Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 25, 2000, 439–456

AUTOMORPHISM GROUPS OF ORIENTABLE ELLIPTIC-HYPERELLIPTIC KLEIN SURFACES

Beatriz Estrada

Universidad Nacional de Ecucacion Distancia, Depto de Matemáticas Fundamentales
c/. Senda del Rey 9, E-28040 Madrid, Spain; bestra@mat.uned.es

Abstract. A compact Klein surface X is a compact surface with a dianalytic structure. Such a surface is said to be elliptic-hyperelliptic if it admits an involution ϕ, that is an order two automorphism, such that $X/\langle\phi\rangle$ has algebraic genus 1. Klein surfaces can be seen as quotients of the hyperbolic plane by the action of NEC groups, and their automorphism groups as quotients of NEC groups. Using this, we determine the full automorphism groups of orientable elliptic-hyperelliptic Klein surfaces of algebraic genus $p > 5$.

1. Introduction

Klein surfaces, introduced from a modern point of view by Alling and Greenleaf [1], are surfaces endowed with a dianalytic structure. A compact Klein surface is said to be *elliptic-hyperelliptic* if it admits an involution ϕ, that is an order two automorphism, such that $X/\langle\phi\rangle$ has algebraic genus 1.

Non-Euclidean crystallographic groups (NEC groups in short) where introduced by Wilkie and Macbeath, and they are an important tool in the study of Klein surfaces since the results of Preston and May. Klein surfaces can be seen as quotients of the hyperbolic plane under the action of an NEC group, and the automorphism groups of such surfaces as quotients of NEC groups; hence the relevance of the work about normal subgroups of NEC groups in [2], [3] and [9].

In [6] elliptic-hyperelliptic Klein surfaces (EHKS in short) were characterized by means of NEC groups. In this paper we determine all groups that are the automorphism group of an orientable EHKS of algebraic genus $p > 5$. Similar studies have been made for hyperelliptic and cyclic-trigonal Klein surfaces (see [5], [8]).

2. Preliminaries on NEC groups

Let $\mathscr{D}$ denote the hyperbolic plane and $\mathscr{G}$ its group of isometries. A *non-Euclidean crystallographic group* Γ, is a discrete subgroup of $\mathscr{G}$ with compact quotient $X = \mathscr{D}/\Gamma$. NEC groups were introduced by Wilkie [14], and Macbeath [10]

1991 Mathematics Subject Classification: Primary 30F50, 20H10; Secondary 14J50.
Partially supported by CICYTPB95-0017.

associated to each NEC group a signature that determines its algebraic structure and has the following form:

$$\sigma(\Gamma) = \big(g, \pm, [m_1, \ldots, m_r], \{(n_{i1}, \ldots, n_{is_i}),\ i = 1, \ldots, k\}\big), \tag{2.1}$$

where g, m_i, n_{ij} are integers verifying $g \geq 0$, $m_i \geq 2$, $n_{ij} \geq 2$; g is the topological genus of X. The sign determines the orientability of X. The numbers m_i are the *proper periods* corresponding to cone points in X. The brackets $(n_{i1}, \ldots, n_{is_i})$ are the *period-cycles*. The number k of period-cycles is equal to the number of boundary components of X. Numbers n_{ij} are the periods of the period-cycle $(n_{i1}, \ldots, n_{is_i})$ also called *link-periods*, corresponding to corner points in the boundary of X. The number $p = \alpha g + k - 1$, where $\alpha = 1$ or 2 if the sign of $\sigma(\Gamma)$ is $-$ or $+$, respectively, is called the *algebraic genus* of X.

An NEC group Γ with signature (2.1) has the following presentation [10]:
Generators:

$$\begin{aligned}
&x_i, && i = 1, \ldots, r;\\
&e_i, && i = 1, \ldots, k;\\
&c_{ij}, && i = 1, \ldots, k,\ j = 0, \ldots, s_i;\\
&a_i,\ b_i, && i = 1, \ldots, g,\ \text{(if } \sigma \text{ has the sign } +);\\
&d_i, && i = 1, \ldots, g \ \text{(if } \sigma \text{ has the sign } -).
\end{aligned}$$

Relations:

$$\begin{aligned}
&{x_i}^{m_i}, \qquad i = 1, \ldots, r;\\
&{c_{ij-1}}^2 = {c_{ij}}^2 = (c_{ij-1}c_{ij})^{n_{ij}}, \qquad i = 1, \ldots, k;\ j = 1, \ldots, s_i;\\
&{e_i}^{-1} c_{i0} e_i c_{is_i} = 1, \qquad i = 1, \ldots, k;\\
&x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \qquad \text{(if } \sigma \text{ has the sign } +);\\
&x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1 \qquad \text{(if } \sigma \text{ has the sign } -).
\end{aligned}$$

By [10] cyclic permutations of periods in period-cycles or arbitrary permutations of proper periods in the signature of an NEC group Γ, lead to a signature σ' corresponding to an NEC group isomorphic to Γ. Every NEC group Γ with signature (2.1) has associated to it a fundamental region whose area $\mu(\Gamma)$, called the *area of the group* (see [13]), is

$$\mu(\Gamma) = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right). \tag{2.2}$$

An NEC group with signature (2.1) actually exists if and only if the right-hand side of (2.2) is greater than 0 (see [15]). If Γ is a subgroup of an NEC group Γ' of finite index N, then also Γ is an NEC group and the following Riemann–Hurwitz formula holds

$$\mu(\Gamma) = N\mu(\Gamma'). \tag{2.3}$$

Let X be a Klein surface of topological genus g having k boundary components. Then, by [12], there exists an NEC group Γ with signature

$$\big(g; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\}\big), \tag{2.4}$$

such that $X = \mathscr{D}/\Gamma$, where the sign is "$+$" if X is orientable and "$-$" if not. An NEC group with signature (2.4) is called a *surface group*.

May [11] proved that if G is a group of automorphisms of a surface $X = \mathscr{D}/\Gamma$ of algebraic genus $p \geq 2$, then G can be presented as a quotient Γ'/Γ for some NEC group Γ'. The *full group of automorphisms* of X is $\mathrm{Aut}(X) = N_{\mathscr{G}}(\Gamma)/\Gamma$, where $N_{\mathscr{G}}(\Gamma)$ is the normalizer of Γ in the group $\mathscr{G}$ of isometries of $\mathscr{D}$.

In order to decide if a group G of automorphisms of a Klein surface X equals $\mathrm{Aut}(X)$, the concept of maximality we are going to expose now is very useful.

An NEC group is said to be *maximal* if there does not exist another NEC group containing it properly. In particular if Γ' is maximal (following the notation above) then $G = \mathrm{Aut}(X)$.

An NEC signature τ is said to be *maximal* if for any NEC group Γ with signature τ and for every NEC group Γ' containing Γ, the equality $d(\Gamma) = d(\Gamma')$ (dimensions of the associated Teichmüller spaces [8]) implies $\Gamma = \Gamma'$.

Let σ and σ' be NEC signatures of two NEC groups Γ and Γ', respectively. We will say that $(\sigma,\ \sigma')$ is a *normal pair*, and we write $\sigma \triangleleft \sigma'$, if $\Gamma \triangleleft \Gamma'$ and $d(\Gamma) = d(\Gamma')$. The pair is said to be *proper* if σ' has period-cycles. The list of normal proper pairs can be seen in [4].

The following results from [8] will be very useful.

Theorem 2.1. *Given a maximal NEC signature τ there exists a maximal NEC group Γ with signature τ.*

Theorem 2.2. *Let Λ be an NEC group containing a surface group Γ as a normal subgroup. If $\sigma(\Lambda^+)$ is maximal, the topological surface $\mathscr{D}/\Gamma$ can be endowed with a structure of a Klein surface such that $\mathrm{Aut}(\mathscr{D}/\Gamma) = \Lambda/\Gamma$, where Λ^+ is the normal subgroup of Λ of orientation preserving transformations.*

3. Characterization of EHKS in terms of NEC groups

Definition 3.1. Let X be a Klein surface of algebraic genus $p \geq 2$. We say that X is an *elliptic-hyperelliptic* Klein surface (EHKS in short) if it admits an involution ϕ (an automorphism of order 2), such that $X/\langle\phi\rangle$ has algebraic genus 1.

In the sequel Klein surface will mean compact bordered Klein surface. Now we give some results about EHKS and NEC groups obtained in [6].

Proposition 3.2. *Let $X = \mathscr{D}/\Gamma$ be a bordered Klein surface of algebraic genus $p \geq 2$. X is an EHKS if and only if there exists an NEC group Γ_1 of algebraic genus 1 such that $[\Gamma_1 : \Gamma] = 2$.*

The group Γ_1 is called *the group of the EH character* of X, and $\Gamma_1/\Gamma = \langle\phi\rangle$ the group generated by ϕ.

Proposition 3.3. *If X is an EHKS of algebraic genus $p > 5$, ϕ is unique and central in the full group of automorphisms of X. We will call it the* EH-involution.

Theorem 3.4. *Let $X = \mathscr{D}/\Gamma$ be an orientable bordered EHKS of topological genus g and k boundary components ($p \geq 2$), and let Γ_1 be the group of the EH character of X. Then Γ_1 has one of the following signatures:*

(i) *If $g = 0$, $\big(0; +; [-], \{(2, \overset{2k-4}{\ldots}, 2), (-)\}\big)$.*

(ii) *If $g = 1$, $\big(0; +; [-], \{(2, \overset{k-2}{\ldots}, 2), (-)\}\big)$ or $\big(0; +; [-], \{(2, \overset{s}{\ldots}, 2), (2, \overset{2k-s}{\ldots}, 2)\}\big)$, $s > 0$, s even, or $\big(0; +; [-], \{(2, \overset{2k}{\ldots}, 2)\}\big)$.*

(iii) *If $g \geq 0$, $2 \leq k \leq 4$, $\big(0; +; [2, \overset{p-1}{\ldots}, 2], \{(-), (-)\}\big)$.*

Corollary 3.5. *An orientable EHKS must have one of the following topological types: topological genus 0 and more than 2 boundary components, topological genus 1 and at least 1 boundary component or topological genus greater than 2 and 2, 3 or 4 boundary components.*

Remark 3.6. In [6] it was also proved that if X is an orientable EHKS of algebraic genus $p > 5$, then $|\operatorname{Aut}(X)| \leq 4(p-1)$, except for two special cases:

$$\sigma(\Gamma) = \big(\tfrac{1}{2}(p-3), +, [-], (-)^4\big), \qquad \operatorname{Aut}(\mathscr{D}/\Gamma) = (D_{p-1} \times \mathbf{Z}_2) \propto \mathbf{Z}_2;$$
$$\sigma(\Gamma) = \big(\tfrac{1}{2}(p-1), +, [-], (-)^2\big), \qquad \operatorname{Aut}(\mathscr{D}/\Gamma) = D_{2(p-1)} \propto \mathbf{Z}_2.$$

4. Signatures associated to automorphism groups of EHKS

Let $X = \mathscr{D}/\Gamma$ be an orientable EHKS of topological genus g, k boundary components and algebraic genus $p > 5$; this condition gives unicity and centrality properties for the EH-involution. Let Γ_1 be the group of the EH character. If G is an automorphism group of X containing ϕ (the EH-involution), then $G = \Gamma'/\Gamma$ for a certain NEC group Γ' such that $\Gamma \triangleleft \Gamma_1 \triangleleft \Gamma'$. Let us suppose that $N = [\Gamma' : \Gamma_1]$, then the following three propositions determine a finite set of possible signatures of Γ', for each topological type:

Proposition 4.1. *If X has topological genus 0 then the signature of Γ' is one of the following:*

$$\tau_1 = \big(0, +, [-], \{(2, \overset{2(p-1)/N}{\ldots}, 2), (-)\}\big), \qquad N \text{ some divisor of } 2(p-1);$$
$$\tau_2 = \big(0, +, [-], \{(2, \overset{(2(p-1)/N)+4}{\ldots}, 2)\}\big), \qquad N \text{ some even divisor of } 2(p-1).$$

Proof. Let Γ' be an NEC group of signature of the general form

$$\sigma(\Gamma') = \big(g', \pm, [m_1, \dots, m_r], \{(n_{i1}, \dots, n_{is_i}), i = 1, \dots, k'\}\big) \tag{4.1.1}$$

and let $\theta_1 \colon \Gamma' \to \Gamma'/\Gamma_1$ be the canonical epimorphism such that $\operatorname{Ker}(\theta_1) = \Gamma_1$. Let $G_1 = \Gamma'/\Gamma_1$ and

l_i the order of $\theta_1(e_i)$ in G_1 for $i = 1, \dots, k'$;
p_i the order of $\theta_1(x_i)$ in G_1 for $i = 1, \dots, r$;
q_{ij} the order of $\theta_1(c_{ij-1}c_{ij})$ in G_1 for $i = 1, \dots, k'$, $j = 1, \dots, s_i$;
$n^l(i,j)$ the order of $\theta_1(c_{li-1}c_{lj+1})$ in G_1 for $i = 1, \dots, s_l$, $j = 1, \dots, s_l - 1$, $i \le j$.

Firstly we are looking for signatures for Γ' such that $\mu(\Gamma') = N\mu(\Gamma_1)$, where $\sigma(\Gamma_1) = (0; +; [-], \{(2^{2k-4}), (-)\})$ (see 3.4), and then we will try to obtain θ_1.

By the Riemann–Hurwitz formula

$$N\left(\alpha g' + k' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k'}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right) = \frac{k-2}{2}. \tag{4.1.2}$$

Period-cycles of Γ_1 come from those of Γ' having some reflection in Γ_1. Two possibilities appear ([2], [9]):

(i) If the two period-cycles of Γ_1 come from different period-cycles of Γ', namely C_1 and C_2, then $C_i = (2, \overset{s_i}{\dots}, 2)$, $i = 1, 2$. Let us suppose that all reflections of C_1 and C_2 are in Γ_1, then $Ns_1 = 2k - 4$, $s_2 = 0$, $l_1 = l_2 = N$ and from (4.1.2)

$$N\left(\alpha g' + k' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{4}\left(\frac{2k-4}{N}\right) + \frac{1}{2}\sum_{i=3}^{k'}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right) = \frac{k-2}{2};$$

hence $g' = 0$, $k' = 2$, $r = 0$ and $\sigma(\Gamma') = \big(0, +, [-], \{(2^{(2(p-1))/N}), (-)\}\big)$. The epimorphism θ_1 is defined by

$$\theta_1(e_1) = x, \quad \theta_1(e_2) = x^{-1}, \quad \theta_1(c_{ij}) = 1, \quad \text{for all } i, j, \text{ where } G_1 = \langle x : x^N = 1\rangle.$$

When not all reflections of C_1 or C_2 are in Γ_1, the area of Γ' increases and does not satisfy (4.1.2).

(ii) If the period-cycles of Γ_1 come from the same period-cycle of Γ', namely C_1, then by [9], N is even and $C_1 = (2, \overset{s_1}{\dots}, 2)$, where

$$s_1 \ge \left(\frac{2k-4}{N}\right) + 4,$$
$$n^1\left(1, \left(\frac{2k-4}{N}\right) + 1\right) = n^1\left(\frac{2k-4}{N} + 2, \frac{2k-4}{N} + 4\right) = \frac{N}{2},$$
$$\frac{2(p-1)}{N} \ge 2.$$

Thus, from (4.1.2) we have

$$N\left(\alpha g' + k' - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) + \frac{1}{4}\left(\frac{2k-4}{N}+4\right) + \frac{1}{2}\sum_{j=(2k-4)/N+5}^{s_1}\left(1-\frac{1}{n_{1j}}\right) + \frac{1}{2}\sum_{i=2}^{k'}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{ij}}\right)\right) = \frac{k-2}{2},$$

and therefore $g'=0$, $k'=1$, $r=0$ and $\sigma(\Gamma') = \big(0,+,[-],\big\{(2,\overset{(2(p-1)/N)+4}{\ldots},2)\big\}\big)$. The epimorphism θ_1 is defined by $\theta_1(e_1)=1$, $\theta_1(c_{10}) = \theta_1(c_{1,((2k-4)/N)+4}) = x$, $\theta_1(c_{1,((2k-4)/N)+2}) = y$, $\theta_1(c_{1j}) = 1$ for the remaining reflections, with $G_1 = \langle x,y : x^2=y^2=(xy)^{N/2}=1\rangle \simeq D_{N/2}$.

Proposition 4.2. *If X has topological genus 1 then the signature of Γ' is one of the following, with N some divisor of $2k$:*

If $\sigma(\Gamma_1) = \big(0,+,[-],\big\{(2,\overset{2k}{\ldots},2),(-)\big\}\big)$;

$\tau_3 = \big(0,+,[-],\big\{(2,\overset{2k/N}{\ldots},2),(-)\big\}\big)$;

$\tau_4 = \big(0,+,[-],\big\{(2,\overset{(2k/N)+4}{\ldots}2)\big\}\big)$, *$N$ even.*

If $\sigma(\Gamma_1) = \big(0,+,[-],\big\{(2,\overset{s}{\ldots},2),(2,\overset{2k-s}{\ldots},2)\big\}\big)$, $0<s<2k$, *s even,*

$\tau_5 = \big(0,+,[-],\big\{(2,\overset{(2k-s)/N}{\ldots},2),(2,\overset{s/N}{\ldots},2)\big\}\big)$, *$N|2k-s$ and $N|s$;*

$\tau_6 = \big(0,+,[-],\big\{(2,\overset{2k/N}{\ldots},2),(-)\big\}\big)$, $s=k$;

$\tau_7 = \big(0,+,[2,2],\big\{(2,\overset{2k/N}{\ldots},2)\big\}\big)$, $s=k$;

$\tau_8 = \big(1,-,[-],\big\{(2,\overset{2k/N}{\ldots},2)\big\}\big)$, $s=k$;

$\tau_9 = \big(0,+,[2],\big\{(2,\overset{(2k/N)+2}{\ldots},2)\big\}\big)$, $s=k$, $N\equiv 0 \pmod 4$;

$\tau_{10} = \big(0,+,[-],\big\{(2,\overset{(2k/N)+4}{\ldots},2)\big\}\big)$, *$N$ even.*

If $\sigma(\Gamma_1) = \big(1,-,[-],\big\{(2,\overset{2k}{\ldots},2)\big\}\big)$

$\tau_{11} = \big(0,+,[-],\big\{(2,\overset{2k/N}{\ldots},2),(-)\big\}\big)$, *$N$ even,*

$\tau_{12} = \big(1,-,[-],\big\{(2,\overset{2k/N}{\ldots},2)\big\}\big)$,

$\tau_{13} = \big(0,+,[2],\big\{(2,\overset{2k/N}{\ldots},2)\big\}\big)$, $N\equiv 2 \pmod 4$,

$\tau_{14} = \big(0,+,[-],\big\{(2,\overset{(2k/N)+4}{\ldots},2)\big\}\big)$, $N\equiv 0 \pmod 4$.

Proof. We will use the notation of the proof of Proposition 4.1.

Case 1. If $\sigma(\Gamma_1)=\big(0,+,[-],\{(2,\overset{2k}{\ldots},2),(-)\}\big)$. Then $\sigma(\Gamma')$ can have signature τ_3 or τ_4. These cases are completely analogous to τ_1 and τ_2 (in Proposition 4.1), respectively. The epimorphisms are:

– for τ_3, $\theta_1(e_1)=x_1$, $\theta_1(e_2)={x_1}^{-1}$, $\theta_1(c_{2,0})=1$, $\theta_1(c_{1,j})=1$, for $j=1,\dots,s_1$; with $G_1=\langle x : x^N=1\rangle$;

– for τ_4, we have $\theta_1(e_1)=1$, $\theta_1(c_{2,0})=1$, $\theta_1(c_{1,0})=\theta_1(c_{1,(2k/N)+4})=x$, $\theta_1(c_{1,(2k/N)+1})=y$, $\theta_1(c_{1,j})=1$ for the rest of reflections in C_1; with $G_1=\langle x : x^N=1\rangle$.

Case 2. If $\sigma(\Gamma_1)=\big(0,+,[-],\{(2,\overset{s}{\ldots},2),(2,\overset{2k-s}{\ldots},2)\}\big)$, $0<s<k$, s even, by the Riemann–Hurwitz formula we have

$$\alpha g'+k'-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\frac{1}{2}\sum_{i=1}^{k'}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{ij}}\right)=\frac{k}{2N}, \tag{4.2.1}$$

and we must consider the following cases:

(i) If the two period-cycles of Γ_1 come from different period-cycles C_1, C_2 in Γ', as in 4.1(i), all reflections of those period-cycles must be in Γ_1, and then $C_1=(2,\overset{s/N}{\ldots},2)$ and $C_2=(2,\overset{(2k-s)/N}{\ldots},2)$, $l_1=l_2=N$. This, together with (4.2.1) forces $g'=0$, $k'=2$ having $\sigma(\Gamma')=\tau_5=\big(0,+,[-],\{(2,\overset{s/N}{\ldots},2),(2,\overset{(2k-s)/N}{\ldots},2)\}\big)$.

The epimorphism θ_1 is defined by $\theta_1(e_1)=x_1$, $\theta_1(e_2)={x_1}^{-1}$, $\theta_1(c_{i,j})=1$, for $i=1,2$, $j=1,\dots,s_1$, where $G_1=\langle x : x^N=1\rangle$.

(ii) If the period-cycles of Γ_1 come from the same period-cycle of Γ', namely C_1, then N even (see [9]) and we have to consider $s=k$ or $s\neq k$.

When $s=k$ and all reflections in C_1 are in Γ_1 then $C_i=(2,\overset{s_i}{\ldots},2)$, $l_1=\frac{1}{2}N$, $\frac{1}{2}Ns_1=k$, and (4.1.2) becomes

$$\alpha g'+k'-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\frac{1}{4}\left(\frac{2k}{N}\right)+\frac{1}{2}\sum_{i=2}^{k'}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{ij}}\right)=\frac{k}{2N}. \tag{4.2.2}$$

Then Γ' can have one or two boundary components:

– If $k'=2$, then $r'=0$, $g'=0$, $s_2=0$, and we have $\sigma(\Gamma')=\tau_6=\big(0,+,[-],\{(2,\overset{2k/N}{\ldots},2),(-)\}\big)$. Here θ_1 is defined as

$$\theta_1(e_1)=x_1,\quad \theta_1(e_2)={x_1}^{-1},\quad \theta_1(c_{1,j})=1,\quad \text{for } j=1,\dots,s_1,\ \theta_1(c_{2,0})=y;$$

with $G_1=\langle x_1,y_1 : {x_1}^{N/2}={y_1}^2=[x_1,y_1]=1\rangle$. (The relation $[x_1,y_1]=1$ comes from $e_2c_{2,0}{e_2}^{-1}=c_{2,0}$ in Γ'.)

– If $k'=1$, then $g'=0$ or 1.

If $g'=0$, then by (4.2.2) $r'=2$, $m_1=2$, $m_2=2$, $p_1=2$, $p_2=2$, and $\sigma(\Gamma')=\tau_7=\big(0,+,[2,2],\{(2,\overset{2k/N}{\ldots},2)\}\big)$, with $\theta_1(x_1)=x_1$, $\theta_1(x_2)=y_1$,

$\theta_1(e_1) = y_1x_1$, $\theta_1(c_{1,j}) = 1$, for $j = 1,\dots,s_1$; and $G_1 = \langle x_1, y_1 : {x_1}^2 = {y_1}^2 = (y_1x_1)^{N/2} = 1\rangle$. (The choice of $\theta_1(e_1)$ comes from the relation $x_1x_2e_1 = 1$ in Γ').

If $g' = 1$, by (4.2.2), $r' = 0$ and $\sigma(\Gamma') = \tau_8 = \big(1, -, [-], \big\{(2, \overset{2k/N}{\dots}, 2)\big\}\big)$. Here

$$\theta_1(d_1) = x_1\theta_1(e_1) = {x_1}^{N-2}, \qquad \theta_1(c_{1,j}) = 1, \qquad \text{for } j = 1,\dots,s_1;$$

and $G_1 = \langle x_1 : {x_1}^N = 1\rangle$. (The choice of $\theta_1(e_1)$ comes from the relation ${d_1}^2e_1 = 1$ in Γ'.)

Now, when $k = s$ and not all reflections in C_1 are in Γ_1, we can obtain the two equal period-cycles of Γ_1 in two different ways:

(a) From the same "piece" of $C_1 = (2, \dots, \overset{i}{2}\overset{(2k/N)+2}{\dots}, \overset{j+1}{2}, \dots, 2)$, such that $c_i, \dots, c_j \in \Gamma_1$, c_{i-1}, $c_{j+1} \notin \Gamma_1$ and $n^1(i,j) = \frac{1}{4}N$. Hence (4.2.1) becomes

$$\begin{aligned} &\alpha g' + k' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{4}\left(\frac{2k}{N+2}\right) \\ &\quad + \frac{1}{2}\sum_{l\notin\{i,\dots,j+1\}}\left(1 - \frac{1}{n_{1,l}}\right) + \frac{1}{2}\sum_{i=2}^{k'}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right) = \frac{k}{2N}. \end{aligned} \tag{4.2.3}$$

Then $g' = 0$, $k' = 1$ and

$$\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{l\notin i,\dots,j+1}\left(1 - \frac{1}{n_{1,l}}\right) = \frac{1}{2}. \tag{4.2.4}$$

– If $r' = 1$, we have by (4.2.4), $m_1 = 2$, $p_1 = 2$ and $\sigma(\Gamma') = \tau_9 = \big(0, +, [2], \big\{(2, \overset{(2k/N)+2}{\dots}, 2)\big\}\big)$; θ_1 and G_1 are defined as

$$\theta_1(x_1) = {x_1}^{-1}, \quad \theta_1(e_1) = x_1, \quad \theta_1(c_{1,0}) = y_1, \quad \theta_1(c_{1,(2k/N)+2}) = z_1$$
$$\theta_1(c_{1,j}) = 1, \quad \text{for } j = 1,\dots,\frac{2k}{N} + 1;$$
$$G_1 = \big\langle x_1, y_1 : {y_1}^2 = {z_1}^2 = (y_1z_1)^{N/4} = 1, x_1y_1 = z_1x_1\big\rangle.$$

Firstly, the order of x_1 is not fixed by the epimorphism, but since G_1 has cardinality N, the order of x_1 equals 2. This group is isomorphic to a semidirect product of type $D_{N/4} \propto \mathbf{Z}_2$.

– If $r' = 0$, also by (4.2.4), $\sigma(\Gamma') = \tau_{10} = (0, +, [-], \big\{(2, \overset{(2k/N)+4}{\dots}, 2)\big\})$. Here θ_1 and G_1 are defined by

$$\theta_1(e_1) = 1, \quad \theta_1(c_{1,0}) = \theta_1(c_{1,(2k/N)+4}) = x_1,$$
$$\theta_1(c_{1,(2k/N)+2}) = y_1, \quad \theta_1(c_{1,(2k/N)+3}) = z_1,$$
$$\theta_1(c_{1,j}) = 1, \quad \text{for } j = 1,\dots,(2k/N)+1;$$
$$G_1 = \big\langle x_1, y_1, z_1 : {x_1}^2 = {y_1}^2 = {z_1}^2 = (y_1x_1)^{N/4} = (y_1z_1)^2 = (z_1y_1)^2 = 1\big\rangle.$$

Note that $\operatorname{ord}(y_1x_1) = \operatorname{ord}(y_1z_1) = 2$ is necessary for $\operatorname{Ker}(\theta_1)$ to have no proper periods.

(b) From different "pieces" of C_1 (also works for $k \neq s$). Then $C_1 = (2, \overset{(2k/N)+4}{\ldots}, 2)$, $c_{1,0}$, $c_{1,(2k/N)+2}$, $c_{1,(2k/N)+4} \notin \Gamma_1$, $c_{1j} \in \Gamma_1$ for the remaining reflections, and $n^1(1, (2k/N)+1) = n^1((2k/N)+3, (2k/N)+3) = \frac{1}{2}N$. Hence (4.2.1) becomes

$$(4.2.5)\quad \alpha g' + k' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{4}\left(\frac{2k}{N} + 4\right) + \frac{1}{2}\sum_{i=2}^{k'}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right) = \frac{k}{2N}.$$

Then $g' = 0$, $k' = 1$, $r' = 0$ and $\sigma(\Gamma') = \tau_{10} = \big(0, +, [-], \{(2, \overset{(2k/N)+4}{\ldots}, 2)\}\big)$. We have obtained the same signature as before but a different epimorphism:

$$\begin{aligned}
&\theta_1(e_1) = 1, \quad \theta_1(c_{1,0}) = \theta_1(c_{1,(2k/N)+4}) = x_1, \quad \theta_1(c_{1,(2k/N)+2}) = y_1,\\
&\theta_1(c_{1,j}) = 1 \quad \text{for the remaining reflections;}\\
&G_1 = \langle x_1, y_1 : {x_1}^2 = {y_1}^2 = (y_1x_1)^{N/2}\rangle.
\end{aligned}$$

Case 3. If $\sigma(\Gamma_1) = \big(1, -, [-], \{(2, \overset{2k}{\ldots}, 2)\}\big)$, we proceed in a similar way to obtain the signatures of Γ'.

Proposition 4.3. *If X has topological genus $g \geq 2$ and $2 \leq k \leq 4$, then the signature of Γ' is one of the following*:

$$\begin{aligned}
&\tau_{15} = \big(0, +, [2, \overset{l+1}{\ldots}, 2, 4,], \{(-)\}\big), \quad l \geq 0,\ N = \frac{2(p-1)}{2l+1};\\
&\tau_{16} = \big(0, +, [2, \overset{l}{\ldots}, 2, 4, 4], \{(-)\}\big), \quad l \geq 0,\ N = \frac{p-1}{l+1};\\
&\tau_{17} = \big(0, +, [2, \overset{l_1}{\ldots}, 2, 4], \{(2, \overset{l_2+2}{\ldots}, 2)\}\big), \quad l_1, l_2 \geq 0,\ N = \frac{2(p-1)}{2l_1 + l_2 + 2};\\
&\tau_{18} = \big(0, +, [-], \{(2, \overset{l+4}{\ldots}, 2)\}\big), \quad l > 0,\ N = \frac{2(p-1)}{l};\\
&\tau_{19} = \big(0, +, [-], \{(2, \overset{l+2}{\ldots}, 2, 4, 4)\}\big), \quad l \geq 0,\ N = \frac{2(p-1)}{l+1};\\
&\tau_{20} = \big(0, +, [-], \{(2, \overset{l+3}{\ldots}, 2, 4)\}\big), \quad l \geq 0,\ N = \frac{4(p-1)}{2l+1};
\end{aligned}$$

The proof of this proposition is analogous to the previous one.

5. The automorphism group of EHKS

Let $X = \mathscr{D}/\Gamma$ be an orientable EHKS of topological genus g, k boundary components and algebraic genus $p > 5$. Remember that $p > 5$ makes the EH-involution unique and central in $\mathrm{Aut}(X)$. Let Γ_1 be the group of the EH character. If G is an automorphism group of X containing ϕ (the EH-involution), then $G = \Gamma'/\Gamma$ for a certain NEC group Γ' with signature of one of the types described in Section 4. Let us suppose that $|G| = 2N$ and denote by $\theta\colon \Gamma' \to G$ and $\pi\colon G \to G/\langle\phi\rangle$ the canonical epimorphisms such that $\theta_1 = \pi\theta$.

The method to obtain θ (and then G) is to use the epimorphism θ_1 already defined in Section 4 and consider the diagram

$$\theta_1\colon \Gamma' \xrightarrow{\theta} G \xrightarrow{\pi} G_1 = G/\langle\phi\rangle.$$

Elements in G_1 will be denoted by letters like x_1, y_1, z_1, t_1, ...; and those in G, like $x, y, z, t, \ldots$, subject to $\pi(x) = x_1$, $\pi(y) = y_1, \ldots$.

The study of automorphism groups of orientable EHKS splits into three parts corresponding to the three propositions in Section 4.

Theorem 5.1. *Let X be an orientable EHKS of algebraic genus $p > 5$, topological genus 0 and k boundary components. Then $G \simeq \mathbf{Z}_N \times \mathbf{Z}_2$ or $D_{N/2} \times \mathbf{Z}_2$ for some integer N (even in the last case). Moreover,*

(i) *There exists such an EHKS having $\mathbf{Z}_N \times \mathbf{Z}_2$ as the full group of automorphisms if and only if N is a proper divisor of $p-1$.*

(ii) *There exists such an EHKS having $D_{N/2} \times \mathbf{Z}_2$ as the full group of automorphisms if and only if N is an even divisor of $2(p-1)$.*

Proof. By Theorem 3.4, $\sigma(\Gamma_1) = \big(0; +; [-], \{(2, \overset{2(p-1)}{\ldots}, 2), (-)\}\big)$ and there are two possibilities for the signature of Γ' namely τ_1 and τ_2 described in 4.1.

We begin with $\tau_1 = \big(0, +, [-], \{(2, \overset{2(p-1)/N}{\ldots}, 2), (-)\}\big)$. The epimorphism θ_1 is defined as

$$\theta_1(e_1) = x_1, \quad \theta_1(e_2) = {x_1}^{-1}, \quad \theta_1(c_{ij}) = 1 \text{ for all reflections in } \Gamma'$$

with $G_1 = \langle x_1 : {x_1}^N = 1\rangle$. Let $\theta(e_1) = x$, now as $\mathrm{ord}(x_1) = N$ and $\theta_1 = \pi\theta$ then $\mathrm{ord}(x') = N$ or $2N$:

(i) If $\mathrm{ord}(x) = 2N$, then $G \simeq \mathbf{Z}_{2N}$ and we will show there exists no epimorphism $\theta\colon \Gamma' \to \mathbf{Z}_{2N}$ verifying $\mathrm{Ker}(\theta) = \Gamma$. Empty period-cycles of Γ come from consecutive pairs of period-cycles with associated reflections

$$c_{ij-1},\ c_{ij},\ c_{ij+1}: \quad c_{ij-1}, c_{ij+1} \notin \Gamma,\ c_{ij} \in \Gamma;$$

from each of these pairs we obtain $N/\mathrm{ord}\big(\theta(c_{ij-1}, c_{ij+1})\big)$ period-cycles in $\mathrm{Ker}(\theta)$. Then we will have as maximum $p - 1 = k - 2$ from the non empty period-cycle of Γ'. The other two will come from the empty period-cycle of Γ', but since $\mathrm{ord}(x) = 2N$, we only can obtain $2N/\mathrm{ord}\big(\theta(e_2)\big) = 1$.

(ii) If $\mathrm{ord}(x) = N$, let us consider the EH-involution ϕ, we can see that $\langle x\rangle \cap \langle \alpha\rangle = 1$, otherwise, since ϕ has order 2 in G, we have $\phi = x^{N/2}$ and consequently $\pi(\phi) = \pi(x^{N/2}) = {x_1}^{N/2} = 1$, a contradiction with the order of x_1 in G_1. Furthermore, since ϕ is central in $\mathrm{Aut}(X)$, then $G \simeq \mathbf{Z}_N \times \mathbf{Z}_2$ and θ is defined by

$$\begin{aligned} &\theta(e_1) = x, \quad \theta(e_2) = x^{-1}, \quad \theta(c_{2,0}) = 1,\\ &\theta(c_{1,2j}) = y, \quad j = 0, \ldots, \frac{p-1}{N};\\ &\theta(c_{1,2j+1}) = 1, \quad j = 0, \ldots, \frac{p-1}{N} - 1 \quad (\pi(y) = 1). \end{aligned} \tag{5.1.1}$$

In this case $2(p-1)/N$ is necessarily even, implying that N divides $p-1$. If $2(p-1)/N$ is odd the epimorphism would be impossible.

If the signature of Γ' is $\tau_2 = \big(0, +, [-], \big\{(2, \overset{(2(p-1)/N)+4}{\ldots}, 2)\big\}\big)$, θ_1 is defined as

$$\theta_1(e_1) = 1, \quad \theta_1(c_{10}) = \theta_1(c_{1,((2k-4)/N)+4}) = x_1, \quad \theta_1(c_{1,((2k-4)/N)+2}) = y_1,$$
$$\theta_1(c_{1j}) = 1 \text{ for the remaining reflections;}$$
$$\text{with } G_1 = \langle x_1, y_1 : {x_1}^2 = {y_1}^2 = (x_1y_1)^{N/2} = 1\rangle \simeq D_{N/2}.$$

Considering $\theta_1 = \pi\theta$, since $\mathrm{ord}(x_1y_1) = \frac{1}{2}N$ then $\mathrm{ord}(xy) = \frac{1}{2}N$ or N. If $\mathrm{ord}(xy) = N$, then $G = D_N$ and as in the previous case, there is no epimorphism $\theta\colon \Gamma' \to D_N$ satisfying $\mathrm{Ker}(\theta) = \Gamma$. Let $l = 2(p-1)/N$; the way to obtain a maximal number of empty period-cycles in $\mathrm{Ker}(\theta)$, having in mind that consecutive reflections cannot be in $\mathrm{Ker}(\theta)$, is to define

$\theta(c_{1,l+3}) = 1$ having one period-cycle from c_{l+2}, c_{l+3}, c_{l+4} (see [9]), and

$\theta(c_{1,2j+1}) = z$ for $j = 0, \ldots, \frac{1}{2}(l-2)$, $\theta(c_{1,2j}) = 1$ for $j = 1, \ldots, \frac{1}{2}(l-2)$, where $z^2 = 1$ and $\pi(z) = 1$. From here we obtain $k - 2 - N$ more period-cycles. There remain $N+1$ period-cycles to be obtained by playing with the values of $\theta(c_l)$, and $\theta(c_{l+1})$. In any case we have

$$\frac{N}{\mathrm{ord}\big(\theta(c_{i,l-1}, c_{i,l+1})\big)} \quad \text{or} \quad \frac{N}{\mathrm{ord}\big(\theta(c_{i,l}, c_{i,l+2})\big)}$$

depending on $\theta(c_{1,l}) = 1$ or $\theta(c_{1,l+1}) = 1$, respectively. In the two cases we have at most $k-1$ empty period-cycles in $\mathrm{Ker}(\theta)$. Then $\mathrm{ord}(xy) = \frac{1}{2}N$.

Let us consider ϕ, the EH-involution. We can see that $\langle x, y\rangle \cap \langle\phi\rangle = 1$ because otherwise, since ϕ has order 2 in G, then $\phi = (xy)^{N/4}$ or $\phi = (xy)^k x$, for some k in $\{0, \ldots, \frac{1}{2}N\}$. The first possibility contradicts the order of x_1y_1 in G_1, because

$\pi(\phi) = \pi\big((xy)^{N/4}\big) = x_1y_1{}^{N/4} = 1$. The second implies $\pi\big((xy)^k x\big) = 1$, or equivalently $(c_{1,l+2}c_{1,l+4})^k c_{l+2} \in \Gamma_1$ and since $\Gamma_1 \triangleleft \Gamma'$, $c_{l+4} \in \Gamma_1$, a contradiction. Thus, since $|G| = 2N$, x, y, ϕ generate G, and since ϕ is central in G the epimorphism is defined by

If l even:

$$\begin{gathered}\theta(e_1) = 1, \quad \theta(c_{1,0}) = \theta(c_{1,l+4}) = x \quad \theta(c_{1,l+2}) = y, \quad \theta(c_{1,l+3}) = 1; \\ \theta(c_{1,2j+1}) = z \quad \text{for } j = 0, \dots, \tfrac{1}{2}l, \quad \theta(c_{1,2j}) = 1 \text{ for } j = 1, \dots, \tfrac{1}{2}l.\end{gathered} \tag{5.1.2}$$

If l odd:

$$\begin{gathered}\theta(e_1) = 1, \quad \theta(c_{1,0}) = \theta(c_{1,l+4}) = x, \quad \theta(c_{1,l+2}) = y, \quad \theta(c_{1,l+3}) = 1, \\ \theta(c_{1,2j+1}) = z \text{ for } j = 0, \dots, \tfrac{1}{2}(l-1), \\ \theta(c_{1,2j}) = 1 \text{ for } j = 1, \dots, \tfrac{1}{2}(l+1).\end{gathered} \tag{5.1.3}$$

In any case $G = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^{N/2} = (xz)^2 = (yz)^2 = 1\rangle \simeq D_{N/2} \times \mathbf{Z}_2$, $\phi = z$.

This proves the first part of the theorem. For the second, let X be an EHKS in the conditions of the theorem:

(i) If $\operatorname{Aut}(X) \simeq \mathbf{Z}_N \times \mathbf{Z}_2$, then Γ' must have signature τ_1 and $N \mid p-1$, as we saw before. Let N be a proper divisor of $p-1$ and consider the signature τ_1. If N divides $p-1$ properly, $\tau_1{}^+$ is maximal (see Section 2), and there exists a maximal NEC group Γ' with signature τ_1. Considering the epimorphism θ (5.1.1), the surface $X = \mathscr{D}/\operatorname{Ker}(\theta)$ is elliptic-hyperelliptic, $\phi = y$ being the EH-involution and $\operatorname{Aut}(X) \simeq \mathbf{Z}_N \times \mathbf{Z}_2$. Now if $N = p-1$ we consider Γ', θ, X as before. If τ_1 is not maximal, we will see that whenever $\mathbf{Z}_{p-1} \times \mathbf{Z}_2$ is a group of automorphisms of X, so is $D_{p-1} \times \mathbf{Z}_2$. To do it let us consider Theorem 4.1 in [7] and the following normal proper pair

$$(\tau_1, \tau_2), \ \ \tau_2 = \big(0, +, [-], \{(2,2,2,2,2)\}\big).$$

We only need to argue that the epimorphism θ: $\Gamma' \to \mathbf{Z}_N \times \mathbf{Z}_2$ is unique up to automorphisms of Γ' and $\mathbf{Z}_N \times \mathbf{Z}_2$.

(ii) If $\operatorname{Aut}(X) = D_{N/2} \times \mathbf{Z}_2$ (N even), then Γ' must have signature τ_2 and $N \mid 2(p-1)$, this last condition makes τ_2 maximal. Let N be an even divisor of $2(p-1)$ and consider the signature τ_2. Since $\tau_2{}^+$ is maximal (see Section 2) there exists a maximal NEC group Γ' with signature τ_2; and considering the epimorphism θ (5.1.2), the surface $X = \mathscr{D}/\operatorname{Ker}(\theta)$ is elliptic-hyperelliptic. The EH-involution ϕ is $(xy)^{N/4}$ and $\operatorname{Aut}(X) \simeq D_{N/2} \times \mathbf{Z}_2$.

Theorem 5.2. *Let X be an orientable EHKS of algebraic genus $p > 5$, topological genus 1 and k boundary components. Then* $\mathrm{Aut}(X)$ *is isomorphic to one of the following groups:* $\mathbf{Z}_N \times \mathbf{Z}_2$, $\mathbf{Z}_{N/2} \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $D_{N/2} \times \mathbf{Z}_2$, $(D_{N/4} \times \mathbf{Z}_2) \times \mathbf{Z}_2$ *or a semidirect product of type* $(D_{N/4} \propto \mathbf{Z}_2) \times \mathbf{Z}_2$ *for some even integer N,* $N \equiv 0 \pmod 4$ *in the last two cases.*

Moreover, in all cases there exists such an EHKS having full group of automorphisms isomorphic to

- $\mathbf{Z}_N \times \mathbf{Z}_2$ *if and only if N is a proper divisor of k;*
- $\mathbf{Z}_{N/2} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ *if and only if N is an even proper divisor of k;*
- $D_{N/2} \times \mathbf{Z}_2$ *if and only if N is an even divisor of $2k$;*
- $D_{N/4} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ *if and only if N is a divisor of $2k$,* $N \equiv 0 \pmod 4$*;*
- $(D_{N/4} \propto \mathbf{Z}_2) \times \mathbf{Z}_2$ *if and only if N is a divisor of $2k$,* $N \equiv 0 \pmod 4$.

Proof. Let G be an automorphism group of X as described at the beginning of this section. Since X has topological genus 1, by Proposition 4.2, the group of the EH character can be of three different types:

1. If $\sigma(\Gamma_1) = \big(0, +, [-], \{(2, \overset{2k}{\ldots}, 2), (-)\}\big)$, we have two possibilities for $\sigma(\Gamma')$, to consider, τ_3 and τ_4. The study is completely analogous that of τ_1 and τ_2, respectively, in the previous theorem. We only show the epimorphisms for each case.

For τ_3, $G = \langle x, y : x^N = y^2 = [x, y] = 1\rangle \simeq \mathbf{Z}_N \times \mathbf{Z}_2$, and θ is defined by

$$\theta(e_1) = x, \quad \theta(e_2) = x^{-1}, \quad \theta(c_{2,0}) = \theta(c_{1,2j}) = 1 \quad \text{for } j = 0, \ldots, \frac{k}{N};$$
$$\theta(c_{1,2j+1}) = y \quad \text{for } j = 0, \ldots, \frac{k}{N} - 1 \ (\pi(y) = 1).$$

For τ_4, $G = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^{N/2} = (xz)^2 = (yz)^2 = 1\rangle \simeq D_{N/2} \times \mathbf{Z}_2$ and θ is defined by

$$\theta(e_1) = 1, \quad \theta(c_{1,0}) = \theta(c_{1,l+4}) = x, \ l = \frac{2k}{N}, \quad \theta(c_{1,l+2}) = y, \quad \theta(c_{1,l+3}) = 1,$$
$$\theta(c_{1,2j+1}) = z \text{ for } j = 0, \ldots, \tfrac{1}{2}l, \quad \theta(c_{1,2j}) = 1 \text{ for } j = 1, \ldots, \tfrac{1}{2}l, \ l \text{ even}.$$

2. If $\sigma(\Gamma_1) = \big(0, +, [-], \{(2, \overset{s}{\ldots}, 2), (2, \overset{2k-2}{\ldots}, 2)\}\big)$, $0 < s < 2k$, s even. By Proposition 4.2 we have six possibilities for the signature of $\sigma(\Gamma')$: $\tau_5, \ldots, \tau_{10}$. The following table shows the results obtained for each one.

$\sigma(\Gamma')$	Γ'/Γ
$\tau_5 = (0,+,[-],\{(2,\overset{(2k-s)/N}{\ldots},2),(2,\overset{s/N}{\ldots},2)\})$, $N \mid k$	$\mathbf{Z}_N \times \mathbf{Z}_2$
$\tau_6 = (0,+,[-],\{(2,\overset{2k/N}{\ldots},2),(-)\})$	$\mathbf{Z}_{N/2} \times \mathbf{Z}_2 \times \mathbf{Z}_2$
$N \mid k$	$\mathbf{Z}_N \times \mathbf{Z}_2$, $s = k$
$\tau_7 = (0,+,[2,2],\{(2,\overset{2k/N}{\ldots},2)\})$, $N \mid k$	$D_{N/2} \times \mathbf{Z}_2$, $s = k$
$\tau_8 = (1,-,[-],\{(2,\overset{2k/N}{\ldots},2)\})$, $N \mid k$	$\mathbf{Z}_N \times \mathbf{Z}_2$, $s = k$
$\tau_9 = (0,+,[2],\{(2,\overset{2k/N}{\ldots},2)\})$, $4 \mid N \mid 2k$	$(D_{N/4} \propto \mathbf{Z}_2) \times \mathbf{Z}_2$, $s = k$
$\tau_{10} = (0,+,[-],\{(2,\overset{(2k/N)+4}{\ldots},2)\})$	$D_{N/2} \times \mathbf{Z}_2$
$N \mid k$	$(D_{N/4} \times \mathbf{Z}_2) \times \mathbf{Z}_2$, $s = k$, $4 \mid N$

For the sake of brevity we only show the complete proof for τ_9. The remaining cases are obtained in a similar way.

Let us consider $\tau_9 = \big(0,+,[2],\{(2,\overset{2k/N}{\ldots},2)\}\big)$. Here θ_1 was defined as

$$\theta_1(x_1) = {x_1}^{-1}, \qquad \theta_1(e_1) = x_1, \qquad \theta_1(c_{1,0}) = y_1, \qquad \theta_1(c_{1,2k/N+2}) = z_1,$$
$$\theta_1(c_{1,j}) = 1, \qquad \text{for } j = 1,\ldots,\frac{2k}{N}+1,$$

where $G_1 = \big\langle x_1,y_1,z_1 : {x_1}^2 = {y_1}^2 = {z_1}^2 = (y_1z_1)^{N/4} = 1,\ x_1y_1 = z_1x_1\big\rangle \simeq D_{N/4} \propto \mathbf{Z}_2$.

To obtain θ let us define $\theta(x_1) = x^{-1}$, $\theta(e_1) = x$, $\theta(c_{1,0}) = y$, $\theta(c_{1,2k/N+2}) = z$, $\theta(c_{1,2l}) = 1$, $\theta(c_{1,2l+1}) = t$, $2l$, $2l+1 \in \{1,\ldots,(2k/N)+1\}$. Here x,y,z,t are order two elements of G, that must verify, for $\mathrm{Ker}(\theta)$ to be a surface group, and θ to be an epimorphism, $(yt)^2 = (tz)^2 = 1$, $xyx^{-1} = z$.

Moreover, since we want $\theta_1 = \pi\theta$ and the order of $\pi(xy) = x_1y_1$ is $\frac{1}{4}N$, then the order of xy has to be $\frac{1}{4}N$ or $\frac{1}{2}N$. If the order is $\frac{1}{2}N$, we will have $\mathrm{Ker}(\theta)$ non orientable because $(c_{1,0}c_{1,(2k/N)+2})^{N/4}c_{1,1}$ would belong to it. Then the order of xy is $\frac{1}{4}N$. Now, since $\pi(t) = 1$ (in other words $t\langle\phi\rangle = \langle\phi\rangle$), we have $t = \phi$, and since ϕ is central in G the following presentation holds:

$$G = \big\langle x,y,z,t : x^2 = y^2 = z^2 = t^2 = (yz)^{N/4} = (yt)^2 = (zt)^2 = (xt)^2 = 1, xyx = z\big\rangle.$$

We can see that this group is isomorphic to a semidirect product $(D_{N/4} \propto_\varphi \mathbf{Z}_2) \times \mathbf{Z}_2$, φ being the automorphism

$$\begin{aligned} \varphi\colon \mathbf{Z}_2 = \langle a : a^2\rangle &\longrightarrow \mathrm{Aut}(D_{N/4}), \qquad D_{N/4} = \langle b,c : b^2 = c^2 = (bc)^{N/4} = (bd)^2\rangle \\ 1 &\longrightarrow \mathrm{Id}_{D_{N/4}} \\ a &\longrightarrow \psi\colon D_{N/4} \longrightarrow D_{N/4}, \qquad b \longrightarrow c, \qquad c \longrightarrow b. \end{aligned}$$

3. If $\sigma(\Gamma_1) = \big(1,-,[-],\{(2,\overset{2k}{\ldots},2)\}\big)$, the following table shows, for each

signature of Γ', the obtained quotient $G = \Gamma'/\Gamma$

$\sigma(\Gamma')$	Γ'/Γ
τ_{11}	$\mathbf{Z}_N \times \mathbf{Z}_2$, N even divisor of k
τ_{12}	There is no epimorphism with $\operatorname{Ker}\theta = \Gamma$
τ_{13}	$D_{N/2} \times \mathbf{Z}_2$, N divides $2k$, $N \equiv 2 \pmod 4$
τ_{14}	$D_{N/2} \times \mathbf{Z}_2$, N divides $2k$, $N \equiv 0 \pmod 4$.

This proves the first part of the theorem.

For the second part, let us suppose that $\operatorname{Aut}(X) = \mathbf{Z}_N \times \mathbf{Z}_2$, N even, then $\operatorname{Aut}(X) = \Gamma'/\Gamma$ were Γ' has signature τ_3, τ_5, τ_8 or τ_{11}. In any case N divides k, as we saw in the first part of the theorem. Now, if N is a proper even divisor k we can consider the signature τ_5. Since $\tau_5{}^+$ is maximal (see Section 2) there exist a maximal NEC group Γ' with signature τ_5. Define θ as follows:

$$\theta(e_1) = x, \quad \theta(e_2) = x^{-1}, \quad \theta(c_{i,j}) = y \quad \text{for } i = 1,2,\ j \text{ even},$$
$$\theta(c_{i,j}) = 1 \quad \text{for } i = 1,2,\ j \text{ odd},$$

where the EH-involution $\phi = y$ and $\Gamma'/\Gamma = \langle x, y : x^N = y^2 = (xy)^2 = 1\rangle$. Then, the surface $X = \mathscr{D}/\operatorname{Ker}(\theta)$ is elliptic-hyperelliptic and $\operatorname{Aut}(X) \simeq \mathbf{Z}_N \times \mathbf{Z}_2$.

If $N = k$, we have no maximality conditions for the signatures of Γ', and we can prove that $\mathbf{Z}_k \times \mathbf{Z}_2$ can be extended to $D_k \times \mathbf{Z}_2$ as an automorphism group.

The remaining cases are proved in a similar way.

For the sequel, we set the following group notation:

$$U_N = \langle x, y, z : x^4 = y^2 = z^N = xyz = 1, x^2 = z^{N/2}\rangle,$$
$$V_N = \langle x, y, z : x^4 = y^4 = z^N = xyz = 1, y^2 = x^2 = z^{N/2}\rangle,$$
$$W_N = \langle x, y, z : x^4 = y^4 = z^{N/2} = xyz = 1, y^2 = x^2\rangle,$$
$$Q_N = \langle x, y, z : x^2 = y^2 = z^4 = (xy)^{N/2} = 1, zxz^{-1} = y, (xy)^{N/4} = z^2\rangle.$$

Theorem 5.3. *Let X be an orientable EHKS of algebraic genus $p > 5$, topological genus $g \geq 2$ and k boundary components, $2 \leq k \leq 4$. Then,* $\operatorname{Aut}(X)$ *is isomorphic in one of the following groups:*

(i) U_N, Q_N, $N = \big(2(p-1)\big)/(2l+1)$ *and* $l > 0$, *or* V_N, $N = (p-1)/(l+1)$ *and* $l > 0$, *or* $D_{N/2} \propto \mathbf{Z}_2$, $N = \big(2(p-1)\big)/(2l+1)$ *and* $l \geq 0$ *if* $k = 2$.

(ii) W_N, $N = (p-1)/(l+1)$ *and* $l > 0$, 4,
or $D_{N/4} \propto \mathbf{Z}_4$, $N = \big(2(p-1)\big)/(2l+1)$ *and* $l > 0$,
or $(D_{N/4} \times \mathbf{Z}_2) \propto \mathbf{Z}_2$, $N = \big(4(p-1)\big)/(2l+1)$ *and* $l \geq 0$ *if* $k = 4$.

(iii) $\mathbf{Z}_2$ *if* $k = 3$.

Proof. Let G be an automorphism group of X containing the EH involution ϕ, such that $|G| = 2N$. Then $G = \Gamma'/\Gamma$ for a certain NEC group Γ' with signature of one of the types described in Proposition 4.3: $\tau_{15}, \dots, \tau_{20}$. The following table summarizes the groups obtained in each case:

$\sigma(\Gamma')$	k	$G = \Gamma'/\Gamma$
$(0, +, [2, \overset{l+1}{\ldots}, 2, 4], \{(-)\})$	2	U_N
$N = \dfrac{2(p-1)}{2l+1}$ even	4	D_4, $l = \frac{1}{4}(p-3)$
$(0, +, [2, \overset{l}{\ldots}, 2, 4, 4], \{(-)\})$	2	V_N
$N = \dfrac{p-1}{l+1}$ even	4	W_N
$(0, +, [2, \overset{l}{\ldots}, 2, 4], \{(2,2)\})$	2	Q_N
$N = \dfrac{2(p-1)}{2l+1} \equiv 0 \pmod 4$	4	$D_{N/4} \propto \mathbf{Z}_4$
$(0, +, [-], \{(2,2,4,4)\})$	2	$D_{N/2} \propto \mathbf{Z}_2$
$N = \dfrac{2(p-1)}{l+1} \equiv 0 \pmod 4$	4	$(D_{N/4} \times \mathbf{Z}_2) \propto \mathbf{Z}_4$
$(0, +, [-], \{(2,2,2,4)\})$	2	$D_{N/2} \propto \mathbf{Z}_2$
$N = \dfrac{4(p-1)}{2l+1} \equiv 0 \pmod 4$	4	$(D_{N/4} \times \mathbf{Z}_2) \propto \mathbf{Z}_2$

Remark. It is easy to see that there exists no epimorphism from Γ', with signature τ_{18}, onto G, such that $\mathrm{Ker}(\theta)$ is a surface group. The same occurs for τ_{19} and τ_{20} if $l > 0$.

When $k = 2$ and $l = 0$, we can see that whenever U_N, V_N, Q_N or $D_{p-1} \propto \mathbf{Z}_2$ is a group of automorphisms of an EHKS in the conditions of the theorem, it can be extended to $D_{2(p-1)} \propto \mathbf{Z}_2$. When $k = 4$ and $l = 0$, the automorphism groups W_N, $D_{p-1} \propto \mathbf{Z}_4$ or $(D_{p-1} \times \mathbf{Z}_2) \propto \mathbf{Z}_2$ can be extended to $(D_{p-1} \times \mathbf{Z}_2) \propto \mathbf{Z}_2$. To do it, we consider Theorem 4.1 in [7] and the following normal proper pairs (see Section 1):

$$(\tau_{15}, \tau_{20}),\ (\tau_{16}, \tau_{20}),\ (\tau_{17}, \tau_{20}),\ \text{where } l = 0,\ \text{and } (\tau_{19}, \tau_{20}).$$

In each case we only need to prove that the epimorphism $\theta\colon \Gamma' \to G$, having kernel Γ, is unique up to automorphisms of Γ' and G. This situation occurs when Γ' has signature τ_{15}, τ_{16}, τ_{17} (with $l = 0$) or τ_{16}. We show the epimorphisms for each one:

– if $\sigma(\Gamma') = \tau_{12}$, $G = U_N$, then $\theta(x_1) = x$, $\theta(x_2) = y$, $\theta(e_1) = z$, $\theta(c_{1,0}) = 1$;

– if $\sigma(\Gamma') = \tau_{13}$, $G = V_N$ or W_N, then $\theta(x_1) = x$, $\theta(x_2) = y$, $\theta(e_1) = z$, $\theta(c_{1,0}) = 1$;

– if $\sigma(\Gamma') = \tau_{14}$, $G = Q_N$ or $D_{p-1} \propto \mathbf{Z}_4 = \langle x, y, z : x^2 = y^2 = z^4 = (xy)^{N/2} = 1, zxz^{-1} = y \rangle$, then $\theta(x_1) = z$, $\theta(e_1) = z^{-1}$, $\theta(c_{1,0}) = x$, $\theta(c_{1,1}) = 1$, $\theta(c_{1,2}) = y$;

– if $\sigma(\Gamma') = \tau_{16}$, G has one of the following presentations

$$\begin{aligned} D_{p-1} \propto \mathbf{Z}_2 = \bigl\langle x, y, z : x^2 = y^2 = z^2 = (xy)^{p-1} \\ = (yz)^4 = (zx)^4 = 1, (xy)^{(p-1)/2} = (yz)^2 = (zx)^2 \bigr\rangle, \quad k = 2, \end{aligned}$$

or

$$\begin{aligned} (D_{p-1} \times \mathbf{Z}_2) \propto \mathbf{Z}_2 = \bigl\langle x, y, z : x^2 = y^2 = z^2 \\ = (xy)^{(p-1)/2} = (yz)^4 = (zx)^4 = 1, (yz)^2 = (zx)^2 \bigr\rangle, \quad k = 4. \end{aligned}$$

The epimorphism is

$$\theta(e_1) = 1;\ \theta(c_{1,0}) = x;\ \theta(c_{1,1}) = 1;\ \theta(c_{1,2}) = y;\ \theta(c_{1,3}) = z;\ \theta(c_{1,4}) = x.$$

Corollary. *Let X be an orientable EHKS with algebraic genus $p > 5$ and topological genus $g \geq 2$. If X has a non trivial automophism different from the EH-involution, then it has 2 or 4 boundary components.*

Acknowledgments. I would like to express my acknowledgment to Professors José A. Bujalance and Ernesto Martínez who proposed me this work, and Emilio Bujalance for his helpful suggestions. Also, I would thank the referee for his comments.

References

[1] Alling, N.L., and N. Greanleaf: Foundations of the Theory of Klein Surfaces. - Lecture Notes in Math. 219, Springer-Verlag, 1971.

[2] Bujalance, E.: Normal subgroups of NEC groups. - Math. Z. 178, 1981, 331–341.

[3] Bujalance, E.: Proper periods of normal NEC subgroups with even index. - Rev. Mat. Hisp-Amer. (4) 41, 1981, 121–127.

[4] Bujalance, E.: Normal NEC signatures. - Illinois J. Math. 26, 1982, 519–530.

[5] Bujalance, E., J.A. Bujalance, G. Gromadzki, and E. Martínez: Cyclic trigonal Klein surfaces. - J. Algebra 159:2, 1993, 436–458.

[6] Bujalance, E., J.J. Etayo, and J.M. Gamboa: Superficies de Klein elípticas hiperelípticas. - Memorias de la Real Academia de Ciencias, Tomo XIX, 1985.

[7] Bujalance, E., J.J. Etayo, and J.M. Gamboa: Groups of automorphisms of hyperelliptic Klein surfaces of genus three. - Michigan Math. J. 33, 1986, 55–74.

[8] Bujalance, E., J.J. Etayo, J.M. Gamboa, and G. Gromadzki: A Combinatorial Approach to Automorphisms Groups of Compact Bordered Klein Surfaces. - Lecture Notes in Math. 1439, Springer-Verlag, New York–Berlin, 1990.

[9] Bujalance, J.A.: Normal subgroups of even index in an NEC group. - Arch. Math. (Basel) 49, 1987, 470–478.

[10] Macbeath, A.M.: The classification of non-Euclidean crystallographic groups. - Canad. J. Math. 6, 1967, 1192–1205.

[11] May, C.L.: Large automorphism groups of compact Klein surfaces with boundary. - Glasgow Math. J. 18, 1977, 1–10.

[12] Preston, R.: Projective structures and fundamental domains on compact Klein surfaces. - Ph.D. Thesis, University of Texas, 1975.

[13] Singerman, D.: On the structure of non-euclidean crystallographic groups. - Proc. Cambridge Phil. Soc. 76, 1974, 233–240.

[14] Wilkie, H.C.: On non-Euclidean crystallographic groups. - Math. Z. 91, 1966, 87–102.

[15] Zieschang, H., E. Vogt, and H.D. Coldewey: Surfaces and Planar Discontinuous Groups. - Lecture Notes in Math. 835, Springer-Verlag, New York–Berlin, 1980.

Received 12 February 1999