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LOGARITHMIC COEFFICIENTS OF UNIVALENT FUNCTIONS

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Abstract. We prove that if $n \geq 2$ there exists a close-to convex function f in S whose n-th logarithmic coefficient γ_n satisfies $|\gamma_n| > 1/n$. Also, we prove some results related to a conjecture of Milin on the logarithmic coefficients of functions in the class S and give some applications of them to obtain upper bounds on the integral means of these functions.

1. Introduction and statement of results

Let S be the class of functions f analytic and univalent in the unit disc

$$\Delta = \{z \in \mathbf{C} : |z| < 1\}$$

with $f(0) = 0$, $f'(0) = 1$. Let $S^\star$ denote the subset of S consisting of those functions f in S for which $f(\Delta)$ is starlike with respect to 0. It is well known (see [6] or [20]) that if f is analytic in Δ, with $f(0) = 0$, $f'(0) = 1$, then $f \in S^\star$ if and only if $\operatorname{Re}\bigl(zf'(z)/f(z)\bigr) > 0$, for all z in Δ. Finally, we let $\mathscr{C}$ denote the set of those functions f in S for which there exists a real number α and a function g in $S^\star$ such that

$$\operatorname{Re}\frac{zf'(z)}{e^{i\alpha}g(z)} > 0, \qquad z \in \Delta.$$

The elements of $\mathscr{C}$ are called close-to-convex functions. Clearly, $S^\star \subset \mathscr{C}$.

Associated with each f in S is a well defined logarithmic function

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n, \qquad z \in \Delta.$$

The numbers γ_n are called the logarithmic coefficients of f. Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = 1/n$.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ then $\gamma_1 = \frac{1}{2}a_2$. Hence, since $|a_2| \leq 2$, $|\gamma_1| \leq 1$.

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The inequality $|\gamma_n| \leq 1/n$ holds for functions f in $S^\star$, but is false for the full class S, even in order of magnitude. Indeed, (see Theorem 8.4 on p. 242 of [6]) there exists a bounded function $f \in S$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$.

In a recent paper [7] it is presented that the inequality $|\gamma_n| \leq 1/n$ holds also for close-to-convex functions. However, it is pointed out in [19] that there are some errors in the proof and, hence, the result is not substantiated. We will prove that actually the result is false for $n \geq 2$.

Following [10], $\mathrm{E}\mathscr{C}$ will denote the set of the extreme points of the closed convex hull of the class $\mathscr{C}$. Brickman, MacGregor and Wilken proved in [4] (see also [10, p. 56]) that

$$\mathrm{E}\mathscr{C} = \{f_{x,y} : x, y \in \mathbf{C},\ |x| = |y| = 1,\ x \neq y\}, \tag{1.1}$$

where

$$f_{x,y}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \qquad z \in \Delta. \tag{1.2}$$

Each element of $\mathrm{E}\mathscr{C}$ belongs to $\mathscr{C}$ and it maps Δ onto the complement of a half-line. It is also known that the set $\mathrm{E}\mathscr{C}$ coincides with the set of support points of $\mathscr{C}$ [9], [13] (see also [10, p. 98–100]).

Now we can state our first result.

Theorem 1. *If $n \geq 2$ there exists a function f in $\mathrm{E}\mathscr{C}$ (and, hence, f in $\mathscr{C}$) with*

$$\log \frac{f(z)}{z} = 2 \sum_{j=1}^{\infty} \gamma_j z^j$$

such that $|\gamma_n| > 1/n$.

The relevance of the logarithmic coefficients comes from the fact that, by means of the so called Lebedev–Milin inequalities ([6, p. 142–146], [16, Chapter 2]), estimates on the logarithmic coefficients γ_j of f can be transferred to bounds on the coefficients of f and related functions. Milin conjectured the inequality

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0, \qquad n = 1, 2, \ldots, \tag{1.3}$$

which implies Robertson's conjecture and, hence, Bieberbach's conjecture. L. de Branges [3] (see also [11]) proved (1.3) and thus established the Bieberbach conjecture. In Section 2, we shall draw our attention to another conjecture of Milin relative to the logarithmic coefficients.

For an arbitrary function g, analytic in Δ, we shall set

$$M(r,g)=\max_{|z|=r}|g(z)|,\qquad 0<r<1. \tag{1.4}$$

Lebedev [15] (see also [16, p. 55–57]) proved that if f in S has the logarithmic coefficients $\{\gamma_j\}_{j=1}^{\infty}$ then

$$\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j}\le \frac{1}{2}\log\frac{M(r,f)}{r},\qquad 0<r<1. \tag{1.5}$$

I.M. Milin conjectured in [17] that the right hand side of (1.5) can be changed to $\frac{1}{2}\log(M(r^2,f)/r^2)$, that is, that the inequality

$$\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j}\le \frac{1}{2}\log\frac{M(r^2,f)}{r^2}, \tag{1.6}$$

should hold for arbitrary f in S and $0<r<1$.

Milin [17] proved that (1.6) holds if $f\in S^\star$ and $0<r<1$ and that given f in S there exists r_f, with $0<r_f<1$, such that (1.6) holds for f and $0<r<r_f$. Using different ideas, Milin proved in [18] that (1.6) also holds for $0<r<1$ if f belongs to a certain subclass of those f in S that have real coefficients and are such that both f and f' have a continuous extension to the closed unit disc.

Let D be a domain in $\mathbf{C}$ with $0\in D$. We shall say that D is circularly symmetric if, for every R with $0<R<\infty$, $D\cap\{|z|=R\}$, is either empty, is the whole circle $|z|=R$, or is a single arc on $|z|=R$ which contains $z=R$ and is symmetric with respect to the real axis. Following [14], we shall denote by Y the class of those functions f in S which map Δ onto a circularly symmetric domain. The elements of Y will be called circularly symmetric functions. Our first result in Section 2 will be observing that the method of Milin [18] can be used to prove that (1.6) holds for $0<r<1$ if f belongs to either $S^\star$ or Y.

Theorem 2. *Suppose that $f\in S^\star\cup Y$ and that f has logarithmic coefficients $\{\gamma_j\}_{j=1}^{\infty}$. Then* (1.6) *holds for* $0<r<1$.

We can also prove the following.

Theorem 3. *Suppose that $f\in \mathrm{EC}\mathscr{C}$ and that f has logarithmic coefficients $\{\gamma_j\}_{j=1}^{\infty}$. Then* (1.6) *holds for* $0<r<1$.

In Section 4 we shall prove that the results obtained in Section 3 can be used to obtain upper bounds on the integral means of f. In particular, we can prove the following result.

Theorem 4. *Let f belong to any of the classes $S^\star$, Y or $\mathrm{EC}\mathscr{C}$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \le M(r^2,f)^{1/2}, \qquad 0<r<1. \tag{1.7}$$

We remark that, by the distortion theorem, the inequality (1.7) is stronger than the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \le \frac{r}{1-r^2}, \qquad 0<r<1,$$

which holds for every f in S [2] (see also [6, Chapter 7]).

2. Logarithmic coefficients of close-to-convex functions

Before embarking on the proof of Theorem 1, let us fix some notation. Given x,y in $\mathbf{C}$ with $|x|=|y|=1$ and $x\neq y$, we shall denote the logarithmic coefficients of $f_{x,y}$ by $\{\gamma_j(x,y)\}_{j=1}^\infty$, that is, we set

$$\log\frac{f_{x,y}(z)}{z} = 2\sum_{j=1}^\infty \gamma_j(x,y)z^n, \qquad z\in\Delta. \tag{2.1}$$

Also, we shall write $\gamma_j(x)$ for $\gamma_j(x,1)$, $j=1,2,\dots$, that is

$$\log\frac{f_{x,1}(z)}{z} = 2\sum_{j=1}^\infty \gamma_j(x)z^n, \qquad z\in\Delta. \tag{2.2}$$

Notice that if $|x|=|y|=1$ and $x\neq y$, we have that $f_{x,y}(z)=y^{-1}f_{xy^{-1},1}(yz)$ and, hence,

$$\gamma_j(x,y)=\gamma_j(xy^{-1})y^j, \qquad j=1,2,\dots\,. \tag{2.3}$$

Proof of Theorem 1. Take x in $\mathbf{C}$ with $|x|=1$ and $x\neq 1$ then

$$f_{x,1}(z)=\frac{z-\frac12(x+1)z^2}{(1-z)^2}, \qquad z\in\Delta.$$

Then, if we set

$$b=b(x)=\tfrac12(x+1),$$

we have that $|b-\frac12|=\frac12$ and $b\neq 1$. Hence, b can be written in the form

$$b=e^{i\theta}\cos\theta \qquad \text{with} \qquad |\theta|\le\tfrac12\pi,\ \theta\neq 0. \tag{2.4}$$

Now,

$$\log \frac{f_{x,1}(z)}{z} = \log \frac{1-bz}{(1-z)^2} = 2\sum_{j=1}^{\infty} \frac{1}{j}\left(1 - \frac{1}{2}b^j\right) z^j.$$

That is, we have that

$$\gamma_j(x) = \frac{1}{j}\left(1 - \frac{1}{2}b^j\right), \qquad j = 1, 2, \dots . \tag{2.5}$$

Then, using (2.4), we obtain

$$\begin{aligned} |\gamma_j(x)|^2 &= \frac{1}{j^2}\left(1 + \frac{1}{4}|b|^{2j} - \operatorname{Re} b^j\right) \\ &= \frac{1}{j^2}\left(1 + \frac{1}{4}\cos^{2j}\theta - \cos^j\theta\cos j\theta\right), \qquad j = 1, 2, \dots . \end{aligned} \tag{2.6}$$

Given $n \geq 2$ if, with the above notation, we take x so that $\theta = \pi/2n$ we have that

$$|\gamma_n(x)|^2 = \frac{1}{n^2}\left(1 + \frac{1}{4}\cos^{2n}\frac{\pi}{2n}\right)$$

and, hence $|\gamma_n(x)| > 1/n$. This finishes the proof.

Remark 1. Using (2.5) and having in mind that $|b - \frac{1}{2}| = \frac{1}{2}$, we easily see that for every x with $|x| = 1$ and $x \neq 1$ we have

$$|\gamma_j(x)| \leq \frac{3}{2}\frac{1}{j}, \qquad j = 1, 2, \dots .$$

This and (2.3) implies that

$$|\gamma_j(x,y)| \leq \frac{3}{2}\frac{1}{j}, \quad j = 1, 2, \dots, \quad \text{for every } x, y \text{ with } |x| = |y| = 1 \text{ and } x \neq y.$$

That is, we have proved the following result.

Proposition 1. *If $f \in \mathrm{EC}\mathscr{C}$ then its logarithmic coefficients γ_j satisfy*

$$|\gamma_j| \leq \frac{3}{2}\frac{1}{j}, \qquad j = 1, 2, \dots . \tag{2.7}$$

Remark 2. Since the functional J defined by $J(f) = \gamma_n$ where

$$\log \frac{f(z)}{z} = 2\sum_{j=1}^{\infty} \gamma_j z^j$$

is not linear, Proposition 1 does not imply that (2.7) should hold for arbitrary f in $\mathscr{C}$. In fact, it is an open question whether or not given f in $\mathscr{C}$ its logarithmic coefficients γ_j satisfy $\gamma_j = O(1/j)$, as $j \to \infty$.

3. Some results related to a conjecture of Milin

We start by introducing some notation. Given a function Ψ, analytic in Δ, we set

$$\sigma(r,\Psi)=\frac{1}{\pi}\iint_{|z|<r}|\Psi'(z)|^2\,dx\,dy,\qquad 0<r<1. \tag{3.1}$$

Notice that if $\Psi(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ then

$$\sigma(r,\Psi)=\sum_{n=1}^{\infty}n|\alpha_n|^2r^{2n},\qquad 0<r<1. \tag{3.2}$$

Using (3.2), we easily see that if f in S has the logarithmic coefficients $\{\gamma_j\}_{j=1}^{\infty}$ then

$$\sum_{j=1}^{\infty}j|\gamma_j|^2r^{2j}=\frac{1}{4}\sigma\left(r,\log\frac{f(z)}{z}\right),\qquad 0<r<1.$$

Hence (1.6) is equivalent to

$$\sigma\left(r,\log\frac{f(z)}{z}\right)\le 2\log\frac{M(r^2,f)}{r^2}. \tag{3.3}$$

As we mentioned above, Milin proved in [18, Theorem 2] the validity of (1.6) for $0<r<1$ if f belongs to a certain subclass of those functions in S with real coefficients. The following simple lemma was a key ingredient in the proof of this result.

Lemma A (Milin, [18, p. 136]). *Let Ψ be a function which is analytic in Δ, $u=\operatorname{Re}\Psi$ and $v=\operatorname{Im}\Psi$. If $r\in(0,1)$ and $r_1,r_2\in(0,1)$ and are such that $r_1r_2=r^2$ then*

$$\sigma(r,\Psi)=\frac{1}{\pi}\int_0^{2\pi}u(r_1e^{i\theta})\frac{\partial v(r_2e^{i\theta})}{\partial\theta}\,d\theta=-\frac{1}{\pi}\int_0^{2\pi}v(r_1e^{i\theta})\frac{\partial u(r_2e^{i\theta})}{\partial\theta}\,d\theta.$$

Let us notice that Lemma A can be easily proved writing u and v in terms of the Taylor series expansion of Ψ. Our proof of Theorem 2 will also be based on Lemma A. Hence, in a unified way, we shall obtain

(i) a new proof of the validity of (1.6) for $0<r<1$ if $f\in S^\star$,

and,

(ii) an extension of Theorem 2 of [18] to the class Y of circularly symmetric functions.

In the following lemma we point out a certain property which is satisfied both for the elements of $S^\star$ and those of Y and which will play a key role in the proof of Theorem 2.

Lemma 1. *Suppose that $f \in S^\star \cup Y$ and that $0 < r_1 \le r_2 < 1$. Then the curve with parametrization*

$$|f(r_1 e^{i\theta})| e^{i \arg f(r_2 e^{i\theta})}, \qquad 0 \le \theta \le 2\pi,$$

is a Jordan curve.

Proof. Let $0 < r_1 \le r_2 < 1$ and f in $S^\star \cup Y$. For simplicity, set

$$\Gamma(\theta) = |f(r_1 e^{i\theta})| e^{i \arg f(r_2 e^{i\theta})}, \qquad 0 \le \theta \le 2\pi. \tag{3.4}$$

Notice that

$$\Gamma(\theta) = f(r_2 e^{i\theta}) \frac{|f(r_1 e^{i\theta})|}{|f(r_2 e^{i\theta})|}.$$

Hence it is clear that Γ is the parametrization of a closed curve. Consequently, it suffices to prove that Γ is injective in $[0, 2\pi)$.

Suppose first that $f \in S^\star$. Then (see [6, p. 41–42]) $\arg f(r_2 e^{i\theta})$ is a strictly increasing function of θ in $[0, 2\pi]$ and the difference between its value at 2π and its value at 0 is 2π. Then it is clear that the function $\theta \mapsto e^{i \arg f(r_2 e^{i\theta})}$ is injective in $[0, 2\pi)$. Clearly, this implies that Γ is injective in $[0, 2\pi)$.

On the other hand, if $f \in Y$ then (see [14]), unless $f(z) \equiv z$ in which case the result is obvious, the function $\theta \mapsto |f(r_1 e^{i\theta})|$ is strictly decreasing in $[0, \pi]$ and strictly increasing in $[\pi, 2\pi]$. This implies that Γ is injective in $[0, \pi]$ and in $[\pi, 2\pi]$. This and the fact that $\operatorname{Im} f(z) > 0$ if $\operatorname{Im} z > 0$ and $\operatorname{Im} f(z) < 0$ if $\operatorname{Im} z < 0$ easily imply that Γ is injective in $[0, 2\pi)$. This finishes the proof.

Proof of Theorem 2. Suppose that $f \in S^\star \cup Y$ and that f has the logarithmic coefficients $\{\gamma_j\}_{j=1}^{\infty}$. Let (R, Φ) be the polar coordinates in the w-plane (i.e., in the image plane). If $r \in (0,1)$ and $\varepsilon \in (0,1)$ we set

$$r_1 = r^{2-\varepsilon}, \qquad r_2 = r^{\varepsilon}, \tag{3.5}$$

so that $r_1 r_2 = r^2$. For simplicity, set

$$R_j(\theta) = |f(r_j e^{i\theta})|, \qquad \Phi(\theta) = \arg f(r_j e^{i\theta}), \qquad 0 \le \theta \le 2\pi,\ j = 1, 2. \tag{3.6}$$

Let Γ be the Jordan curve considered in Lemma 1. Using Lemma A and having in mind that $\int_0^{2\pi} (\log(R_1(\theta)/r_1))\, d\theta = 0$, we see that

$$\begin{aligned} \sigma\left(r, \log \frac{f(z)}{z}\right) &= \frac{1}{\pi} \int_0^{2\pi} \left(\log \frac{R_1(\theta)}{r_1}\right) \frac{\partial\big(\Phi_2(\theta) - \theta\big)}{\partial\theta}\, d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} \left(\log \frac{R_1(\theta)}{r_1}\right) \frac{\partial \Phi_2(\theta)}{\partial\theta}\, d\theta = \frac{1}{\pi} \int_\Gamma \log \frac{R}{r_1}\, d\Phi. \end{aligned} \tag{3.7}$$

Let L be the circle $|w| = M(r_1, f)$. Clearly, Γ lives in the closed disc $\{|w| \le M(r_1, f)\}$. Let G be the region bounded by Γ and L. By Green's theorem, we have

$$\iint_G \frac{dR\,d\Phi}{R} = \int_L \log\left(\frac{R}{r_1}\right) d\Phi - \int_\Gamma \log\left(\frac{R}{r_1}\right) d\Phi.$$

Then, using (3.7) and the definition of r_1, we deduce that

$$\begin{aligned}\sigma\left(r, \log\frac{f(z)}{z}\right) &= 2\log\frac{M(r_1, f)}{r_1} - \frac{1}{\pi}\iint_G \frac{dR\,d\Phi}{R} \\ &\le 2\log\frac{M(r_1, f)}{r_1} = 2\log\frac{M(r^{2-\varepsilon}, f)}{r^{2-\varepsilon}}.\end{aligned}$$

Since ε is arbitrary subject to $0 < \varepsilon < 1$, letting ε tend to zero we obtain (3.3) and, hence, (1.6). This finishes the proof.

Remark 3. The argument used in the proof of Theorem 2 can be used to give a short proof of the above mentioned result of Lebedev which asserts that (1.5) holds for arbitrary f in S (compare [16, p. 55–57]). Indeed, let f be in S and $0 < r < 1$. Argue as in the proof of Theorem 2 but taking $r_1 = r_2 = r$. Then we obtain

$$\iint_G \frac{dR\,d\Phi}{R} = 2\pi\log\frac{M(r, f)}{r} - \pi\sigma\left(r, \log\frac{f(z)}{z}\right)$$

and, hence,

$$\sigma\left(r, \log\frac{f(z)}{z}\right) \le 2\log\frac{M(r, f)}{r}.$$

This gives (1.5).

Proof of Theorem 3. In view of the facts that we stated at the beginning of the proof of Theorem 1, it is clear that Theorem 3 is equivalent to the following result.

Proposition 2. *Let b belong to* $\mathbf{C}$ *and suppose that $|b - \frac{1}{2}| = \frac{1}{2}$ and $b \ne 1$. If*

$$f(z) = \frac{z - bz^2}{(1 - z)^2} \qquad \text{and} \qquad \log\frac{f(z)}{z} = 2\sum_{j=1}^{\infty} \gamma_j z^j, \qquad z \in \Delta,$$

then

$$\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} \le \frac{1}{2}\log\frac{M(r^2, f)}{r^2}, \qquad 0 < r < 1.$$

Proof. We recall from (2.5) that

$$\gamma_j = \frac{1}{j}\left(1 - \frac{1}{2}b^j\right), \qquad j = 1, 2, \dots .$$

Then, for $0 < r < 1$,

$$\begin{aligned}\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} &= \sum_{j=1}^{\infty} \frac{1}{j}\left(1-\frac{1}{2}b^j\right)\left(1-\frac{1}{2}\bar{b}^j\right) r^{2j} \\ &= \sum_{j=1}^{\infty} \frac{1}{j} r^{2j} + \frac{1}{4}\sum_{j=1}^{\infty}\frac{1}{j}|b|^{2j} r^{2j} - \operatorname{Re}\left(\sum_{j=1}^{\infty}\frac{1}{j} b^j r^{2j}\right) \\ &= \log\frac{1}{1-r^2} + \frac{1}{4}\log\frac{1}{1-|b|^2 r^2} - \log\frac{1}{|1-br^2|}.\end{aligned} \tag{3.8}$$

We recall from (2.4) that b can be written in the form $b = e^{i\theta}\cos\theta$ with $\theta \in \mathbf{R}$ and then

$$\begin{aligned}|1-br^2|^2 &= 1 + |b|^2 r^4 - 2\operatorname{Re}(br^2) = 1 + r^4\cos^2\theta - 2r^2\cos^2\theta \\ &\le 1 - r^2\cos^2\theta = 1 - |b|^2 r^2\end{aligned}$$

and, hence

$$\log\frac{1}{1-|b|^2 r^2} \le 2\log\frac{1}{|1-br^2|},$$

which implies

$$\frac{1}{4}\log\frac{1}{1-|b|^2 r^2} - \log\frac{1}{|1-br^2|} \le -\frac{1}{2}\log\frac{1}{|1-br^2|}.$$

Then, using (3.8), we obtain

$$\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} \le \log\frac{1}{1-r^2} - \frac{1}{2}\log\frac{1}{|1-br^2|}. \tag{3.9}$$

On the other hand,

$$\frac{1}{2}\log\left|\frac{f(r^2)}{r^2}\right| = \frac{1}{2}\log\frac{|1-br^2|}{(1-r^2)^2} = \log\frac{1}{1-r^2} - \frac{1}{2}\log\frac{1}{|1-br^2|},$$

which, with (3.9), gives

$$\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} \le \frac{1}{2}\log\left|\frac{f(r^2)}{r^2}\right| \le \frac{1}{2}\log\frac{M(r^2,f)}{r^2}.$$

This finishes the proof of Proposition 2. Hence, Theorem 3 is proved.

Remark 4. Just as in Remark 2, even though Theorem 3 is true, it is an open question whether or not (1.6) holds for arbitrary f in $\mathscr{C}$ and $0 < r < 1$.

Remark 5. There are some other subclasses of S for which Milin's conjecture (1.6) can be proved easily. For instance, we can prove the following result.

Proposition 3. *Suppose that $f \in S$ and that the logarithmic coefficients $\{\gamma_j\}$ of f are real and satisfy $0 \le \gamma_j \le 1/j$ for all j. Then (1.6) holds for $0 < r < 1$.*

Proof. For simplicity, set $g(z) = \log\bigl(f(z)/z\bigr)$. Then

$$\begin{aligned}4\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} &= \frac{1}{\pi}\iint_{|z|<r} |g'(z)|^2\,dx\,dy \\ &= \frac{1}{\pi}\int_0^r\int_0^{2\pi} \rho|g'(\rho e^{i\theta})|^2\,d\theta\,d\rho = 2\int_0^r \rho I_2(\rho, g')\,d\rho,\end{aligned} \tag{3.10}$$

where, for h analytic in Δ,

$$I_2(\rho, h) = \frac{1}{2\pi}\int_0^{2\pi} |h(\rho e^{i\theta})|^2\,d\theta.$$

Then, we have

$$\begin{aligned}\frac{d}{dr}\left(4\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} - 2\log\frac{f(r^2)}{r^2}\right) &= \frac{d}{dr}\left(4\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} - 2g(r^2)\right) \\ &= 2rI_2(r, g') - 4rg'(r^2) \\ &= 2r\bigl(I_2(r, g') - 2g'(r^2)\bigr).\end{aligned} \tag{3.11}$$

Since $0 \le \gamma_j \le 1/j$, then $0 \le j^2\gamma_j^2 \le j\gamma_j$, which implies

$$I_2(r, g') = 4\sum_{j=1}^{\infty} j^2\gamma_j^2 r^{2(j-1)} \le 4\sum_{j=1}^{\infty} j\gamma_j r^{2(j-1)} = 2g'(r^2).$$

This and (3.11) show that

$$\frac{d}{dr}\left(4\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} - 2\log\frac{f(r^2)}{r^2}\right) \le 0,$$

whenever $r \in (0, 1)$ and, hence the function

$$r \mapsto \left(4\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} - 2\log\frac{f(r^2)}{r^2}\right)$$

is decreasing in $[0,1)$. Since the value of this function at $r=0$ is 0, it follows that

$$4\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} \le 2\log\frac{f(r^2)}{r^2}$$

whenever $r\in(0,1)$. Hence, the proposition is proved.

We remark that the author proved in [8] that there exist functions f in Y and satisfying $|\gamma_n|>1/n$ for some n. Hence, the class Y is not contained in the class of those f considered in Proposition 3.

4. Bounds for the integral means

In this section we shall show that the estimates on $\sigma\big(r,\log\big(f(z)/z\big)\big)$ obtained in Section 3 can be applied to obtain upper bounds for the integral means of f. Our results will be obtained using the following well-known inequality of Lebedev and Milin (see [6, p. 143] or [16, Theorem 2.3]).

First Lebedev–Milin inequality. *Let* $F(z)=\sum_{j=1}^{\infty}A_j z^j$ *be a function which is analytic in* Δ *with* $F(0)=0$ *and write*

$$G(z)=\exp\big(F(z)\big)=\sum_{k=0}^{\infty}D_k z^k, \qquad z\in\Delta.$$

Then,

$$\sum_{k=0}^{\infty}|D_k|^2 \le \exp\bigg(\sum_{k=1}^{\infty}k|A_k|^2\bigg). \tag{4.1}$$

Notice that the right hand side of (4.1) is

$$\|G\|_{H^2}^2=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\theta})|^2\,d\theta,$$

(see e.g. [5]) and hence (4.1) can be written as

$$\|G\|_{H^2}^2 \le \exp\bigg(\sum_{k=1}^{\infty}k|A_k|^2\bigg). \tag{4.2}$$

Now we can prove Theorem 4.

Proof of Theorem 4. Take a function f which belongs to $S^\star \cup Y \cup \mathrm{EC}\mathscr{C}$ and let $\{\gamma_j\}$ be its logarithmic coefficients. Take $0 < r < 1$. Theorem 2 and Theorem 3 imply that

$$\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k} \le \frac{1}{2}\log\frac{M(r^2,f)}{r^2}. \tag{4.3}$$

Set

$$F(z) = \frac{1}{2}\log\frac{f(rz)}{rz} = \sum_{k=1}^{\infty}\gamma_k r^k z^k, \quad G(z) = \exp\bigl(F(z)\bigr) = \left(\frac{f(rz)}{rz}\right)^{1/2}, \quad z \in \Delta.$$

Then F and G satisfy the conditions of the first Lebedev–Milin inequality. Consequently, writing (4.2) for this choice of F and G, we obtain

$$\|G\|_{H^2}^2 \le \exp\left(\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k}\right). \tag{4.4}$$

Now

$$\|G\|_{H^2}^2 = \frac{1}{2\pi}\int_0^{2\pi}|G(e^{i\theta})|^2\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(re^{i\theta})}{r}\right|\,d\theta.$$

Then (4.3) and (4.4) give

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(re^{i\theta})}{r}\right|\,d\theta &\le \exp\left(\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k}\right)\\ &\le \exp\left(\frac{1}{2}\log\frac{M(r^2,f)}{r^2}\right) = \left(\frac{M(r^2,f)}{r^2}\right)^{1/2},\end{aligned}$$

which clearly implies that

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \le M(r^2,f)^{1/2}.$$

This finishes the proof.

Remark 6. It is an open question whether or not the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \le M(r^2,f)^{1/2}$$

holds for arbitrary f in S and $0 < r < 1$. However, we remark that the proof of Theorem 4 shows that this is true whenever (1.6) is true.

Remark 7. If f is an arbitrary element of the class S and $0<r<1$ then if we argue as in the proof of Theorem 4 but using Lebedev's estimate (1.5) instead of (1.6), we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \le \big(rM(r,f)\big)^{1/2}, \qquad 0<r<1,\ f\in S.$$

Remark 8. Andreev and Duren [1, p. 722] proved as a consequence of de Branges' theorem that if f in S has logarithmic coefficients $\{\gamma_j\}$ then

$$\sum_{j=1}^{\infty} j|\gamma_j|^2 r^{2j} \le \sum_{j=1}^{\infty}\frac{1}{j}r^{2j} = \log\frac{1}{1-r^2}, \qquad 0<r<1. \tag{4.5}$$

Using this inequality and the argument of the proof of Theorem 4 we obtain Baernstein's result

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \le \frac{r}{1-r^2}, \qquad 0<r<1. \tag{4.6}$$

We should mention that Holland [12] proved (4.6) as a consequence of the truth of Robertson's conjecture.

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