h_p(\tau) + 1/k$ and the supremum is over all sequences $\{\varphi_n\}$ in $A(\Omega)$ degenerating towards p. In keeping with standard notation, we put

$$G_p = \frac{1+g_p}{1-g_p}.$$

Note that β_p is the infinitesimal version of g_p and b_p is the infinitesimal version of h_p. The following theorem is the analogue of Theorem 4 and the local version of Theorem 2.

Theorem 5. *For all* $\tau \in T(\Omega)$,

$$g_p(\tau) = h_p(\tau).$$

Proof. It is easy to see that $g_p \leq h_p$. In order to estimate $h_p - g_p$, select μ representing the class τ in $T(\Omega)$ such that $h_p^*(\mu)$ is arbitrarily close to $h_p(\tau)$. Clearly, $h_p^*(\mu) = b_p^*(\mu)$. Moreover, if we let v be the linear functional in $Z(\Omega)$ represented by μ, then Theorem 5 and the existence of the limit in the definition of $g_p(\tau)$ follow from the next lemma.

Lemma 1. *For every* Ω,

$$b_p^*(\mu) - \beta_p(v) \leq H_p^*(\mu) - H_p(\tau).$$

Proof. Pick a neighborhood $U = \{z : |z-p| < r\}$ of p such that $K(\mu\chi_{\Omega\cap U}) \leq H_p^*(\mu) + 1/n$ and $b_q(v) \leq b_p(v) + 1/n$ for all $q \in \partial\Omega \cap U$. Define a new Beltrami coefficient η on Ω by letting

$\eta(z) = \mu(z)$ for all z in Ω with $|z-p| < \frac{1}{2}r$,
$\eta(z) = 0$ for all $z \in \Omega \setminus U$, and
$\eta(z) = t\mu$ for all $t \in (0,1)$ and all z in Ω with $|z-p| = r\bigl(1-\frac{1}{2}t\bigr)$.

Then by Theorem 3, $b([\eta]) = \sup_{q\in\partial\Omega} b_q([\eta]) \le b_p(v) + 1/n$. Moreover,

$$\begin{aligned} H^*(\eta) - H([\eta]) &\le H^*(\eta) - H_p([\eta]) = H^*(\eta) - H_p(\tau) \\ &\le K(\eta) - H_p(\tau) \le H_p^*(\mu) - H_p(\tau) + 1/n. \end{aligned}$$

Therefore, the inequalities for boundary dilatation in [4] yield

$$b^*(\eta) - b([\eta]) \le H_p^*(\mu) - H_p(\tau) + 1/n.$$

Combining these inequalities we obtain

$$\begin{aligned} b_p^*(\mu) - \beta_p(v) = b_p^*(\eta) - b_p(v) &\le b^*(\eta) - b_p(v) \\ &\le b^*(\eta) - b([\eta]) + 1/n \le H_p^*(\mu) - H_p(\tau) + 2/n. \ \square \end{aligned}$$

6. Substantial points for non-Strebel classes

Now suppose that τ is a non-Strebel point in $T(\Omega)$. Let μ be an extremal Beltrami coefficient representing τ. Then there exists a degenerating Hamilton sequence for μ and we have $h(\tau) = \|\mu\|_\infty = \beta([\mu]) = b([\mu])$ (see [7], [4]). We now prove the local version of this result. Note that μ also represents a linear functional $v = [\mu]$ in $Z(\Omega)$.

Theorem 6. *The following five conditions are equivalent for every boundary point p of Ω and every extremal representative μ of a non-Strebel point τ in $T(\Omega)$:*

(1) $H(\tau) = H_p(\tau)$,
(2) $G(\tau) = G_p(\tau)$,
(3) $b(v) = b_p(v)$,
(4) $\beta(v) = \beta_p(v)$,
(5) *there exists a Hamilton sequence for μ degenerating towards p.*

Proof. Let p be a boundary point of Ω and let μ be an extremal representative of a non-Strebel point τ in $T(\Omega)$. Also let v be a functional in $Z(\Omega)$ represented by the Beltrami differential μ. It is shown in [4] that $b(v) = \beta(v)$. Furthermore $b_p(v) = \beta_p(v)$ by Theorem 4. Thus, (3) is equivalent to (4). The equivalence of (4) and (5) follows from the definitions of the semi-norms β and β_p and the equivalence of (1) and (2) follows from Theorems 2 and 5.

We now show that (1) is equivalent to (5). Let f be a quasiconformal mapping with domain Ω and Beltrami coefficient μ. Assume first that there exists a Hamilton sequence φ_n such that $\int_\Omega \varphi_n\mu \to \|\mu\|_\infty$ and φ_n is degenerating towards p. Suppose that $H_p(\tau) < (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$. Then there exists a neighborhood U of p and a quasiconformal mapping g with domain Ω, range $f(\Omega)$ and Beltrami coefficient ν such that $g^{-1}\circ f$ is homotopic to identity relative to the boundary of Ω and $\|\nu\,|_{g^{-1}\circ f(U\cap\Omega)}\|_\infty < \|\mu\|_\infty$. Let $\sigma = \nu\chi_{g^{-1}\circ f(U\cap\Omega)}$. Then σ

is equivalent to a Beltrami coefficient ζ so that $\zeta(z) = \mu(z)$ for all $z \in U \cap \Omega$. Thus, the first fundamental inequality of Reich and Strebel yields

$$\begin{aligned}\frac{1 - \|\nu\,|_{g^{-1}\circ f(U\cap\Omega)}\,\|_\infty}{1 + \|\nu\,|_{g^{-1}\circ f(U\cap\Omega)}\,\|_\infty} &\le \frac{1}{K_0([\sigma])} \le \liminf_{n\to\infty} \int_\Omega |\varphi_n| \frac{\left|1 - \zeta\varphi_n/|\varphi_n|\right|^2}{1 - |\zeta|^2} \\ &= \liminf_{n\to\infty} \int_{\Omega\cap U} |\varphi_n| \frac{\left|1 - \mu\varphi_n/|\varphi_n|\right|^2}{1 - |\mu|^2} \\ &\le \frac{1 + \|\mu\|_\infty^2 - 2\limsup_{n\to\infty} \operatorname{Re}\int_{\Omega\cap U} \mu\varphi_n}{1 - \|\mu\|_\infty^2} \\ &= \frac{1 + \|\mu\|_\infty^2 - 2\limsup_{n\to\infty} \int_{\Omega} \mu\varphi_n}{1 - \|\mu\|_\infty^2} = \frac{1 - \|\mu\|_\infty}{1 + \|\mu\|_\infty},\end{aligned}$$

a contradiction.

Finally, assume that $H_p(\tau) = H(\tau)$ and let $U_n = \{z \in \Omega : |z - p| < 1/n\}$. If $\mu_n = \mu\chi_{U_n}$, then

$$K_0([\mu_n]) \ge H_p([\mu_n]) = H_p(\tau) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

Thus, by the second fundamental inequality of Reich and Strebel, there exists a unit vector φ_n in $A(\Omega)$ such that

$$\int_\Omega |\varphi_n| \frac{\left|1 + \mu_n\varphi_n/|\varphi_n|\right|^2}{1 - |\mu_n|^2} \ge \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} - \frac{1}{n}.$$

Therefore,

$$\begin{aligned}\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} &\le \liminf_{n\to\infty} \frac{1 + \|\mu_n\|_\infty^2 + 2\operatorname{Re}\int_\Omega \mu_n\varphi_n}{1 - \|\mu_n\|_\infty^2} \\ &\le \frac{1 + \|\mu\|_\infty^2 + 2\liminf_{n\to\infty} \operatorname{Re}\int_\Omega \mu_n\varphi_n}{1 - \|\mu\|_\infty^2}, \\ &\liminf_{n\to\infty} \operatorname{Re}\int_{U_n} \mu\varphi_n \ge \|\mu\|_\infty.\end{aligned}$$

Therefore, $\int_{U_n} |\varphi_n| \to 1$ as $n \to \infty$ and φ_n is degenerating towards p. Moreover,

$$\liminf_{n\to\infty} \operatorname{Re}\int_\Omega \varphi_n\mu \ge \liminf_{n\to\infty} \operatorname{Re}\int_{U_n} \varphi_n\mu - \|\mu\|_\infty \limsup_{n\to\infty} \int_{\Omega\setminus U_n} |\varphi_n| \ge \|\mu\|_\infty. \quad \square$$

Corollary 1. *With the same hypotheses as in the previous theorem, substantial points exist.*

Proof. From this theorem and from Theorems 3 and 4, there exist points p for which the maximum in (1), (2), (3) and (4) are simultaneously achieved. □

Remark. Using the kernel

$$S(z, w) = \frac{(1 - |z|^2)^3}{2\pi i(1 - \bar{z}w)^3(w - z)},$$

Reich partially proved Theorem 6 in the case of the unit disc (see [16]) and hence provided another proof of Fehlmann's theorem for non-Strebel points. Theorem 6 together with the results in Chapters 2 and 4 provides two new proofs of the same result and it also provides the generalization to the plane domain case. We generalize Fehlmann's theorem for Strebel points in Theorem 8.

We say that a boundary point p of the plane domain Ω is a substantial point for a Beltrami coefficient μ if $H([\mu]) = H_p(\mu)$. Also, we say that p is an infinitesimally substantial point for a Beltrami differential μ if $b([\mu]) = b_p([\mu])$. The set of all substantial points is called the substantial set, and the set of all infinitesimally substantial points is called the infinitesimally substantial set. These sets are clearly closed subsets of Ω. Theorem 6 shows that the substantial set coincides with the infinitesimally substantial set for any plane domain and any extremal representative of a non-Strebel point.

Theorem 7. *Let Ω be a plane domain and let μ be an extremal representative of a non-Strebel point in $T(\Omega)$. Then every degenerating Hamilton sequence for μ degenerates towards a subset of the set of substantial points.*

Proof. Let φ_n be a degenerating Hamilton sequence for an extremal representative μ of a non-Strebel point $\tau \in T(\Omega)$. Fix a neighborhood U of the set of all substantial points for μ. Let ε be a small positive number. Let p be a point in $\partial\Omega/U$. By Theorem 6, p is not an infinitesimally substantial point for μ. Thus, there exists a neighborhood $V = \{z \in \Omega : |z - p| < \delta\}$ of p in Ω, a subsequence ψ_n of φ_n and a Beltrami differential ν infinitesimally equivalent to μ such that $\|\nu\,|_V\,\|_\infty < \|\mu\|_\infty$ and

$$\limsup_{n\to\infty} \int_W |\psi_n| < \varepsilon,$$

where W is a thin annulus consisting of those points z for which $\delta - \alpha < |z - p| < \delta + \alpha$ for sufficiently small $\alpha > 0$. By the proof of Theorem 3, the Beltrami differential $\mu\chi_V$ is infinitesimally equivalent to the Beltrami differential $\eta + \nu\chi_V$ where $\eta(z) \to 0$ as z converges to a point in $\partial\Omega \setminus W$, and $\|\eta\|_\infty \le 2C\|\mu\|_\infty$. Therefore,

$$\begin{aligned}\|\mu\|_\infty &= \limsup_{n\to\infty} \int_\Omega \psi_n\mu \\ &\le \limsup_{n\to\infty}\left(\|\nu\,|_V\,\|_\infty \int_V |\psi_n| + \|\mu\|_\infty \int_{\Omega\setminus V} |\psi_n|\right) + (2C)\|\mu\|_\infty\varepsilon.\end{aligned}$$

Thus, $\int_V |\psi_n| \to 0$ as $n \to \infty$. Since $\partial\Omega \setminus U$ is compact, by passing to a subsequence we conclude that $\int_{\Omega/U} |\varphi_n| \to 0$ as $n \to \infty$. □

7. Substantial points for Strebel classes

The following theorem generalizes to an arbitrary plane domain Fehlmann's theorem on the existence of substantial boundary points for points in the Teichmüller space of the unit disc, and so it answers affirmatively Gardiner's conjecture.

Theorem 8. *For all points τ in $T(\Omega)$*

$$H(\tau) = \max_{p \in \partial\Omega} H_p(\tau).$$

Proof. Let τ be a point in $T(\Omega)$. If τ is a non-Strebel point, then the theorem follows from Theorem 6. Suppose that τ is a Strebel point. We may assume $H(\tau) > 1$. Let μ_n be a sequence of Beltrami coefficients representing τ such that $h^*(\mu_n) \le h(\tau) + 1/n$. Let f be a quasiconformal mapping with domain Ω and Beltrami coefficient μ_n. Then, by the inequalities for boundary dilatation which led to Teichmüller's contraction principle in [4] and [12], $h^*(\mu_n) - \beta([\mu_n]) \to 0$ as $n \to \infty$. By Theorems 3 and 4, there exists a point p_n on the boundary of Ω such that $\beta_{p_n}([\mu_n]) = \beta([\mu_n])$. Thus, $h(\tau) - \beta_{p_n}([\mu_n]) \to 0$ as $n \to \infty$. Take a sequence φ_k degenerating towards p_n such that

$$\mathrm{Re} \int_\Omega \varphi_k \mu_n \to \beta_{p_n}([\mu_n]).$$

There exists a neighborhood U of p_n and a quasiconformal mapping g with domain Ω, range $f(\Omega)$ and Beltrami coefficient ν such that $g^{-1} \circ f$ is homotopic to identity relative to the boundary of Ω, $\|\mu_n\ |_{U\cap\Omega}\ \|_\infty < h^*(\mu_n) + 1/n$ and $\|\nu\ |_{g^{-1}\circ f(U\cap\Omega)}\ \|_\infty < h_{p_n}([\mu_n]) + 1/n$. Let $\sigma = \nu\chi_{g^{-1}\circ f(U\cap\Omega)}$. Then σ is equivalent to a Beltrami coefficient ζ so that $\zeta(z) = \mu_n(z)$ for all $z \in U \cap \Omega$. Thus, the first fundamental inequality for boundary dilatation yields

$$\begin{aligned}
\frac{1 - h_{p_n} - 1/n}{1 + h_{p_n} + 1/n} &\le \frac{1 - \|\nu\ |_{g^{-1}\circ f(U\cap\Omega)}\ \|_\infty}{1 + \|\nu\ |_{g^{-1}\circ f(U\cap\Omega)}\ \|_\infty} \le \frac{1}{K_0([\sigma])} \\
&\le \liminf_{k\to\infty} \int_\Omega |\varphi_k| \frac{\left|1 - \zeta\varphi_k/|\varphi_k|\right|^2}{1 - |\zeta|^2} \\
&= \liminf_{k\to\infty} \int_{\Omega\cap U} |\varphi_k| \frac{\left|1 - \mu_n\varphi_k/|\varphi_k|\right|^2}{1 - |\mu_n|^2} \\
&\le \frac{1 + \|\mu_n\ |_{U\cap\Omega}\ \|_\infty^2 - 2\limsup_{k\to\infty} \mathrm{Re} \int_{\Omega\cap U} \mu_n\varphi_k}{1 - \|\mu\ |_{U\cap\Omega}\ \|_\infty^2}
\end{aligned}$$

$$= \frac{1 + \|\mu_n\ |_{U\cap\Omega}\ \|_\infty^2 - 2\limsup_{k\to\infty} \mathrm{Re}\int_\Omega \mu_n\varphi_k}{1 - \|\mu_n\ |_{U\cap\Omega}\ \|_\infty^2}$$

$$\le \frac{1 + \left(h^*(\mu_n) + 1/n\right)^2 - 2\beta([\mu_n])}{1 - \left(h^*(\mu_n) + 1/n\right)^2} \to \frac{1-h}{1+h}.$$

Therefore,

$$h_{p_n}(\tau) \to h(\tau) \qquad \text{as } n \to \infty.$$

Finally, since Λ is compact and H_p is an upper-semicontinuous function of p, there exists a point p in Λ for which $H_p(\tau) = H(\tau)$. □

Remark. Note that Theorem 3 is the infinitesimal version of Theorem 9.

Appendix

One way to construct interesting Teichmüller classes of mappings is to consider the stretch map $f_K(z) = x + iy$ defined on different plain domains. Since f_K is the Teichmüller mapping associated to the quadratic differential dz^2 and since the norm of this quadratic differential is just the Euclidean area of the domain, these examples are uniquely extremal when the domains have finite Euclidean area. As studied in many papers by Reich and Strebel (see [19] for further references), when the domains have infinite area, the stretch map may be either uniquely extremal or just extremal or not extremal. One of the most important domains in this study is Strebel's chimney domain S. The chimney domain S is the union of the chimney C and the lower half plane. The chimney C is the region in the upper half plane between the vertical line $x = 0$ and the vertical line $x = 2$. This was the first example of a non-uniquely extremal quasiconformal mapping f_K. Strebel's frame mapping theorem implies that the boundary dilatation $H(f_K) = K$ (see [19], [18] and [20] for more details).

In the chimney domain, the point at infinity is the only substantial boundary point. It is easy to see that the boundary dilatation of f_K at any boundary point of S except a vertex point at the base of the chimney or at the point $i\infty$ is less than K. To see that $H_p < K$ at $p = 0$, consider three triangles, T_1, T_2, and T_3. Let T_1 have vertices at -1, $-i$ and 0, T_2 have vertices at 0, $-i$, and 1, and T_3 have vertices at 0, 1 and i. Consider the piecewise affine map which maps T_1 to $\widetilde{T}_1$ with vertices at $-K$, $-iK^{1/2}$, and 0, T_2 to $\widetilde{T}_2$ with vertices at 0, $-iK^{1/2}$, and 1 and T_3 to $\widetilde{T}_3 = T_3$. Since the dilatation of this piecewise affine map is no more than $K^{1/2}$ and since it agrees with f_K on the boundary of S in a neighborhood of p, we see that $H_p < K$ at this point. Clearly this same estimate of H_p applies at the vertex point $p = 2$ of S. By Theorems 3, 6 and 9 there must be a boundary point for which $H_p = H$ and this point must by $i\infty$. Moreover, by Theorem 7, the support of any Hamilton sequence φ_n must "move up" the chimney C. That is, for every positive number M and every $\varepsilon > 0$, there exists an integer n_0, such that for $n \geq n_0$,

$$\iint_{S\cap\{z:y>M\}} |\varphi_n(z)|\,dx\,dy > 1 - \varepsilon.$$

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