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On Q_p spaces and pseudoanalytic extension

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Abstract. For $0 < p < 1$, Q_p is the space of those functions f which are analytic in the unit disc $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and satisfy $\sup_{|a|<1} \iint_\Delta |f'(z)|^2 (g(z,a)^p \, dx \, dy < \infty$, where $g(\,\cdot\,,\cdot\,)$ is the Green function of Δ. In this paper we obtain a new characterization of Q_p-functions in terms of pseudoanalytic extension and, as a corollary, we prove that Q_p has the K-property of Havin. The latter means that, for any $\psi \in H^\infty$, the Toeplitz operator $T_{\overline{\psi}}$ maps Q_p into itself. This in turn implies (as usual) that Q_p also enjoys the f-property, i.e., division by inner factors preserves membership in Q_p.

1. Introduction and statement of results

We denote by Δ the unit disc $\{z \in \mathbf{C} : |z| < 1\}$ and by H^p $(0 < p \le \infty)$ the classical Hardy spaces of analytic functions in Δ (see [8] and [13]).

For $a \in \Delta$, let φ_a denote the Möbius transformation defined by $\varphi_a(z) = (z-a)/(1-\overline{a}z)$, $z \in \mathbf{C}$, and the Green function $g(\,\cdot\,,\cdot\,)$ of Δ is given by

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|}, \qquad a, z \in \Delta.$$

For $p \ge 0$, we set

$$Q_p = \left\{ f : f \text{ is analytic in } \Delta \text{ and } \sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 g^p(z,a) \, dA(z) < \infty \right\}.$$

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Here and throughout, dA is the Lebesgue measure in $\mathbf{C}$. The Q_p spaces arose in [2] in connection with Bloch and normal functions and have been studied by several authors (see e.g. [2], [3], [4], [12] and [17]). Observe that Q_0 is the Dirichlet space $\mathscr{D}$, while $Q_1 = \mathrm{BMOA}$, the space of functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\Delta$ (see [5] and [13]). Further, Aulaskari and Lappan proved in [2] that for all $p \in (1, \infty)$, the spaces Q_p are the same and equal to the *Bloch space*

$$\mathscr{B} = \Big\{ f : f \text{ is analytic in } \Delta \text{ and } \sup_{z\in\Delta} (1-|z|^2)|f'(z)| < \infty \Big\}.$$

On the other hand Aulaskari, Xiao and Zhao showed in [4] that if $0 \le p < q \le 1$ then $Q_p \subsetneqq Q_q$. In particular, we have

$$\mathscr{D} \subset Q_p \subset \mathrm{BMOA}, \qquad 0 \le p \le 1.$$

The results of [3] (see also [23] and [26] for the case $1 < p < \infty$) show that if $0 < p < \infty$ and f is an analytic function in Δ then

$$(1) \qquad f \in Q_p \quad \Longleftrightarrow \quad \sup_{|a|<1} \iint_\Delta |f'(z)|^2 \big(1 - |\varphi_a(z)|^2\big)^p \, dA(z) < \infty.$$

Our main result, stated as Theorem 1 below, is a new characterization of Q_p-spaces ($0 < p < 1$) in terms of pseudoanalytic continuation. We refer to Dyn'kin's paper [11] for similar descriptions of classical smoothness spaces, as well as for other important applications of the pseudoanalytic extension method.

In what follows, Δ_- denotes the region $\mathbf{C} \setminus \overline{\Delta}$, and we write

$$z^* \stackrel{\text{def}}{=} 1/\overline{z}, \qquad z \in \mathbf{C} \setminus \{0\}.$$

Finally, we need the Cauchy–Riemann operator

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad z = x + iy.$$

Theorem 1. *If* $0 < p < 1$ *and* $f \in \bigcap_{0<q<\infty} H^q$, *then the following conditions are equivalent.*

(i) $f \in Q_p$.

(ii) $\sup_{|a|<1} \iint_\Delta |f'(z)|^2 \big((1/|\varphi_a(z)|^2) - 1\big)^p \, dA(z) < \infty$.

(iii) *There exists a function* $F \in C^1(\Delta_-)$ *satisfying*

$$(2) \qquad F(z) = \mathrm{O}(1), \qquad \text{as } z \to \infty,$$

$$\lim_{r\to 1^+} F(re^{i\theta}) = f(e^{i\theta}), \qquad \text{a.e. and in } L^q([-\pi,\pi]) \text{ for all } q \in [1,\infty), \tag{3}$$

and

$$\sup_{|a|<1} \iint_{\Delta_-} |\overline{\partial} F(z)|^2 \left(|\varphi_a(z)|^2 - 1\right)^p \, dA(z) < \infty. \tag{4}$$

We remark that our proof of Theorem 1 will show that the equivalence (i) $\Longleftrightarrow$ (ii) holds for an arbitrary holomorphic function f (without the a-priori assumption that $f \in \bigcap_{0<q<\infty} H^q$).

To describe some consequences of Theorem 1, we have to introduce further terminology. We recall first that, given a function $v \in L^\infty(\partial\Delta)$, the associated *Toeplitz operator* T_v is defined by

$$(T_v f)(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{v(\zeta) f(\zeta)}{\zeta - z} \, d\zeta \qquad (f \in H^1, \ z \in \Delta).$$

Definition 1. A subspace X of H^1 is said to have the K-property if $T_{\overline{\psi}}(X) \subset X$ for any $\psi \in H^\infty$.

Definition 2. A subspace X of H^1 is said to have the f-property if $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$.

These notions were introduced by Havin in [14]. It was also pointed out in [14] that the K-property implies the f-property: indeed, if $h \in H^1$, I is inner and $h/I \in H^1$ then $h/I = T_{\overline{I}} h$.

Our next result is

Theorem 2. *For $0 < p < 1$, the space Q_p has the K-property.*

In view of the above discussion, this immediately yields

Corollary 1. *For $0 < p < 1$, the space Q_p has the f-property.*

Since, as we have mentioned above, the Q_p spaces $(0 < p < 1)$ are intermediate spaces between the Dirichlet class $\mathscr{D}$ and BMOA, we wish to remark that both of these endpoint spaces do have the K-property (and hence also the f-property). The case of $\mathscr{D}$ is covered by results in [14]; see also [9], [15], [16] and [18] for various extensions dealing with Dirichlet-type spaces. The K-property of BMOA can be established along the lines of [14]: given $\psi \in H^\infty$, the multiplication map T_ψ acts boundedly on H^1, whence the adjoint operator $T_{\overline{\psi}}$ must act boundedly on BMOA.

Since, for $p > 1$, $Q_p = \mathscr{B}$ and $\mathscr{B}$ is not contained in H^1, it does not make sense to ask for this range of p's whether or not Q_p has the K- (or f-)property. However, let us mention that $H^\infty \cap \mathscr{B}_0$ fails to possess the f-property (here, $\mathscr{B}_0$ is the subspace of $\mathscr{B}$ defined by the corresponding "little oh" condition). This result, due to Anderson [1], can be also deduced from the fact that $\mathscr{B}_0$ contains an infinite Blaschke product (see [19]). Some other "Bloch-type" subclasses of H^∞ without the f-property have been exhibited by Vinogradov in [25] (see also [7]).

For further examples of spaces with or without the K- (or f-)property, the reader is referred to [20], [21] and the bibliography therein.

2. Proofs of the results

We begin by showing how Theorem 2 follows from Theorem 1. The proof of Theorem 1 will be presented afterwards.

Proof of Theorem 2. Let $0 < p < 1$, $f \in Q_p$ and $\psi \in H^\infty$. We have to show that $g \overset{\text{def}}{=} T_{\overline{\psi}} f$ is necessarily in Q_p.

Since g is the orthogonal projection of $f\overline{\psi}$ onto H^2, one has

$$f\overline{\psi} = g + \overline{h}$$

for some $h \in H^2_0$. (Actually, both g and h lie in $\bigcap_{0<q<\infty} H^q$. To see why, recall that $f \in \mathrm{BMO}$, $\psi \in L^\infty$ and use the boundedness properties of the Riesz projection.) Thus,

$$g = f\overline{\psi} - \overline{h} \qquad \text{a.e. on } \partial\Delta. \tag{5}$$

Now, since $f \in Q_p$, Theorem 1 says that there is a function $F \in C^1(\Delta_-)$ satisfying (2), (3) and (4). Further, we set, for $z \in \Delta_-$,

$$\Psi(z) \overset{\text{def}}{=} \overline{\psi(z^*)}, \qquad H(z) \overset{\text{def}}{=} \overline{h(z^*)}$$

and finally

$$G(z) \overset{\text{def}}{=} F(z)\Psi(z) - H(z).$$

This done, we claim that

$$G|_{\partial\Delta} = g \tag{6}$$

(the boundary values are again taken in the sense of radial convergence a.e. on $\partial\Delta$ and in each L^q with $q < \infty$) and

$$|\overline{\partial} G(z)| \le \|\psi\|_\infty |\overline{\partial} F(z)|, \qquad z \in \Delta_-. \tag{7}$$

Indeed, (6) follows from (5) and the facts that

$$F|_{\partial\Delta} = f, \qquad \Psi|_{\partial\Delta} = \overline{\psi}, \qquad H|_{\partial\Delta} = \overline{h},$$

while (7) holds because Ψ and H are holomorphic in Δ_-, and so $\overline{\partial} G = \Psi \cdot \overline{\partial} F$ on Δ_-.

Since G is obviously C^1-smooth in Δ_- and bounded at ∞, we now conclude from (6) and (7) that the analogues of (2), (3) and (4) hold true with G and g in place of F and f. Another application of Theorem 1 yields $g \in Q_p$, as desired. □

Now it remains to prove Theorem 1. Before doing so, let us recall that if h is an analytic function in Δ then, as usual, we set

$$M_2(r,h) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(re^{i\theta})|^2\, d\theta\right)^{1/2}, \qquad 0<r<1.$$

Proof of Theorem 1. (i) $\iff$ (ii) Let us borrow the argument used in the proof of Theorem 2.2 of [3]. By (1), it suffices to prove that there exist positive constants A_p and B_p such that

$$\begin{aligned} A_p \iint_\Delta |f'(z)|^2 \left(\frac{1}{|\varphi_a(z)|^2}-1\right)^p dA(z) &\le \iint_\Delta |f'(z)|^2 \big(1-|\varphi_a(z)|^2\big)^p\, dA(z) \\ &\le B_p \iint_\Delta |f'(z)|^2 \left(\frac{1}{|\varphi_a(z)|^2}-1\right)^p dA(z), \end{aligned} \tag{8}$$

for all $a\in\Delta$ and any f. By a change of variables argument, it is enough to check (8) for $a=0$. This is equivalent to

$$\begin{aligned} A_p \int_0^1 M_2(r,f')^2 \left(\frac{1}{r^2}-1\right)^p r\, dr &\le \int_0^1 M_2(r,f')^2 (1-r^2)^p r\, dr \\ &\le B_p \int_0^1 M_2(r,f')^2 \left(\frac{1}{r^2}-1\right)^p r\, dr, \end{aligned}$$

that is, to

$$\begin{aligned} A_p \int_0^1 M_2(r,f')^2 (1-r^2)^p r^{1-2p}\, dr &\le \int_0^1 M_2(r,f')^2 (1-r^2)^p r\, dr \\ &\le B_p \int_0^1 M_2(r,f')^2 (1-r^2)^p r^{1-2p}\, dr. \end{aligned}$$

This follows easily, since $0<p<1$ and $M_2(r,f')$ is an increasing function of r. ◻

Of course, the second inequality in (8) actually holds with $B_p=1$. It is the first inequality that we were mainly concerned with.

(i) $\Rightarrow$ (iii) Let $0<p<1$ and $f\in Q_p$. Set

$$F(z) = f(z^\star), \qquad z\in\Delta_-.$$

It is clear that F is C^1-smooth and satisfies (2) and (3). Now let $a\in\Delta$; making the change of variables $z=w^\star$ in the integral which appears in (4) and noting that $|\overline{\partial}F(z)| = |f'(z^\star)|\,|z^\star|^2$, we obtain

$$\begin{aligned} \iint_{\Delta_-} |\overline{\partial}F(z)|^2 (|\varphi_a(z)|^2-1)^p\, dA(z) &= \iint_\Delta |f'(w)|^2 (|\varphi_a(w)^\star|^2-1)^p\, dA(w) \\ &= \iint_\Delta |f'(w)|^2 \left(\frac{1}{|\varphi_a(w)|^2}-1\right)^p dA(w). \end{aligned}$$

Then, since (i) $\iff$ (ii), (4) follows. ◻

The proof of the remaining implication (iii) $\Rightarrow$ (ii) makes use of Calderón–Zygmund operators and Muckenhoupt weights. We refer to [22] and [24] for the notion of a Calderón–Zygmund operator, as well as for the basic terminology and facts listed below.

If $q > 1$ and ω is a positive measurable function on $\mathbf{C}$, then ω is said to be an A_q-weight if

$$A_q(\omega) \overset{\text{def}}{=} \sup_Q \left[\frac{1}{|Q|} \iint_Q \omega(z)\, dA(z) \right] \left[\frac{1}{|Q|} \iint_Q \bigl(\omega(z)\bigr)^{-q'/q}\, dA(z) \right]^{q/q'} < \infty.$$

Here Q ranges over the discs in $\mathbf{C}$, $|Q|$ denotes the area of Q, and $q' = q/(q-1)$. The A_2-condition has a simpler appearance:

$$A_2(\omega) \overset{\text{def}}{=} \sup_Q \left[\frac{1}{|Q|} \iint_Q \omega(z)\, dA(z) \right] \left[\frac{1}{|Q|} \iint_Q \bigl(\omega(z)\bigr)^{-1}\, dA(z) \right] < \infty.$$

Now if ω is an A_2-weight with $A_2(\omega) \le \alpha$ and if T is a Calderón–Zygmund operator, then we have the weighted inequality

$$\iint_{\mathbf{C}} |Tg(z)|^2 \omega(z)\, dA(z) \le B_{T,\alpha} \iint_{\mathbf{C}} |g(z)|^2 \omega(z)\, dA(z), \quad \text{for all } g \in L^2(\omega), \tag{9}$$

where the constant $B_{T,\alpha}$ depends only on α and $\|T\|_{L^2 \to L^2}$, the norm of T in the unweighted L^2-space.

We are now in a position to complete the proof of Theorem 1.

(iii) $\Rightarrow$ (ii) Suppose (iii) holds. We shall argue as in the proof of Lemma 7 on p. 154 of [10]. Fix $z \in \Delta$ and $R > 1$. In view of (3), the Cauchy–Green formula applied to the function that equals f in Δ and F in Δ_- gives

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=R} \frac{F(\xi)}{\xi - z}\, d\xi - \frac{1}{\pi} \iint_{1<|\xi|<R} \frac{\bar\partial F(\xi)}{\xi - z}\, dA(\xi). \tag{10}$$

Differentiating (10) and noticing that the arising contour integral is $\mathrm{O}(1/R)$, as $R \to \infty$, we obtain

$$f'(z) = -\frac{1}{\pi} \iint_{\Delta_-} \frac{\bar\partial F(\xi)}{(\xi - z)^2}\, dA(\xi). \tag{11}$$

Put

$$\Phi(z) = \begin{cases} \bar\partial F(z), & \text{if } z \in \Delta_-, \\ 0, & \text{if } z \in \Delta. \end{cases} \tag{12}$$

Let S be the Calderón–Zygmund operator defined by

$$Sg(z) = \text{p.v.} \iint_{\mathbf{C}} \frac{g(\xi)}{(\xi - z)^2}\, dA(\xi). \tag{13}$$

Using (11), (12) and (13), we see that

$$f'(z) = -\frac{1}{\pi}(S\Phi)(z), \qquad z \in \Delta. \tag{14}$$

Given $a \in \Delta$, define

$$U_a(z) = \left|1 - \frac{1}{|\varphi_a(z)|^2}\right|^p = \frac{(1-|a|^2)^p \big||z|^2 - 1\big|^p}{|z-a|^{2p}}, \qquad z \in \mathbf{C}. \tag{15}$$

We shall prove the following result.

Proposition 1. *There exists a positive constant α such that, for every $a \in \Delta$, U_a is an A_2-weight with*

$$A_2(U_a) \le \alpha, \qquad \textit{for all } a \in \Delta. \tag{16}$$

Once Proposition 1 is established, we can proceed as follows. Taking (12)–(15) into account and using inequality (9), with suitable replacements and in conjunction with (16), we get

$$\begin{aligned}
\iint_{\Delta} |f'(z)|^2 \left(\frac{1}{|\varphi_a(z)|^2} - 1\right)^p dA(z) &= \iint_{\Delta} |f'(z)|^2 U_a(z)\, dA(z) \\
&= \frac{1}{\pi^2} \iint_{\Delta} |(S\Phi)(z)|^2 U_a(z)\, dA(z) \\
&\le \frac{1}{\pi^2} \iint_{\mathbf{C}} |(S\Phi)(z)|^2 U_a(z)\, dA(z) \\
&\le C \iint_{\mathbf{C}} |\Phi(z)|^2 U_a(z)\, dA(z) \\
&= C \iint_{\Delta_-} |\overline{\partial} F(z)|^2 U_a(z)\, dA(z) \\
&\le C \iint_{\Delta_-} |\overline{\partial} F(z)|^2 \big(|\varphi_a(z)|^2 - 1\big)^p\, dA(z),
\end{aligned} \tag{17}$$

where $C > 0$ is a constant independent of $a \in \Delta$. To verify the last step, note that $|\varphi_a(z)| > 1$ for $z \in \Delta_-$. The resulting inequality from (17) shows that (ii) follows from (4). Consequently, it only remains to prove Proposition 1.

Proof of Proposition 1. Given $a \in \Delta$, set

$$V_a(z) = \frac{\big||z|^2-1\big|^p}{|z-a|^{2p}}, \qquad z \in \mathbf{C}. \tag{18}$$

It is clear that

$$A_2(U_a) = A_2(V_a), \qquad \text{for all } a \in \Delta. \tag{19}$$

We can write

$$V_a(z) = W(z)Y_a(z),$$

where

$$W(z) = \big||z|^2-1\big|^p, \qquad Y_a(z) = \frac{1}{|z-a|^{2p}}.$$

It is well known (see e.g. [22, p. 218]) that, since $0 < p < 1$, the weight $W_0(z) = |z|^{-2p}$ satisfies the A_s-condition for all $s > 1$. Since the Y_a are translates of W_0, it follows that for every $s > 1$ there exists a constant $\alpha_s > 0$ such that

$$A_s(Y_a) \le \alpha_s, \qquad \text{for all } a \in \Delta. \tag{20}$$

Take and fix $r \in (1, 1/p)$, and let Q be any disc. Then, for every $a \in \Delta$, we have

$$\begin{aligned}\left[\frac{1}{|Q|}\iint_Q V_a(z)\,dA(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{V_a(z)}\,dA(z)\right] &= \left[\frac{1}{|Q|}\iint_Q W(z)Y_a(z)\,dA(z)\right] \\ &\quad\times\left[\frac{1}{|Q|}\iint_Q \frac{1}{W(z)Y_a(z)}\,dA(z)\right] \\ &\le \left[\sup_{z\in Q} W(z)\right]\left[\iint_Q Y_a(z)\frac{dA(z)}{|Q|}\right] \\ &\quad\times\left[\iint_Q \frac{1}{W(z)^r}\frac{dA(z)}{|Q|}\right]^{1/r} \\ &\quad\times\left[\iint_Q \frac{1}{Y_a(z)^{r'}}\frac{dA(z)}{|Q|}\right]^{1/r'}.\end{aligned} \tag{21}$$

Now it can be easily proved by direct calculation that there exists a positive constant C such that

$$\left[\sup_{z\in Q} W(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{W(z)^r}\,dA(z)\right]^{1/r} \le C, \qquad \text{for any } Q.$$

Then (21) implies that, for every disc Q and every $a \in \Delta$,

$$\begin{aligned} &\left[\frac{1}{|Q|}\iint_Q V_a(z)\,dA(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{V_a(z)}\,dA(z)\right] \\ &\qquad \le C\left[\frac{1}{|Q|}\iint_Q Y_a(z)\,dA(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{Y_a(z)^{r'}}\,dA(z)\right]^{1/r'}. \end{aligned} \tag{22}$$

Next, we set

$$s = 1 + \frac{1}{r'},$$

(so that $s/s' = 1/r'$) and rewrite (22) as

$$\begin{aligned} &\left[\frac{1}{|Q|}\iint_Q V_a(z)\,dA(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{V_a(z)}\,dA(z)\right] \\ &\qquad \le C\left[\frac{1}{|Q|}\iint_Q Y_a(z)\,dA(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{Y_a(z)^{s'/s}}\,dA(z)\right]^{s/s'}. \end{aligned}$$

Together with (20) this yields

$$\left[\frac{1}{|Q|}\iint_Q V_a(z)\,dA(z)\right]\left[\frac{1}{|Q|}\iint_Q \frac{1}{V_a(z)}\,dA(z)\right] \le C\alpha_s,$$

for every disc Q and every $a \in \Delta$. Hence

$$A_2(V_a) \le C\alpha_s, \qquad \text{for every } a \in \Delta.$$

In view of (19), this gives (16) and finishes the proof. □

After this work had been completed, Professor Jie Xiao kindly informed us of his (unpublished) proof of Corollary 1, based on ideas different from ours.

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