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ON THE VISIBILITY OF INVISIBLE SETS

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Abstract. A set $A \subset \mathbf{R}^2$ is called invisible, if its orthogonal projection is of measure 0 in almost every direction; A is visible, if it is not invisible. We say that A is invisible from a point, if almost all lines through that point do not hit the set, except possibly for this point itself. P. Mattila raised the question whether the set of points from which an invisible set is visible is invisible. We answer this question in the negative.

Let $\mathscr{D}$ denote the set of the directions, i.e., the set of the lines through the origin with the usual topology.

A set $A \subset \mathbf{R}^2$ is called *invisible*, if its orthogonal projection is of measure 0 in almost every direction; A is *visible*, if it is not invisible. We say that A is *invisible from a point*, if almost all lines through that point do not hit the set, except possibly for this point itself. By a theorem of A.S. Besicovitch (see [2]) for sets with finite length invisibility is equivalent to pure unrectifiability; the latter means that it intersects every rectifiable curve in a set of length 0.

In 1954 J.M. Marstrand proved that the set of the points of the plane from which a given purely unrectifiable set of finite length is visible has Hausdorff dimension at most 1. He also gave an example that it can be 1 (see [4]).

P. Mattila in [5] proved that for an arbitrary invisible set (without assuming finite length) the set of the points from which this set is visible must be purely unrectifiable and it has capacity 0 related to the kernel $1/\|x\|$. He raised the question whether the set of points from which an invisible set is visible is invisible. We answer this question in the negative by proving the following theorem:

Theorem 1. *For every closed subset of directions $F \subset \mathscr{D}$ there exist a compact set $K \subset \mathbf{R}^2$ and a closed set $L \subset \mathbf{R}^2$ such that*

(A) *for every $e \in F$ the projection of K in the direction e is of positive measure;*
(B) *for every $e \notin F$ the projection of K in the direction e is of measure 0;*
(C) *the projection of L is of measure 0 in every direction;*
(D) *for every point $P \in K$ the set of the directions of lines connecting P and L is exactly $\mathscr{D} \setminus F$.*

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Hence, if both F and $\mathscr{D} \setminus F$ have positive measure then K is visible, L is invisible, and L is visible from every point of K.

The proof of the theorem is based on the so-called 'venetian blind' construction: for a given parallelogram T we can put narrow parallel parallelograms inside such that the measure of the projection of the union of these narrow parallelograms is small in some directions but it is the same as the projection of T in many other directions. (In this note 'parallelogram' always means a non-degenerate parallelogram or a point, it cannot be a segment.)

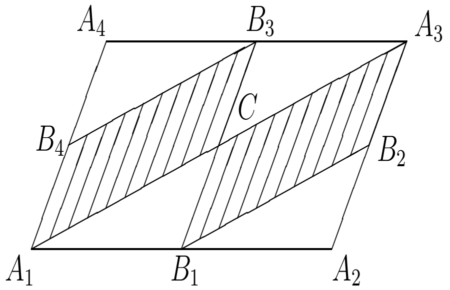

Figure 1. First step

By this method M. Talagrand proved in [8] that for every u.s.c. function f on $\mathscr{D}$ there exists a compact set $K \subset \mathbf{R}^2$ such that the measure of the projection of K in direction e is $f(e)$ for every $e \in \mathscr{D}$. The first part of our theorem (Corollary 1) is a special case of this theorem.

Similar parallelogram constructions were used by A.S. Besicovitch, P. Mattila and in higher dimensions by K.J. Falconer (see [1], [3], [6] and [7]).

Let $A_1A_2A_3A_4$ be a non-degenerate parallelogram, and let the midpoints of its edges be B_1, B_2, B_3, B_4, as in Figure 1. Let C denote the midpoint of the parallelogram. We consider the parallelograms $A_1CB_3B_4$ and $B_1B_2A_3C$, and we do the same for these parallelograms and iterate the process (see Figure 2).

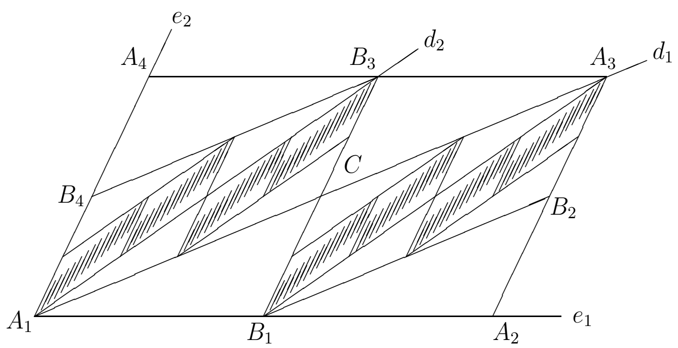

Figure 2. First three steps

Let e_1, e_2 denote the directions of the edges, and let $d_1, d_2, \ldots$ be the direction of the diagonals A_1A_3, A_1B_3, etc. It is clear that $d_n \to e_2$. Let the set of directions of lines through A_4 which intersect A_1A_3 be I, the set between e_1 and d_1 be II, the set between d_1 and d_2 be III, etc; (see Figure 3).

Now, every line of a direction in I which intersects $A_1A_2A_3A_4$ must intersect the diagonal A_1A_3, thus it intersects one of the parallelograms $A_1CB_3B_4$ and $B_1B_2A_3C$. Notice that the measure of the (non-orthogonal) projection of each parallelogram to the line A_1A_4 is at most $|A_1A_4|$ in every direction in II, thus the measure of the projection of the union of the two shaded parallelograms is at most $2 \cdot |A_1A_4|$.

Similarly, every line of a direction in I or II which intersects $A_1CB_3B_4$ or $B_1B_2A_3C$ must intersect one of the 4 parallelograms we get in the second step, and the measure of the projection of the union of these 4 parallelograms to the line A_1A_4 in the directions in III is at most $4 \cdot |A_1B_4| = 2 \cdot |A_1A_4|$.

In the nth step, lines of directions in the first n sets will intersect the union of the small parallelograms whenever they intersect one of the parallelograms in the previous step, and the measure of the (non-orthogonal) projection of the union of the small parallelograms to the line A_1A_4 is at most $2 \cdot |A_1A_4|$ in every direction in the $(n+1)$th set.

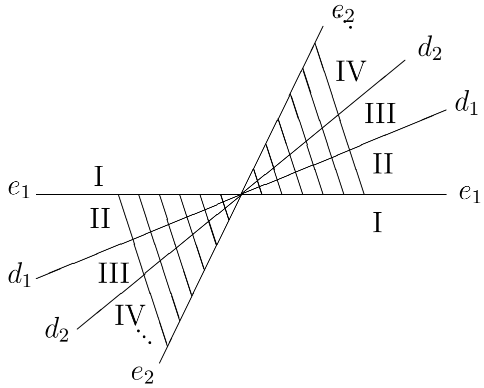

Figure 3. Directions

After these remarks it is easy to prove the following lemma:

Lemma 1. *Let* $\mathscr{I} = \{I^1, I^2, \ldots, I^m\}$ *be a finite set of pairwise disjoint intervals in* $\mathscr{D}$, *and let* $\mathscr{J} = \{J^1, J^2, \ldots, J^m\}$ *be a set of strict subintervals (i.e.,* $\mathrm{cl}\,(J^j) \subset \mathrm{int}\, I^j$ *for every* j*). Then for every parallelogram* T *and* $\varepsilon > 0$ *there exist non-overlapping parallelograms* $T_1, T_2, \ldots \subset T$ *such that*

(i) $U \overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} T_n$ *is compact;*
(ii) *for every* $e \in \cup \mathscr{J}$ *the orthogonal projection of* U *in direction* e *is of measure less than* ε;

(iii) *for every line l of direction $e \notin \cup\mathscr{I}$, if l intersects T then l intersects U, as well.*

Proof. We can assume that $m = 1$, i.e., we have only one interval I and a subinterval J. If T is a point, then the lemma is trivial. If T is a non-degenerate parallelogram, then we choose an interval (a, b) for which $\mathrm{cl}\,(J) \subset (a, b) \subset [a, b] \subset \mathrm{int}\, I$. Then it is possible to choose vertices of T and sequences of pairwise disjoint non-degenerate parallelograms in T (see Figure 4) such that

(i) the directions of the edges of all the sub-parallelograms are a and b;
(ii) every line of direction in $\mathscr{D} \setminus I$ which intersects T intersects some of our (degenerate or non-degenerate) sub-parallelograms, as well;
(iii) the sum of the edges of direction a is less than $\frac{1}{2}\delta$, where δ is a fixed small positive number.

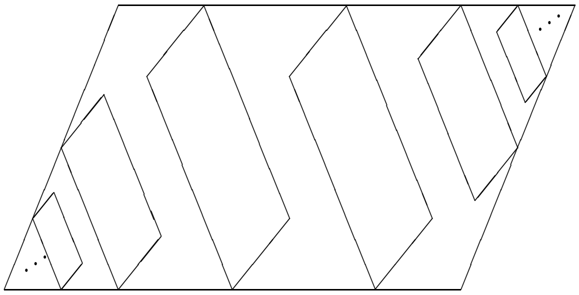

Figure 4. Sub-parallelograms

Then we apply our 'venetian blind' procedure in every sub-parallelogram. The result is a new system of sub-parallelograms, and the measure of the (non-orthogonal) projection of the union of these sub-parallelograms to a line of direction a in every direction from J is less than $2 \cdot \frac{1}{2}\delta = \delta$. Thus if δ is small enough then the measure of the orthogonal projection in every direction in J is less than ε. It is easy to see that it satisfies the other requirements, too. □

Now, the following two corollaries prove our theorem:

Corollary 1. *For every closed subset $F \subset \mathscr{D}$ there exists a compact set $K \subset \mathbf{R}^2$ satisfying assumptions* (A) *and* (B).

Proof. We choose a sequence of positive numbers $\{\varepsilon_n\}$ and sequences of finite families of intervals $\mathscr{I}_n = \{I_n^1, I_n^2, \dots, I_n^{m(n)}\}$, $\mathscr{J}_n = \{J_n^1, J_n^2, \dots, J_n^{m(n)}\}$ in $\mathscr{D}$, for which $\varepsilon_n \to 0$, $\cup\mathscr{J}_n \to F^c$, $I_n^j \cap F = \emptyset$ and $\mathrm{cl}\,(J_n^j) \subset \mathrm{int}\, I_n^j$ for every j and n. We apply our lemma for an arbitrary parallelogram T and ε_1, $\mathscr{I}_1$, $\mathscr{J}_1$. Then we apply our lemma for the sub-parallelograms with $\frac{1}{2}\varepsilon_2, \frac{1}{4}\varepsilon_2, \dots$ and $\mathscr{I}_2$, $\mathscr{J}_2$; then we apply the lemma for the sub-parallelograms with $\frac{1}{2}\varepsilon_3, \frac{1}{4}\varepsilon_3, \dots$ and $\mathscr{I}_3$, $\mathscr{J}_3$; etc. □

Corollary 2. *Let F be a closed subset in $\mathscr{D}$, and let K be a compact set satisfying assumptions* (A) *and* (B). *Then there exists a closed set $L \subset \mathbf{R}^2$ satisfying assumptions* (C) *and* (D).

Proof. We construct L by induction.

Let $\{\varepsilon_n\}$ be a sequence of positive numbers for which $\varepsilon_n \to 0$. In the 0th step we choose countably many pairwise disjoint non-degenerate parallelograms far away from K, such that for every point $P \in K$ the set of the directions of the lines connecting P and the parallelograms is exactly $\mathscr{D} \setminus F$.

In the nth step we divide all parallelograms we constructed in the previous step to finitely many parallelograms of diameter at most ε_n arbitrarily, let these small parallelograms be $T_1, T_2, \dots$. Let E_k be the closed set of the directions of the lines connecting K and T_k, and let the ε_n neighbourhood of E_k in $\mathscr{D}$ be E_k^*. Then the complement of E_k^* is the union of finitely many intervals, say $J_k^1, J_k^2, \dots, J_k^{m(k)}$. We choose pairwise disjoint intervals I_k^j for every $1 \le j \le m(k)$ for which $\operatorname{cl}(J_k^j) \subset \operatorname{int} I_k^j$ and $I_k^j \cap E_k = \emptyset$. Then we apply our lemma for every k with T_k, $\mathscr{I}_k = \{I_k^1, I_k^2, \dots, I_k^{m(k)}\}$, $\mathscr{J}_k = \{J_k^1, J_k^2, \dots, J_k^{m(k)}\}$ and $\varepsilon_n/2^k$.

Let U_n be the union of the parallelograms defined in the nth step, then it is immediate that $L \overset{\text{def}}{=} \bigcap_n U_n$ satisfies (D). For a given direction e let L^* denote the set of the points of L for which the line through the point in direction e does not hit K (thus if $e \in F$ then $L^* = L$). It is clear from the construction that the projection of L^* in direction e is of measure 0, thus if $e \in F$ then we are done. If $e \notin F$ then the projection of K in direction e is of measure 0, hence the projection of $L \setminus L^*$ is also of measure 0, as required. $\square$

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