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DEFINITIONS OF QUASIREGULARITY

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Abstract. We show that if $D \subset \mathbf{R}^n$ is open, $f\colon D \to \mathbf{R}^n$ is continuous, open and discrete such that for some $h > 0$ the linear dilatation satisfies $h(x, f) < h$ for every $x \in D$, then f is $K(h, n)$-quasiregular. Here "lim inf" is used in the definition for $h(x, f)$. A removability result on quasiregular mappings is obtained, and we enlarge the notion of the discrete modulus introduced by J. Heinonen and P. Koskela in [7]. We also give some bounds for the discrete modulus.

1. Introduction

In a recent paper [7], J. Heinonen and P. Koskela showed that if $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism such that for some $h > 0$, $h(x, f) < h$ holds for every $x \in \mathbf{R}^n$, then f is $K(h, n)$-quasiconformal. Here

$$\begin{aligned}
L(x, f, r) &= \sup_{|y-x|=r} |f(y) - f(x)|, \\
l(x, f, r) &= \inf_{|y-x|=r} |f(y) - f(x)|, \\
h(x, f) &= \liminf_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)}, \\
H(x, f) &= \limsup_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)}.
\end{aligned}$$

This generalizes the classical result which says that f is quasiconformal if $H(x, f)$ is uniformly bounded. In [3] we introduced the inferior linear dilatation $h(x, f)$ for mappings $f\colon D \to \mathbf{R}^n$, $D \subset \mathbf{R}^n$ open, f continuous, open and discrete, and we studied some properties of such mappings. We show now that if $D \subset \mathbf{R}^n$ is open, $n \ge 2$, $f\colon D \to \mathbf{R}^n$ is continuous, open and discrete such that there exists $h > 0$ such that $h(x, f) < h$ for every $x \in D$, then f is $K(h, n)$-quasiregular and $H(x, f) = h(x, f)$ a.e. in D. In this way, the problem of finding the best constant of quasiregularity $K(h, n)$ is reduced to the classical case when $H(x, f) < H$ for every $x \in D$.

A removability result of J. Heinonen and P. Koskela [7] shows that if $n \ge 2$, $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism such that there exists a closed set $H \subset \mathbf{R}^n$ such that f is K-quasiconformal on $\mathbf{R}^n \setminus H$ and $a > 1$ such that for every $x \in H$ there

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exists a sequence $r_j \to 0$ depending on x such that $\big(B(x, ar_j)\backslash \overline{B}(x, r_j)\big) \cap H = \emptyset$ for every $j \in \mathbf{N}$, then it follows that f is K-quasiconformal. We shall show that if $D \subset \mathbf{R}^n$ is open, $n \geq 2$, $f\colon D \to \mathbf{R}^n$ is continuous such that there exists $H \subset D$ closed in D such that f is K-quasiregular on $D \setminus H$ and there exists $0 < a < 1$ and $\mathscr{B} = (B_i)_{i \in \mathbf{N}}$ an a-porous base of H, then, if f is open and discrete or if $\operatorname{int} \mathrm{f(H)} = \emptyset$, it follows that f is K-quasiregular on D. Here, if $D \subset \mathbf{R}^n$ is open, $n \geq 2$, $A \subset D$, $0 < a < 1$, and $\mathscr{B} = (B_i)_{i \in \mathbf{N}}$ is a covering of A, we say that $\mathscr{B}$ is an a-porous covering of A if $(B_i \setminus a\overline{B}_i) \cap A = \emptyset$ for every $i \in \mathbf{N}$ and we say that $\mathscr{B}$ is an a-porous base of A if $\mathscr{B}$ is an a-porous covering of A and for every $x \in A$ and $U \in V(x)$, there exists $i \in \mathbf{N}$ such that $x \in B_i \subset U$.

We showed in [5] that if $D \subset \mathbf{R}^n$ is open, $n \geq 2$, $f\colon D \to \mathbf{R}^n$ is continuous, open and discrete such that there exists $0 < a \leq 1$ and $H > 0$ with $\limsup_{r \to 0} d\big(f\big(B(x, ar)\big)\big)^n / \mu_n\big(f\big(B(x, r)\big)\big) \leq H$ for every $x \in D$, then f is $K(H, n, a)$-quasiregular. We shall partially generalize this result, giving at the same time a generalization of the removability result of J. Heinonen and P. Koskela from [7]. We show that if $D \subset \mathbf{R}^n$ is open, $n \geq 2$, $f\colon D \to \mathbf{R}^n$ is continuous, open and discrete, there exists $K \subset D$ and $h, t > 0$ such that for every $x \in D \backslash K$, $\liminf_{r \to 0} d\big(f\big(B(x, r)\big)\big)^n / \mu_n\big(f\big(B(x, r)\big)\big) < h$, and there exists $0 < a < 1$ and $\mathscr{B} = (B_i)_{i \in \mathbf{N}}$ an a-porous base of K such that $d\big(f(aB_i)\big)^n / \mu_n\big(f(B_i)\big) < t$ for every $i \in \mathbf{N}$, then it follows that f is $K(h, t, n)$-quasiregular and $K_0(f) < V_n \cdot H$, where $H = \max\{h, t\}$.

As in [7], the principal instrument used in proving such theorems is the new discrete modulus introduced by J. Heinonen and P. Koskela. We give a slightly modified version of this notion, and obtain bounds for this class of discrete modulus. For these bounds we modify some covering theorems from [10] and, for the sake of completeness, we give a detailed proof of one of them (Theorem 3).

We use the notation from [15], [9] and [12]. If $f\colon D \to \mathbf{R}^n$ is a map, $D \subset \mathbf{R}^n$ is open, $x \in D$, $U(x, f, r)$ is the component of $f^{-1}\big(B\big(f(x), r\big)\big)$ containing x. For $D \subset \mathbf{R}^n$ open and $f\colon D \to \mathbf{R}^n$ a map, we say that f is discrete if $f^{-1}(y)$ is isolated for every $y \in \mathbf{R}^n$, and we say that f is light if $f^{-1}(y)$ is a set of topological dimension zero for every $y \in \mathbf{R}^n$.

For $x \in \mathbf{R}^n$ we denote by $|x| = \big(\sum_{i=1}^n x_i^2\big)^{1/2}$, and if $A, B \subset \mathbf{R}^n$, we denote by $d(A) = \sup_{x, y \in A} |x - y|$ and by $d(A, B) = \inf_{x \in A,\, y \in B} |x - y|$. If $A \in L(\mathbf{R}^n, \mathbf{R}^n)$, let $|A| = \sup_{|h|=1} |A(h)|$, and $l(A) = \inf_{|h|=1} |A(h)|$.

If $D \subset \mathbf{R}^n$ is a domain, $n \geq 2$, and $f\colon D \to \mathbf{R}^n$ is a map, we say that f is quasiregular if f is ACL^n and there exists $K \geq 1$ such that $|f'(x)|^n \leq K \cdot J_f(x)$ a.e. in D. For a quasiregular map we also have $J_f(x) \leq K' \cdot l\big(f'(x)\big)^n$ a.e. in D for some $K' \geq 1$, and we denote by $K_0(f)$ the smallest $K \geq 1$ such that $|f'(x)|^n \leq K \cdot J_f(x)$ a.e. in D, and by $K_I(f)$ the smallest $K \geq 1$ such that $J_f(x) \leq K \cdot l\big(f'(x)\big)^n$ a.e. in D. We say that a quasiregular map $f\colon D \to \mathbf{R}^n$, $n \geq 2$, is K-quasiregular if $K_0(f) \leq K$, $K_I(f) \leq K$.

If Γ is a path family, we denote by

$$F(\Gamma) = \left\{\rho\colon \overline{\mathbf{R}}^n \to [0,\infty] \text{ a Borel map : } \int_\gamma \rho\, ds \geq 1 \text{ for every } \gamma \in \Gamma\right\}$$

and by $M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbf{R}^n} \rho^n(x)\, dx$.

If $E, F \subset \mathbf{R}^n$, $D \subset \mathbf{R}^n$ is open such that $E \cup F \subset \overline{D}$, let $\Delta(E,F,D)$ be those paths, open or not, which join E with F in D. If $f\colon D \to \mathbf{R}^n$ is a map, we let $N(f,D) = \sup_{y \in \mathbf{R}^n} \operatorname{Card} f^{-1}(y)$. A domain $A \subset \overline{\mathbf{R}}^n$ is a ring if CA has exactly two components, C_0 and C_1, and we denote $A = R(C_0, C_1)$, C_0 being bounded, and $\Gamma_A = \Delta(C_0, C_1, A)$. For $r > 0$, let $\Phi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\overline{\mathbf{R}}^n$ such that $0 \in C_0$ and there exists $a \in C_0$ with $|a| = 1$ and such that $\infty \in C_1$, and there exists $b \in C_1$ with $|b| = r$, and we let $\mathscr{H}(r) = \inf M(\Gamma_A)$, the infimum being taken over all rings $A \in \Phi_n(r)$. We denote by V_n the volume of the unit ball in $\mathbf{R}^n$, by μ_n the Lebesgue measure in $\mathbf{R}^n$ and by ω_{n-1} the area of the unit sphere from $\mathbf{R}^n$. For $0 < a < \lambda < 1$, we write $N(a,\lambda) = 2K \cdot \big(4/\lambda \cdot (\lambda - a)\big)^n$, where $K = [\log \frac{1}{2}(1-a)/\log\lambda] + 1$. For $\varepsilon > 0$ we let $N(\varepsilon) = (3+\varepsilon)^n \cdot \big(\frac{1}{2}(2+\varepsilon)\big)^n$. In [10, p. 29] the following lemma is proved: There exists a positive integer $B(n)$ such that if $a_1, \dots, a_k \in \mathbf{R}^n$, $r_1, \dots, r_k \in (0,\infty)$, $a_i \notin B(a_j, r_j)$ for $i \neq j$, $i,j = 1, \dots, k$ and $\bigcap_{i=1}^k B(a_i, r_i) \neq \emptyset$, then it follows that $k \leq B(n)$. Throughout this paper we shall denote by $B(n)$ the number from this lemma.

2. Preliminaries

Theorem 1. *Let $D \subset \mathbf{R}^n$ be open, $A \subset D$ closed in D and such that A is bounded or $A = D$, $\mathscr{B}$ a collection of balls from D such that every ball from $\mathscr{B}$ is of the form $B(x,r)$ with $x \in A$ and $r \in I_x \subset \mathbf{R}$ with*

(a) *$0 \in I_x'$ for every $x \in A$,*

(b) *if $x_k \in A$, $r_{x_k} \in I_{x_k}$ for $k \in \mathbf{N}$, $x_k \to x$, $r_{x_k} \to r$ and $\overline{B}(x,r) \subset D$, then $r \in I_x$.*

Then there exists an at most countable collection of balls $(B_i)_{i \in \mathbf{N}}$ such that $B_i = B(x_i, r_i) \in \mathscr{B}$ for $i \in \mathbf{N}$, $A \subset \bigcup_{i=1}^\infty B_i$, $r_k \geq r_{k+1}$ and $x_k \notin \bigcup_{i=1, i\neq k}^\infty B_i$ for $k \geq 1$. Hence every point from A belongs to at most $B(n)$ balls and $\frac{1}{2}B_i \cap \frac{1}{2}B_j = \emptyset$ for $i,j \in \mathbf{N}$, $i \neq j$.

Theorem 2. *Let $D \subset \mathbf{R}^n$ be open, $A \subset D$ such that either A is bounded or $A = D$, $\mathscr{B}$ a collection of balls from D such that every ball $B \in \mathscr{B}$ is of the form $B = B(x,r)$ with $x \in A$ and $r \in I_x \subset \mathbf{R}_+$ such that $0 \in I_x'$ for every $x \in A$. Then, for every $\varepsilon > 0$ there exists an at most countable collection of balls $(B_i)_{i \in \mathbf{N}}$ such that $B_i = B(x_i, r_i) \in \mathscr{B}$ for $i \in \mathbf{N}$, $A \subset \bigcup_{i=1}^\infty B_i$, $B_i/(2+\varepsilon) \cap B_j/(2+\varepsilon) = \emptyset$ for $i,j \in \mathbf{N}$, $i \neq j$ and every point from A belongs to at most $N(\varepsilon) \cdot B(n)$ balls B_i.*

Theorem 3. *Let $D \subset \mathbf{R}^n$ be open, $A \subset D$, $0 < a < 1$ and $\mathscr{B} = (B_i)_{i\in\mathbf{N}}$ an a-porous covering of A. Then, if A is bounded or if $A = D$ and $\mathscr{B}$ is an a-porous base of A, for every $0 < a < \lambda < 1$ we can find $\mathscr{C} = (B_{i_k})_{k\in\mathbf{N}}$ a subcovering of A such that every point from A belongs to at most $N(a,\lambda)$ balls B_{i_k} and $\frac{1}{2}(\lambda - a)B_{i_k} \cap \frac{1}{2}(\lambda - a)B_{i_l} = \emptyset$ for $k, l \in \mathbf{N}$, $k \neq l$, where $N(a,\lambda) = 2k \cdot (4/\lambda(\lambda - a))^n$, with $k = [\log \frac{1}{2}(1-a)/\log\lambda] + 1$.*

Proof. We suppose first that A is bounded. Hence we can suppose that $M = \sup_{i\in\mathbf{N}} r_i < \infty$, where $B_i = B(x_i, r_i)$ for $i \in \mathbf{N}$. Let $0 < a < \lambda < 1$ and

$$A_1 = \{x \in A \,|\text{there exists } i \in \mathbf{N} \text{ and } \lambda M < r_i \leq M \\ \text{such that } B(x_i, r_i) \in \mathscr{B} \text{ and } x \in B(x_i, r_i)\}.$$

If $A_1 \neq \emptyset$, we take $x_{11} \in D$ and $\lambda M < \rho_{11} \leq M$ such that $B(x_{11}, \rho_{11}) \in \mathscr{B}$ and $A \cap B(x_{11}, \rho_{11}) \neq \emptyset$. If $A_1 \not\subset B(x_{11}, \rho_{11})$, we take $x_{12} \in D$ and $\lambda M < \rho_{12} \leq M$ such that $B(x_{12}, \rho_{12}) \in \mathscr{B}$ and $\big(A_1 \setminus B(x_{11}, \rho_{11})\big) \cap B(x_{12}, \rho_{12}) \neq \emptyset$. If $A_1 \not\subset B(x_{11}, \rho_{11}) \cup B(x_{12}, \rho_{12})$, we continue the process and we show that this process must end in a finite number of steps. Indeed, let $B(x_i, r_i)$, $B(x_j, r_j)$ be two balls which cover A_1 obtained as before. Then there exists a point $x \in A$ such that $x \in B(x_j, r_j) \setminus B(x_i, r_i)$, hence $|x - x_j| \leq a r_j \leq aM$ (because $\big(B(x_j, r_j) \setminus \overline{B}(x_j, a r_j)\big) \cap A = \emptyset$) and $|x - x_i| > r_i \geq \lambda M$, which implies that

$$|x_i - x_j| \geq |x - x_i| - |x - x_j| > (\lambda - a)M.$$

Let $\rho = \frac{1}{2}(\lambda - a)$ and suppose that there exists a point $z \in B(x_j, \rho r_j) \cap B(x_i, \rho r_i)$. Then $2\rho M \geq \rho r_j + \rho r_i \geq |x_j - z| + |x_i - z| \geq |x_i - x_j| > (\lambda - a)M$, which represents a contradiction. It follows that $B\big(x_i, \frac{1}{2}(\lambda - a)r_i\big) \cap B\big(x_j, \frac{1}{2}(\lambda - a)r_j\big) = \emptyset$. Using the boundedness of A we can find $m(1) \in \mathbf{N}, x_{11}, \ldots, x_{1m(1)} \in D$, real numbers $\rho_{11}, \ldots, \rho_{1m(1)} \in (\lambda M, M]$ such that $B(x_{1l}, \rho_{1l}) \in \mathscr{B}$ for $l = 1, \ldots, m(1)$, $\big(A_1 \setminus \bigcup_{l=1}^{k-1} B(x_{1l}, \rho_{1l})\big) \cap B(x_{1k}, \rho_{1k}) \neq \emptyset$ for $k = 2, \ldots, m(1)$ and $A_1 \subset \bigcup_{l=1}^{m(1)} B(x_{1l}, \rho_{1l})$. Also, a point from A_1 may belong to at most $m = \big(4/\lambda(\lambda - a)\big)^n$ balls $B(x_{1l}, \rho_{1l})$. Indeed, let $B(x_{1i}, \rho_{1i})$ be a fixed ball of this type and suppose that it is intersected by m balls $B(x_{1j}, r_{1j})$, $j \in C \subset \{1, \ldots, m(1)\}$. Then every such ball is contained in $B(x_{1i}, 2M)$; hence

$$\begin{aligned} V_n \cdot (2M)^n = \mu_n\big(B(x_{1i}, 2M)\big) &\geq \sum_{j\in C} \mu_n\bigg(\Big(B(x_{1j}, \frac{1}{2}(\lambda - a)\rho_{1j}\Big)\bigg) \\ &\geq V_n \cdot m \cdot \frac{1}{2^n}\big((\lambda - a) \cdot \lambda M)\big)^n \end{aligned}$$

and this implies that $m \leq \big(4/\lambda(\lambda - a)\big)^n$. We have completed the first step of our inductive process.

At step j, we take

$$A_j = \left\{ x \in A \setminus \bigcup_{k=1}^{j-1} \bigcup_{l=1}^{m(k)} B(x_{1l}, \rho_{1l}) \mid \text{ there exists } i \in \mathbf{N} \right.$$
$$\left. \text{and } \lambda^j M < r_i \leq \lambda^{j-1} \cdot M \text{ such that } B(x_i, r_i) \in \mathscr{B} \text{ and } x \in B(x_i, r_i) \right\}.$$

If $A_j \neq \emptyset$, we take $x_{j1} \in A_j$ and $\lambda^j \cdot M < \rho_{j1} \leq \lambda^{j-1} \cdot M$ such that $B(x_{j1}, \rho_{j1}) \in \mathscr{B}$ and $A_j \cap B(x_{j1}, \rho_{j1}) \neq \emptyset$. If $A_j \not\subset B(x_{j1}, \rho_{j1})$, we continue the process from step j, and if $A_j \not\subset \bigcup_{l=1}^{k-1} B(x_{jl}, \rho_{jl})$, we take $x_{jk} \in D$ and $\lambda^j \cdot M < \rho_{jk} \leq \lambda^{j-1} \cdot M$ such that $B(x_{jk}, \rho_{jk}) \in \mathscr{B}$ and $\bigl(A_j \setminus \bigcup_{l=1}^{k-1} B(x_{jl}, \rho_{jl})\bigr) \cap B(x_{jk}, \rho_{jk}) \neq \emptyset$.

Using the boundedness of A_j, we see as in step 1 that this process must end in a finite number of steps. Hence we find $m(j) \in \mathbf{N}$, $x_{j1}, \dots, x_{jm(j)} \in D$, $\rho_{j1}, \dots, \rho_{jm(j)} \in (\lambda^j \cdot M, \lambda^{j-1} \cdot M]$ such that $B(x_{jl}, \rho_{jl}) \in \mathscr{B}$, $l = 1, \dots, m(j)$,

$$\left(A_j \setminus \bigcup_{l=1}^{k-1} B(x_{jl}, \rho_{jl}) \right) \cap B(x_{jk}, \rho_{jk}) \neq \emptyset$$

for $k = 1, \dots, m(j)$ and $A_j \subset \bigcup_{l=1}^{m(j)} B(x_{jl}, \rho_{jl})$.

The process will have an infinite number of steps. We show that $A \subset \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{m(j)} B(x_{jl}, \rho_{jl})$. Indeed, if this is not true, we can find a point $x \in A \setminus \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{m(j)} B(x_{jl}, \rho_{jl})$. Since $(B_i)_{i \in \mathbf{N}}$ is a covering of A, we can find $i \in \mathbf{N}$ such that $x \in B_i$ and $j \in \mathbf{N}$ with $\lambda^j \cdot M < r_i \leq \lambda^{j-1} \cdot M$. Using the definition of A_j, we obtain that $x \in A_j$, which represents a contradiction, since $x \notin \bigcup_{l=1}^{m(j)} B(x_{jl}, \rho_{jl})$ and we proved that $A_j \subset \bigcup_{l=1}^{m(j)} B(x_{jl}, \rho_{jl})$. It follows that $A \subset \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{m(j)} B(x_{jl}, \rho_{jl})$.

Let now $k \in \mathbf{N}$ be such that $a \leq -2\lambda^k + 1$. We show that two balls $B(x_{il}, \rho_{il})$, $B(x_{jq}, \rho_{jq})$, $l \in \{1, \dots, m(i)\}$, $q \in \{1, \dots, m(j)\}$ with $j - i - 1 \geq k$ cannot have a common point from A. Indeed, suppose that this is not true and pick two balls as before, $B(x_{il}, \rho_{il}), B(x_{jq}, \rho_{jq})$, with $j - i - 1 \geq k$ such that there exists a point $x \in A \cap B(x_{il}, \rho_{il}) \cap B(x_{jq}, \rho_{jq})$. From the construction of the balls $B(x_{kl}, \rho_{kl})$ we can find a point $z \in \bigl(A \setminus B(x_{il}, \rho_{il})\bigr) \cap B(x_{jq}, \rho_{jq})$, and since $A \cap \bigl(B(x_{il}, \rho_{il}) \setminus \overline{B}(x_{il}, a \cdot \rho_{il})\bigr) = \emptyset$, we see that the point $x \in B(x_{il}, a \cdot \rho_{il})$. Then $2M \cdot \lambda^{j-1} \geq 2\rho_{jq} \geq |x - z| \geq (1 - a)\rho_{il} > (1 - a)M \cdot \lambda^i$, and hence $\frac{1}{2}(1 - a) < \lambda^{j-i-1} < \lambda^k$, i.e., $a > -2\lambda^k + 1$, which contradicts the way we chose k. It follows that if $j - i - 1 \geq k$, then any two balls $B(x_{il}, \rho_{il})$, $B(x_{jq}, \rho_{jq})$ with $j - i - 1 \geq k$ cannot have a common point with A. Since we showed that any point from A_j can belong to at most $\bigl(4/\lambda(\lambda - a)\bigr)^n$ balls $B(x_{jl}, \rho_{jl})$, $j \in \mathbf{N}$, $l = 1, \dots, m(j)$, it follows that any point from A can belong to at most $2k \cdot \bigl(4/\lambda(\lambda - a)\bigr)^n$ balls $B(x_{il}, \rho_{il})$, $i \in \mathbf{N}$, $l \in \{1, \dots, m(i)\}$, where $k = [\log \frac{1}{2}(1 - a)/\log \lambda] + 1$. As in step 1, we show that $\frac{1}{2}(\lambda - a) \cdot B(x_{il}, \rho_{il}) \cap \frac{1}{2}(\lambda - a) \cdot B(x_{jq}, \rho_{jq}) = \emptyset$, for $i \neq j$, $l \in \{1, \dots, m(i)\}$, $q \in \{1, \dots, m(j)\}$ and the theorem is proved if A is bounded.

If $A = D$ and $\mathscr{B}$ is an a-porous base of A, we leave the proof to the reader.

3. Discrete modulus. Bounds for discrete modulus

We shall use, now in a slightly enlarged form, the concept of the discrete modulus introduced by J. Heinonen and P. Koskela in [7]. If $D \subset \mathbf{R}^n$ is open and $\mathscr{B} = (B_i)_{i\in\mathbf{N}}$ is a covering with balls of D, we say that $\mathscr{B}$ is a (λ,μ)-covering of D if every point of D belongs to at most λ balls B_i and $\mu B_i \cap \mu B_j = \emptyset$ for $i,j \in \mathbf{N}$, $i \neq j$ and $0 < \mu < 1$. From Theorem 1, we always have $(B(n), \frac{1}{2})$ coverings of an open set $D \subset \mathbf{R}^n$; hence we always have (λ,μ)-coverings of an open set $D \subset \mathbf{R}^n$ if $\lambda \geq B(n)$, $0 < \mu \leq \frac{1}{2}$, and we denote by $E(\lambda,\mu,D)$ all the (λ,μ)-coverings of the open set D from $\mathbf{R}^n$. If $\mathscr{B} = \mathscr{B}^1 \cup \cdots \cup \mathscr{B}^m$ is a covering with balls of D, $0 < \mu < 1$, D open in $\mathbf{R}^n$, $\mathscr{B}^k = (B_i^k)_{i\in\mathbf{N}}$, $k = 1,\dots,m$, such that every point from D belongs to at most λ balls B_i^k for $k = 1,\dots,m$, and for every $k \in \{1,\dots,m\}$, we have $\mu B_i^k \cap \mu B_j^k = \emptyset$, $i,j \in \mathbf{N}$, $i \neq j$, we say that $\mathscr{B}$ is a (m,λ,μ)-covering of D and we denote by $E(m,\lambda,\mu,D)$ all the (m,λ,μ)-coverings of D. In this definition it is possible that some families of balls $\mathscr{B}^i$, $\mathscr{B}^j$ are the same. If Γ is a path family such that $\operatorname{Im}\gamma \subset D$ for every $\gamma \in \Gamma$ and $\mathscr{B} = (B_i)_{i\in\mathbf{N}}$ is a collection of balls from D, a subcollection $\mathscr{A} = (B_i)_{i\in I}$ with $I \subset \mathbf{N}$ will be called a chain of balls from the collection $\mathscr{B}$ along the path γ if $\operatorname{Im}\gamma \cap B_i \neq \emptyset$ for every $i \in I$ and $\operatorname{Im}\gamma \subset \bigcup_{i\in I} B_i$.

Let now $m,\lambda \in \mathbf{N}$, $0 < \mu < 1$, $\mathscr{B} = \mathscr{B}^1 \cup \cdots \mathscr{B}^m, \mathscr{B} \in E(m,\lambda,\mu,D)$ and $v_k \colon \mathscr{B}^k \to [0,\infty)$ be set functions which assign to each ball $B_i^k \in \mathscr{B}^k$ a positive number for $k = 1,\dots,m$. We say that $v = (v_1,\dots,v_m)$ is (m,λ,μ,D)-admissible for the (m,λ,μ) covering of D,

$$\mathscr{B} = (B_i)_{i\in\mathbf{N}}, \quad \mathscr{B} = \mathscr{B}^1 \cup \cdots \cup \mathscr{B}^m, \quad \mathscr{B}^k = (B_i)_{i\in I_k}, \quad k = 1,\dots,m$$

and for the path family Γ such that $\operatorname{Im}\gamma \subset D$ for every $\gamma \in \Gamma$, if

$$\sum_{k=1}^m \sum_{i\in I_k\cap I} v_k(B_i) \geq 1,$$

for every $\mathscr{A} = (B_i)_{i\in I}$ chain of balls along some path $\gamma \in \Gamma$. We denote by $F_{(m,\lambda,\mu,D)}(\mathscr{B},\Gamma)$ the class of all (m,λ,μ,D)-admissible m-tuples of set functions $v = (v_1,\dots,v_m)$ for the collection $\mathscr{B} \in E(m,\lambda,\mu,D)$ and the path family Γ. For $\delta > 0$ we denote by

$$\delta - \operatorname{Mod}_{(m,\lambda,\mu)}(\Gamma) = \inf \sum_{\mathscr{B}\in E(m,\lambda,\mu,D)} \sum_{v=(v_1,\dots,v_m)\in F_{(m,\lambda,\mu,D)}(\mathscr{B},\Gamma)} \sum_{k=1}^m \sum_{i\in I_k} v_k(B_i)^n,$$

where the infimum is taken along all the collections

$$\mathscr{B} \in E(m,\lambda,\mu,D), \quad \mathscr{B} = \mathscr{B}^1 \cup \cdots \cup \mathscr{B}^m, \quad \mathscr{B} = (B_i)_{i\in\mathbf{N}}, \quad \mathscr{B}^k = (B_i)_{i\in I_k},$$

$k = 1,\dots,m$, and all the (m,λ,μ,D)-admissible m-tuples of set functions $v = (v_1,\dots,v_m) \in F_{(m,\lambda,\mu,D)}(\mathscr{B},\Gamma)$, D being any open set in $\mathbf{R}^n$ such that $\operatorname{Im}\gamma \subset D$

for every $\gamma \in \Gamma$ and $d(B_i) < \delta$ for $i \in \mathbf{N}$. The function $\delta \to \delta - \mathrm{Mod}_{(m,\lambda,\mu)}(\mathscr{B}, \Gamma)$ is clearly decreasing; hence it has a limit in 0 and we denote by $\mathrm{Mod}_{(m,\lambda,\mu)}(\Gamma) = \lim_{\delta \to 0} \delta - \mathrm{Mod}_{(m,\lambda,\mu)}(\Gamma)$ and call the number $\mathrm{Mod}_{(m,\lambda,\mu)}(\Gamma)$ the (m, λ, μ) discrete modulus of the path family Γ. Of course, this notion has sense if $E(m, \lambda, \mu, D) \neq \emptyset$. For $\lambda \geq B(n)$ and $0 < \mu \leq \frac{1}{2}$, $m \in \mathbf{N}$, the (m, λ, μ) discrete modulus is defined. For $m = 1$ we note $\mathrm{Mod}_{(\lambda,\mu)}(\Gamma)$ instead of $\mathrm{Mod}_{(1,\lambda,\mu)}(\Gamma)$ and by $F_{(\lambda,\mu,D)}(\mathscr{B}, \Gamma)$ instead of $F_{(1,\lambda,\mu,D)}(\mathscr{B}, \Gamma)$ for $\mathscr{B} \in E(\lambda, \mu, D)$.

Using the method from [7], we can prove the following relation between the classical modulus and the (m, λ, μ) discrete modulus:

Theorem 4. *Let Γ be a path family in $\mathbf{R}^n$. Then there exists a constant $R(n, m, \mu)$ such that $M(\Gamma) \leq R(n, m, \mu) \cdot \mathrm{Mod}_{(m,\lambda,\mu)}(\Gamma)$, where $R(n, m, \mu) = V_n \cdot 2^n \cdot n^{n^2} \big(3^n / (\mu(n-1))^n \cdot m \cdot n\big)^{n-1}$.*

Theorem 5. *Let $n \geq 2$, $D \subset \mathbf{R}^n$ be open, $K \subset D$, $f \colon D \to \mathbf{R}^n$ continuous such that there exists $h > 0$ with*

$$\liminf_{r \to 0} \frac{d\big(f(B(x, r))\big)^n}{\mu_n\big(f(B(x, r))\big)} < h$$

for every $x \in D \setminus K$ and $N(f, D) < \infty$. If $K = \emptyset$, for $E, F \subset \overline{D}$, we have

$$\mathrm{Mod}_{(B(n),1/2)}\big(\Delta(E, F, D)\big) \leq h \cdot N(f, D) \cdot B(n) \cdot \frac{\mu_n\big(f(D)\big)}{d\big(f(E), f(F)\big)^n}.$$

If $K \neq \emptyset$, D is bounded and there exists $t > 0$ and $0 < \alpha \leq 1$ such that for an α-porous base of K, $\widetilde{\mathscr{B}} = \big(B(x_i, r_i)\big)_{i \in \mathbf{N}}$ the inequality

$$\frac{d\big(f(\alpha \cdot B(x_i, r_i))\big)^n}{\mu_n\big(f(B(x_i, r_i))\big)} < t$$

holds for $i \in \mathbf{N}$, then, for every $\varepsilon > 0$ and $0 < \alpha < \lambda < 1$ it follows that

$$\mathrm{Mod}_{(2,M,P)}\big(\Delta(E, F, D)\big) \leq 2H \cdot N(f, D) \cdot M \cdot \frac{\mu_n\big(f(D)\big)}{d\big(f(E), f(F)\big)^n},$$

where

$$P = \min\left\{\frac{1}{2+\varepsilon}, \frac{\lambda - \alpha}{2}\right\}, \quad H = \max\{h, t\}, \quad M = \max\{B(n) \cdot N(\varepsilon), N(\alpha, \lambda)\}.$$

Proof. Suppose first that $K = \emptyset$. We may assume that $r = d\big(f(E), f(F)\big) > 0$, since otherwise the theorem is clear. If $x \in D$, let

$$I_x = \left\{ r \in \mathbf{R}_+ \;\middle|\; \frac{d\big(f(B(x, r))\big)^n}{\mu_n\big(f(B(x, r))\big)} < h \right\}.$$

Then $0 \in I'_x$, and let $\mathscr{B}^1 = \bigcup_{x \in D, r \in I_x} B(x,r)$. We define for $B \in \mathscr{B}^1$ a set function v by $v(B) = d\big(f(B)\big)/d\big(f(E), f(F)\big)$. Now $\mathscr{B}^1$ satisfies the conditions from Theorem 1, hence, if $\delta > 0$ is fixed, we can find $\mathscr{B} \in E\big(B(n), \frac{1}{2}, D\big)$, $\mathscr{B} = (B_i)_{i \in \mathbf{N}}$, a subcollection of $\mathscr{B}^1$ such that $d(B_i) < \delta$, $\overline{B}_i \subset D$, $d\big(f(B_i)\big)^n/\mu_n\big(f(B_i)\big) < h$ for $i \in \mathbf{N}$.

Let $\gamma \in \Delta(E,F,D)$ and $\mathscr{A} = (B_i)_{i \in I}$ be a chain of balls from the collection $\mathscr{B}$ along the path γ. Then $\bigcup_{i \in I} f(B_i)$ is a connected set which covers the path $\Gamma = f \circ \gamma$, hence $\sum_{i \in I} d\big(f(B_i)\big) \geq l(\Gamma) \geq d\big(f(E), f(F)\big)$, and this implies that $\sum_{i \in I} v(B_i) \geq 1$ for every $\mathscr{A} = (B_i)_{i \in I}$ chain of balls from the collection $\mathscr{B}$ along every path $\gamma \in \Delta(E,F,D)$, i.e., $v \in F_{(B(n),1/2,D)}\big(\mathscr{B}, \Delta(E,F,D)\big)$.

We have

$$\text{(1)} \qquad \delta - \mathrm{Mod}_{(B(n),1/2)}\big(\Delta(E,F,D)\big) \leq \sum_{i=1}^{\infty} v(B_i)^n \leq \sum_{i=1}^{\infty} \frac{d\big(f(B_i)\big)^n}{d\big(f(E), f(F)\big)^n}.$$

Now every point $y \in f(D)$ has at most $N(f,D)$ points $x_k \in D$ such that $f(x_k) = y$, and since such a point x_k may belong to at most $B(n)$ balls B_i, it follows that every point $y \in f(D)$ belongs to at most $B(n) \cdot N(f,D)$ sets $f(B_i)$. This implies that

$$\begin{aligned} \sum_{i=1}^{\infty} \mu_n\big(f(B_i)\big) &= \sum_{i=1}^{\infty} \int_{f(D)} \mathscr{X}_{f(B_i)}(x)\, dx = \int_{f(D)} \Big(\sum_{i=1}^{\infty} \mathscr{X}_{f(B_i)}\Big)(x)\, dx \\ &\leq \int_{f(D)} B(n) \cdot N(f,D)\, dx = B(n) \cdot N(f,D) \cdot \mu_n\big(f(D)\big), \end{aligned}$$

and using (1), we have

$$\begin{aligned} \delta - \mathrm{Mod}_{(B(n),1/2)}\big(\Delta(E,F,D)\big) &\leq \sum_{i=1}^{\infty} \frac{d\big(f(B_i)\big)^n}{d\big(f(E), f(F)\big)^n} \\ &\leq h \cdot \sum_{i=1}^{\infty} \frac{\mu_n\big(f(B_i)\big)}{d\big(f(E), f(F)\big)^n} \\ &\leq h \cdot B(n) \cdot N(f,D) \cdot \frac{\mu_n\big(f(D)\big)}{d\big(f(E), f(F)\big)^n}. \end{aligned}$$

Letting δ tend to zero, we obtain that

$$\mathrm{Mod}_{(B(n),1/2)}\big(\Delta(E,F,D)\big) \leq h \cdot B(n) \cdot N(f,D) \cdot \frac{\mu_n\big(f(D)\big)}{d\big(f(E), f(F)\big)^n}.$$

Suppose now that $K \neq \emptyset$ and let $0 < \alpha < \lambda < 1$ and $\varepsilon > 0$ and let $\delta > 0$. Applying Theorem 2 to the base of balls $\mathscr{D} = \bigcup_{x \in D \setminus K, r \in I_x} B(x,r)$, we can consider $\mathscr{B}^1 = (B_i^1)_{i\in\mathbf{N}}$ a subcovering of $\mathscr{D}$ which covers $D\setminus K$ such that $d(B_i^1)<\delta$, $\overline{B}_i^1\subset D$, $d\big(f(B_i^1)\big)^n/\mu_n\big(f(B_i^1)\big)<h$ for $i\in\mathbf{N}$ and every point from $D\setminus K$ belongs to at most $N(\varepsilon)\cdot B(n)$ balls B_i^1 and $B_i^1/(2+\varepsilon)\cap B_j^1/(2+\varepsilon)=\emptyset$ for $i\neq j$, $i,j\in\mathbf{N}$.

Applying Theorem 3 to the α-porous covering of K, $\mathscr{C}=\big(B(x_i,r_i)\big)_{i\in\mathbf{N}}$, we can consider $\mathscr{B}^2=(B_i^2)_{i\in\mathbf{N}}$ a subcollection of $\mathscr{C}$ which covers K such that

$$d(B_i^2)<\delta,\qquad \overline{B}_i^2\subset D,\qquad \frac{d\big(f(\alpha B_i^2)\big)^n}{\mu_n\big(f(B_i^2)\big)}<t$$

for $i\in\mathbf{N}$, and every point from K belongs to at most $N(\alpha,\lambda)$ balls B_i^2 and $\frac{1}{2}(\lambda-\alpha)\cdot B_i^2\cap\frac{1}{2}(\lambda-\alpha)\cdot B_j^2=\emptyset$ for $i\neq j$, $i,j\in\mathbf{N}$. Since $(B_i^2-\alpha\overline{B}_i^2)\cap K=\emptyset$ for $i\in\mathbf{N}$, we see that $(\alpha B_i^2)_{i\in\mathbf{N}}$ is also a covering of K; hence $\mathscr{B}=\mathscr{B}^1\cup\alpha\mathscr{B}^2\in E(2,M,P,D)$. We define set functions v_k on the balls B_i^k, $k=1,2$, $i\in\mathbf{N}$ by $v_1(B_i^1)=d\big(f(B_i^1)\big)/d\big(f(E),f(F)\big)$ for $i\in\mathbf{N}$ and $v_2(\alpha B_i^2)=d\big(f(\alpha B_i^2)\big)/d\big(f(E),f(F)\big)$ for $i\in\mathbf{N}$. Let us show that

$$v=(v_1,v_2)\in F_{(2,M,P,D)}\big(\mathscr{B},\Delta(E,F,D)\big).$$

Indeed, let $\gamma\in\Delta(E,F,D)$ and let $\mathscr{A}=(B_i)_{i\in\mathbf{N}}$ be a chain of balls from $\mathscr{B}$ along the path γ, where $I=I_1\cup I_2$ and $B_i\in\mathscr{B}^1$ for $i\in I_1$, $B_i\in\alpha\mathscr{B}^2$ for $i\in I_2$. Then $\bigcup_{i\in I_1}f(B_i)\cup\bigcup_{i\in I_2}f(\alpha B_i)$ is a connected set which covers the path $\Gamma=f\circ\gamma$, hence $\sum_{i\in I_1}d\big(f(B_i^1)\big)+\sum_{i\in I_2}d\big(f(\alpha B_i^2)\big)\geq l(\Gamma)\geq d\big(f(E),f(F)\big)$ and this implies that $v=(v_1,v_2)\in F_{(2,M,P,D)}\big(\mathscr{B},\Delta(E,F,D)\big)$. We have

$$\begin{aligned}
\delta-\mathrm{Mod}_{(2,M,P)}\big(\Delta(E,F,D)\big)&\leq\sum_{i\in I_1}v_1(B_i^1)^n+\sum_{i\in I_2}v_2(\alpha B_i^2)^n\\
&=\sum_{i\in I_1}\frac{d\big(f(B_i^1)\big)^n}{d\big(f(E),f(F)\big)^n}+\sum_{i\in I_2}\frac{d\big(f(\alpha B_i^2)\big)^n}{d\big(f(E),f(F)\big)^n}\\
&\leq\frac{H}{d\big(f(E),f(F)\big)^n}\left(\sum_{i\in I_1}\mu_n\big(f(B_i^1)\big)+\sum_{i\in I_2}\mu_n\big(f(B_i^2)\big)\right)\\
&\leq\frac{H\cdot M\cdot N(f,D)}{d\big(f(E),f(F)\big)^n}\left(\mu_n\Big(f\Big(\bigcup_{i\in I_1}B_i^1\Big)\Big)+\mu_n\Big(f\Big(\bigcup_{i\in I_2}B_i^2\Big)\Big)\right)\\
&\leq\frac{2\cdot H\cdot M\cdot N(f,D)}{d\big(f(E),f(F)\big)^n}\cdot\mu_n\big(f(D)\big).
\end{aligned}$$

Letting δ tend to zero, we complete the proof.

4. Conditions of quasiregularity and a removability result

We shall first prove the following theorem.

Theorem 6. *Let $D \subset \mathbf{R}^n$ be open, $n \geq 2$, $a > 1$, $x \in D$ and $r > 0$ such that $\overline{B}(x, a.r) \subset D$ and $f\colon D \to \mathbf{R}^n$ continuous, open and discrete. If f is K-quasiregular on $B(x, a.r) \setminus \overline{B}(x, r)$, then it follows that*

$$\frac{d\big(f\big(B(x,r)\big)\big)^n}{\mu_n\big(f\big(B(x,a.r)\big)\big)} \leq C(a, K),$$

where $C(a, K) = 1/L(a, K)$ and

$$L(a,K) = \min\left\{\frac{V_n}{2^n}\cdot\exp\left(\frac{-2K\cdot n\cdot\omega_{n-1}}{c_n\cdot(\log a)^{n-1}}\right),\ \frac{V_n\cdot n\cdot K\cdot\omega_{n-1}}{2^n\cdot c_n\cdot(\log a)^{n-1}}\cdot\exp\left(\frac{-2K\cdot n\cdot\omega_{n-1}}{c_n\cdot(\log a)^{n-1}}\right)\right\}.$$

Proof. Suppose first that $\mu_n\big(f\big(S(x,a.r)\big)\big) = 0$. Let $s = L(x, f, r), \delta = K\cdot\omega_{n-1}/(\log a)^{n-1}$ and let $\alpha > 0$ be such that $2\delta = c_n \cdot \log(1/\alpha)$, where c_n is the constant from Theorem 10.12, [15, p. 31].

Now $\alpha = \exp\big(-2K\cdot\omega_{n-1}/c_n\cdot(\log a)^{n-1}\big)$; hence $0 < \alpha < 1$.

$$\begin{aligned}\frac{\mu_n\big(f\big(B(x,a.r)\big)\big)}{d\big(f\big(B(x,r)\big)\big)^n} &\geq \frac{\mu_n\big(f\big(B(x,a.r)\big)\big)}{2^n\cdot L(x,f,r)^n}\\ &\geq V_n\cdot\frac{l(x,f,r)^n}{2^n\cdot L(x,f,r)^n} \geq V_n\cdot\frac{\alpha^n}{2^n} \geq L(a,K),\end{aligned}$$

if $L(x,f,r)/l(x,f,r) \leq 1/\alpha$. Suppose that $L(x,f,r)/l(x,f,r) > 1/\alpha$. If $Q = f\big(S(x,r)\big)$, then $Q \cap S\big(f(x),t\big) \neq \emptyset$ for every $t \in [\alpha s, s]$ and let $Q_p, p \in P$ be the components of $Cf\big(\overline{B}(x,a.r)\big)$ which intersect $B\big(f(x),s\big) \setminus \overline{B}\big(f(x),\alpha s\big)$. Since $S(x,r) \subset \operatorname{Int}\overline{B}(x,ar)$ and f is open, it follows that $Q \cap \partial Q_p = \emptyset$ for $p \in P$, and let $B_p = \big\{t \in (\alpha s, s) \mid S\big(f(x),t\big) \cap Q_p \neq \emptyset\big\}$, $p \in P$, $E = \bigcup_{p\in P} Q_p$ and $B = \big\{t \in (\alpha s, s) \mid S\big(f(x),t\big) \cap E \neq \emptyset\big\}$. Since B is open, $B = \bigcup_{i\in I}(a_i, b_i)$, $I \subset \mathbf{N}$ and $(a_i,b_i) \cap (a_j,b_j) = \emptyset$ for $i,j \in I$, $i \neq j$. When $i \in I$, we denote $D_i = B\big(f(x),b_i\big) \setminus \overline{B}\big(f(x),a_i\big)$ and

$$J_i = \big\{j \in P \mid \text{there exists } t \in (a_i,b_i) \text{ such that } S\big(f(x),t\big) \cap Q_j \neq \emptyset\big\}$$

and let $U_i = \bigcup_{j\in J_i}\partial Q_j$. Let now $i \in I$ and $t \in (a_i,b_i)$. Then there exists $j \in J_i$ such that $S\big(f(x),t\big) \cap Q_j \neq \emptyset$. Since $S\big(f(x),t\big) \cap Q \neq \emptyset$ and $Q \subset f\big(\overline{B}(x,ar)\big) \subset CQ_j$, we see that $S\big(f(x),t\big) \cap CQ_j \neq \emptyset$; hence $S\big(f(x),t\big) \cap \partial Q_j \neq \emptyset$. If Δ_i is the family of all paths $\gamma\colon [0,1] \to \mathbf{R}^n$ which joins Q with U_i in D_i, then $D_i \cap U_i$ and

$D_i \cap Q$ are nonempty, disjoint sets which intersect $S(f(x),t)$ for every $t \in (a_i,b_i)$. Using Theorem 10.12, [15, p. 31], we obtain $M(\Delta_i) \geq c_n \cdot \log(b_i/a_i)$. Let now $\gamma \in \Delta_i$ be such that $\gamma(0) \in Q$, $\gamma(1) \in U_i$. We can find b_γ a subpath of γ, such that there exists a path $a_\gamma\colon [\alpha_\gamma, \beta_\gamma] \to \mathbf{R}^n$ with $0 \leq \alpha_\gamma \leq \beta_\gamma \leq 1$, $a_\gamma(\alpha_\gamma) \in S(x,r)$, $a_\gamma(\beta_\gamma) \in S(x,ar)$, $a_\gamma(\alpha_\gamma,\beta_\gamma) \subset B(x,ar) \setminus \overline{B}(x,r)$ and $f \circ a_\gamma = b_\gamma$. If $\Gamma_i = \{a_\gamma \mid \gamma \in \Delta_i\}$, $\Gamma_i' = \{b_\gamma \mid \gamma \in \Delta_i\}$, then $\Gamma_i' = f(\Gamma_i)$ and the paths from Γ_i' are shorter than the paths from Δ_i, and let $\Gamma' = \bigcup_{i\in I} \Gamma_i'$, $\Gamma = \bigcup_{i \in I} \Gamma_i$ and A be the spherical ring $B(x,ar) \setminus \overline{B}(x,r)$.

Then $\Gamma' = f(\Gamma)$ and

$$\begin{aligned}\delta = \frac{K \cdot \omega_{n-1}}{(\log a)^{n-1}} &= K \cdot M(\Gamma_A) \geq K \cdot M(\Gamma) \geq M(\Gamma') \geq \sum_{i\in I} M(\Gamma_i') \\ &\geq \sum_{i \in I} M(\Delta_i) \geq \sum_{i\in I} c_n \cdot \log \frac{b_i}{a_i}.\end{aligned}$$

We used here a path inequality of Poleckii [11]. Let now $J \subset I$ be finite and suppose that $a_{i_1} < b_{i_1} \leq a_{i_2} < b_{i_2} \leq \cdots \leq a_{i_p} < b_{i_p}$ is an ordering of the endpoints of the intervals counted by J. We take $b_{i_\circ} = \alpha s$, $a_{i_{p+1}} = s$, and since

$$c_n \cdot \sum_{j=1}^{p} \log \frac{b_{i_j}}{a_{i_j}} + c_n \cdot \sum_{j=0}^{p} \log \frac{a_{i_{j+1}}}{b_{i_j}} = c_n \cdot \log \frac{1}{\alpha} = 2\delta,$$

it follows that

$$c_n \cdot \sum_{j=0}^{p} \log \frac{a_{i_{j+1}}}{b_{i_j}} \geq \delta,$$

and let $A_J = \bigcup_{j=0}^{p} \big(\overline{B}\,(f(x), a_{i_{j+1}}) \setminus B\big(f(x), b_{i_j}\big)\big)$. We take $\theta_j \in (b_{i_j}, a_{i_{j+1}})$ such that $\log(a_{i_{j+1}}/b_{i_j}) = (a_{i_{j+1}} - b_{i_j}) \cdot 1/\theta_j$ for $j = 0, 1, \ldots, p$. Then

$$\begin{aligned}V(A_J) = V_n \cdot \sum_{j=0}^{p} (a_{i_{j+1}}^n - b_{i_j}^n) &= V_n \cdot \sum_{j=0}^{p} (a_{i_{j+1}} - b_{i_j}) \cdot \left(\sum_{k=0}^{n-1} a_{i_{j+1}}^{n-k-1} \cdot b_{i_j}^k \right) \\ &\geq n \cdot V_n \cdot \alpha^{n-1} \cdot s^{n-1} \cdot \sum_{j=0}^{p} (a_{i_{j+1}} - b_{i_j}) \\ &= n \cdot V_n \cdot \alpha^{n-1} \cdot s^{n-1} \cdot \sum_{j=0}^{p} \theta_j \log \frac{a_{i_{j+1}}}{b_{i_j}} \\ &\geq n \cdot V_n \cdot \alpha^n \cdot s^n \cdot \sum_{j=0}^{p} \log \frac{a_{i_{j+1}}}{b_{i_j}} \geq \frac{n \cdot V_n \cdot \delta \cdot \alpha^n \cdot s^n}{c_n}.\end{aligned}$$

Let $A = \bigcap_{J \subset I} A_J$, $J \subset I$ being finite. Then A is Lebesgue measurable and $V(A) \geq n \cdot V_n \cdot \delta \cdot \alpha^n \cdot s^n / c_n$ and $A = \bigcap_{t \in [\alpha s, s] \setminus B} S\big(f(x), t\big)$, and we have

$$V(C) = V(A) \geq \frac{n \cdot V_n \cdot \delta \cdot \alpha^n \cdot s^n}{c_n},$$

if $C = \bigcap_{t \in (\alpha s, s) \setminus B} S\big(f(x), t\big)$. Since $S\big(f(x), t\big) \cap E = \emptyset$ for every $t \in (\alpha s, s) \setminus B$, it follows that $S\big(f(x), t\big) \subset f\big(\overline{B}(x, ar)\big)$ for $t \in (\alpha s, s) \setminus B$; hence $C \subset f\big(\overline{B}(x, ar)\big)$.

We have now that

$$\begin{aligned} \frac{\mu_n\big(f\big(B(x, ar)\big)\big)}{d\big(f\big(B(x, ar)\big)\big)^n} &\geq \frac{\mu_n\big(f\big(\overline{B}(x, ar)\big)\big)}{2^n \cdot L(x, f, r)^n} \geq \frac{V(C)}{2^n \cdot s^n} \\ &\geq n \cdot V_n \cdot \alpha^n \cdot s^n \cdot \frac{\delta}{c_n \cdot 2^n \cdot s^n} = n \cdot V_n \cdot \alpha^n \cdot \frac{\delta}{c_n \cdot 2^n} \geq L(a, K), \end{aligned}$$

i.e., $d\big(f\big(B(x, r)\big)\big)^n / \mu_n\big(f\big(B(x, ar)\big)\big) \leq C(a, K)$.

Finally, if $\mu_n\big(f\big(S(x, ar)\big)\big) \neq 0$, we let $\varepsilon > 0$. Since f is K-quasiregular on $B(x, ar) \setminus \overline{B}(x, r)$, we have $\mu_n\big(f\big(S(x, (a - \varepsilon)r)\big)\big) = 0$ and this implies that $d\big(f\big(B(x, r)\big)\big)^n / \mu_n\big(f\big(B(x, (a - \varepsilon)r)\big)\big) \leq C(a - \varepsilon, K)$. Letting ε tend to zero, we complete the proof.

Theorem 7. *Let $D \subset \mathbf{R}^n$ be open, $n \geq 2$, $f \colon D \to \mathbf{R}^n$ continuous, open and discrete such that there exists $K \subset D$, and $h, t > 0$ such that*

$$\liminf_{r \to 0} \frac{d\big(f\big(B(x, r)\big)\big)^n}{\mu_n\big(f\big(B(x, r)\big)\big)} < h$$

for every $x \in D \setminus K$, and there exists $0 < a < 1$ and $\mathscr{B} = (B_i)_{i \in \mathbf{N}}$ an a-porous base of K with $d\big(f(aB_i)\big)^n / \mu_n\big(f(B_i)\big) < t$ for $i \in \mathbf{N}$. Then f is quasiregular and $K_0(f) \leq V_n \cdot H$ where $H = \max\{h, t\}$.

Proof. Let $g \colon (1, \infty) \to (1, \infty)$ be defined by $g(t) = (t^n - 1)/(t - 1)^n$ for $t \in (1, \infty)$. Then $g'(t) < 0$ for $t > 1$, $\lim_{t \to 1} g(t) = \infty$, $\lim_{t \to \infty} g(t) = 1$, hence g is a bijection of $(1, \infty)$ onto $(1, \infty)$. Let $x \in D$ be fixed. We can find $U \in V(x)$, $V \in V\big(f(x)\big)$ such that $f \mid U \colon U \to V$ is a proper map, $f(\partial U) = \partial V$, $\overline{U}$ is compact and $N(f, U) = |i(f, x)|$. Let $r > 0$ be small enough such that $\overline{B}(x, r) \cup U(x, f, L_\alpha) \subset U$, $0 < \alpha \leq 1$ and $L_\alpha = L(x, f, \alpha r)$, $l = l(x, f, r)$ and suppose that $L_\alpha / l \geq 1$. Let $A = B\big(f(x), L_\alpha\big) \setminus \overline{B}\,\big(f(x), l\big)$ and $B = U(x, f, L_\alpha) \setminus \overline{U}\,(x, f, l)$.

From [15, p. 9], for $r > 0$ small enough, B is also a ring of components $C_0 = \overline{U}\,(x, f, l)$, $C_1 = CU(x, f, L_\alpha)$ and $f(B) = A$. Also, the component C_0 contains the point x and a point a such that $|x - a| = r$ and the component C_1 contains ∞ and a point b such that $|x - b| = \alpha r$, hence from [15, p. 36] it follows that $M(\Gamma_B) \geq \mathscr{H}_n(|x - b| / |x - a|) = \mathscr{H}_n(\alpha)$. Let $\varepsilon > 0$ and $0 < a < \lambda < 1$ and let $M = \max\{B(n) \cdot N(\varepsilon), N(a, \lambda)\}$, $P = \min\{1/(2 + \varepsilon), (\lambda - a)/2\}$.

Using Theorem 4 and Theorem 5, we obtain

$$\begin{aligned}
\mathscr{H}_n(\alpha) &\le M(\Gamma_B) \le R(n,2,P)\cdot \mathrm{Mod}_{(2,M,P)}(\Gamma_B)\\
&\le 2\cdot R(n,2,P)\cdot H\cdot N(f,B)\cdot M\cdot \frac{\mu_n(f(B))}{d\big(\overline{B}\,(f(x),l), CB(f(x),L_\alpha)\big)^n}\\
&\le 2\cdot R(n,2,P)\cdot H\cdot M\cdot N(f,U)\cdot \frac{\mu_n(A)}{(L_\alpha - l)^n}\\
&= 2\cdot R(n,2,P)\cdot H\cdot M\cdot N(f,U)\cdot V_n\cdot \frac{(L_\alpha^n - l^n)}{(L_\alpha - l)^n}\\
&= 2\cdot R(n,2,P)\cdot H\cdot M\cdot N(f,U)\cdot V_n\cdot g(L_\alpha/l).
\end{aligned}$$

Since $\lim_{\alpha\to 0}\mathscr{H}_n(\alpha)=\infty$, we can find $0<\alpha<1$ small enough such that

$$\mathscr{H}_n(\alpha) > 2\cdot R(n,2,P)\cdot H\cdot M\cdot N(f,U)\cdot V_n.$$

Then

$$\frac{L_\alpha}{l} \le g^{-1}\left(\frac{\mathscr{H}_n(\alpha)}{2\cdot R(n,2,P)\cdot H\cdot M\cdot N(f,U)\cdot V_n}\right).$$

We denote

$$C(n,h,t,\alpha,\lambda,\varepsilon,U) = \max\left\{1, g^{-1}\left(\frac{\mathscr{H}_n(\alpha)}{2\cdot R(n,2,P)\cdot H\cdot M\cdot N(f,U)\cdot V_n}\right)\right\}.$$

It follows that

$$\frac{L(x,f,\alpha r)}{l(x,f,r)} \le C(n,h,t,\alpha,\lambda,\varepsilon,U)$$

for every $r>0$ such that $\overline{B}\,(x,r)\cup U(x,f,L_\alpha)\subset U$. This yields that

$$H_\alpha(x,f) = \limsup_{r\to 0}\frac{L(x,f,\alpha r)}{l(x,f,r)} \le C(n,h,t,\alpha,\lambda,\varepsilon,U).$$

This inequality is also valid for every point $z\in U$; hence from Theorem 1 [4] it follows that f is quasiregular on U. We have therefore proved that f is locally quasiregular on D; hence f is ACL^n on D and f is a.e. differentiable on D and $J_f(x)\ne 0$ a.e. in D.

Let us now fix a point $x\in D\setminus K$ such that f is differentiable in x and $J_f(x)\ne 0$ and let $0<\varepsilon<|f'(x)|$ be fixed. Then there exists $r_\varepsilon>0$ such that $|f(z)-f(x)-f'(x)(z-x)|\le \varepsilon\cdot|z-x|$ for $|z-x|\le r_\varepsilon$. Then

$$(|f'(x)|-\varepsilon)\cdot r \le L(x,f,r) \qquad \text{for } 0<r<r_\varepsilon. \tag{2}$$

Since $J_f(x)\ne 0$, we have $l\big(f'(x)\big)>0$, and let $s=\big(l\big(f'(x)\big)+\varepsilon\big)/l\big(f'(x)\big)$ and $\varphi\colon \mathbf{R}^n\to\mathbf{R}^n$ be defined by $\varphi(z)=f(x)+f'(x)(z-x)$ for $z\in\mathbf{R}^n$.

We show that $f\bigl(B(x,r)\bigr) \subset \varphi\bigl(B(x,rs)\bigr)$ for $0 < r \le r_\varepsilon$. Indeed, let $0 < r < r_\varepsilon$ be fixed and $y \in f\bigl(B(x,r)\bigr)$. Then there exists $a \in B(x,r)$ such that $y = f(a)$, and since $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ is a bijection, we can find $z \in \mathbf{R}^n$ such that $y = \varphi(z)$.

We have

$$\begin{aligned} |z-a| &\le |f'(x)(z-a)|/l\bigl(f'(x)\bigr) = |\varphi(z)-\varphi(a)|/l\bigl(f'(x)\bigr) \\ &= \bigl|f(a) - \bigl(f(x) - f'(x)(x-a)\bigr)\bigr|/l\bigl(f'(x)\bigr) \\ &\le \varepsilon \cdot |x-a|/l\bigl(f'(x)\bigr) < \varepsilon \cdot r/l\bigl(f'(x)\bigr); \end{aligned}$$

hence $|z-x| \le |x-a| + |z-a| \le r + r\varepsilon/l\bigl(f'(x)\bigr) = rs$, and this shows that $z \in B(x,rs)$. We proved that $y = f(a) = \varphi(z) \in \varphi\bigl(B(x,rs)\bigr)$, and since y was arbitrary in $f\bigl(B(x,r)\bigr)$, it follows that $f\bigl(B(x,r)\bigr) \subset \varphi\bigl(B(x,rs)\bigr)$; hence $\mu_n\bigl(f\bigl(B(x,r)\bigr)\bigr) \le \mu_n\bigl(\varphi\bigl(B(x,rs)\bigr)\bigr) = \mu_n\bigl(f'(x)\bigl(B(x,rs)\bigr)\bigr) = V_n \cdot (rs)^n \cdot |J_f(x)|$. From (2) we have

$$\begin{aligned} \frac{l\bigl(f'(x)\bigr)^n \cdot \bigl(|f'(x)| - \varepsilon\bigr)^n}{\bigl(l\bigl(f'(x)\bigr) + \varepsilon\bigr)^n \cdot V_n \cdot |J_f(x)|} &= \frac{\bigl(|f'(x)| - \varepsilon\bigr)^n \cdot r^n}{V_n \cdot (rs)^n \cdot |J_f(x)|} \\ &\le \frac{L(x,f,r)^n}{\mu_n\bigl(f\bigl(B(x,r)\bigr)\bigr)} \le \frac{d\bigl(f\bigl(B(x,r)\bigr)\bigr)^n}{\mu_n\bigl(f\bigl(B(x,r)\bigr)\bigr)} \end{aligned}$$

for $0 < r \le r_\varepsilon$ and we obtain that

$$\text{(3)} \qquad \frac{\bigl(|f'(x)| - \varepsilon\bigr)^n}{|J_f(x)|} \le V_n \cdot \left(\frac{l\bigl(f'(x)\bigr) + \varepsilon}{l\bigl(f'(x)\bigr)}\right)^n \cdot \frac{d\bigl(f\bigl(B(x,r)\bigr)\bigr)^n}{\mu_n\bigl(f\bigl(B(x,r)\bigr)\bigr)}$$

for $0 < r \le r_\varepsilon$. Since $\liminf_{r\to 0} d\bigl(f\bigl(B(x,r)\bigr)\bigr)^n/\mu_n\bigl(f\bigl(B(x,r)\bigr)\bigr) \le h$, we can find $r_p \to 0$ such that $d\bigl(f\bigl(B(x,r_p)\bigr)\bigr)^n/\mu_n\bigl(f\bigl(B(x,r_p)\bigr)\bigr) \le h$ for every $p \in \mathbf{N}$; hence replacing r by r_p in (3) and letting p tend to infinite, we obtain that

$$\frac{\bigl(|f'(x)| - \varepsilon\bigr)^n}{|J_f(x)|} \le V_n \cdot h \cdot \left(\frac{l\bigl(f'(x)\bigr) + \varepsilon}{l\bigl(f'(x)\bigr)}\right)^n,$$

and letting now ε tend to zero, we find that $|f'(x)|^n/|J_f(x)| \le V_n \cdot h$. Since $\mu_n(K) = 0$ (no point of K can be a point of density of K), f is a.e. differentiable in D and $J_f(x) \ne 0$ a.e. in D, it follows that $|f'(x)|^n \le V_n \cdot h \cdot |J_f(x)|$ a.e. in D and since f is ACL^n, it follows from [9, p. 9] that f is quasiregular on D and $K_0(f) \le V_n \cdot h$.

Corollary 1. *Let $D \subset \mathbf{R}^n$ be open, $K \subset D$, $n \ge 2$, $f\colon D \to \mathbf{R}^n$ continuous, open and discrete such that there exists $h > 0$ such that $h(x,f) < h$ for every $x \in D \setminus K$, and there exists $t > 0$, $0 < a < 1$ and $\mathscr{B} = (B_i)_{i\in\mathbf{N}}$ an a-porous base of K with $d\bigl(f(aB_i)\bigr)^n/\mu_n\bigl(f(B_i)\bigr) < t$ for every $i \in \mathbf{N}$. Then f is quasiregular and $H(x,f) = h(x,f) < h$ a.e. in D.*

Proof. From Theorem 10 it follows that f is quasiregular; hence f is a.e. differentiable and $J_f(x) > 0$ a.e. in D. Let $x \in D$ be fixed such that f is differentiable in x and $J_f(x) > 0$ and let $0 < \varepsilon < l\bigl(f'(x)\bigr)$. Since f is differentiable in x, there exists $r_\varepsilon > 0$ such that $|f(z) - f(x) - f'(x)(z-x)| \le \varepsilon \cdot |z-x|$ for $|z-x| \le r_\varepsilon$. Then

$$\frac{|f'(x)| - \varepsilon}{l\bigl(f'(x)\bigr) + \varepsilon} \le \frac{L(x,f,r)}{l(x,f,r)} \le \frac{|f'(x)| + \varepsilon}{l\bigl(f'(x)\bigr) - \varepsilon} \tag{4}$$

for $0 < r \le r_\varepsilon$. We can find $r_p \to 0$ such that $L(x,f,r_p)/l(x,f,r_p) \to h(x,f)$, and let $p_\varepsilon \in \mathbf{N}$ be such that $0 < r_p \le r_\varepsilon$ for $p \ge p_\varepsilon$. From (4) we have for $p \ge p_\varepsilon$ that

$$\frac{|f'(x)| - \varepsilon}{l\bigl(f'(x)\bigr) + \varepsilon} \le \frac{L(x,f,r_p)}{l(x,f,r_p)} \le \frac{|f'(x)| + \varepsilon}{l\bigl(f'(x)\bigr) - \varepsilon},$$

and letting first p tend to ∞ and then letting ε tend to zero, we obtain that $|f'(x)|/l\bigl(f'(x)\bigr) = h(x,f)$. Using (4) again, we see that

$$\frac{|f'(x)| - \varepsilon}{l\bigl(f'(x)\bigr) + \varepsilon} \le H(x,f) \le \frac{|f'(x)| + \varepsilon}{l\bigl(f'(x)\bigr) - \varepsilon},$$

and letting ε tend to zero we obtain that $H(x,f) = |f'(x)|/l\bigl(f'(x)\bigr)$; hence $H(x,f) = h(x,f)$. We have therefore proved that $H(x,f) = h(x,f)$ a.e. in D.

Remark. In 1994, at a seminar held in Helsinki, J. Väisälä raised the following problem:

It is known that if $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism such that there exists $h > 0$ such that $h(x,f) < h$ for every $x \in \mathbf{R}^n$, then f is $K(h,n)$-quasiconformal (it is the result of J. Heinonen and P. Koskela from [7]). Find the best estimation of the constant of quasiconformality $K(h,n)$. Corollary 1 shows that this problem is reduced to the classical problem when the inferior linear dilatation $h(x,f)$ is replaced by the classical linear dilatation $H(x,f)$. Indeed, we proved that $H(x,f) = h(x,f)$ a.e. in D. Since f is ACL^n, hence a.e. differentiable in D, the problem of finding the best constant of quasiregularity $K = K(h,n)$ is reduced in this way to the known case when we have that $H(x,f) < H$ for every $x \in D$. For interesting estimations of this kind, see the papers of M. Vuorinen [1], [2], [13], [14].

As an open problem, we raise the following question: If $D \subset \mathbf{R}^n$ is open, $n \ge 2$, $f\colon D \to \mathbf{R}^n$ is continuous, open and discrete such that there exists $h > 0$ with $h(x,f) < h$ for every $x \in D$, does it follow that $H(x,f) < h$ for every $x \in D$? We can now prove the following generalization of a removability result of J. Heinonen and P. Koskela from [7]:

Theorem 8. *Let $D \subset \mathbf{R}^n$ be open, $n \ge 2$, $f\colon D \to \mathbf{R}^n$ continuous, such that there exists $H \subset D$ closed in D such that f is K-quasiregular on $D \setminus H$ and there exists $0 < a < 1$, and $\mathscr{B} = (B_i)_{i \in \mathrm{N}}$, an a-porous base of H. Then, if f is open and discrete, or if $\operatorname{int} \mathrm{f(H)} = \emptyset$, it follows that f is K-quasiregular on D.*

Proof. We see that $\mu_n(H)=0$ and $\dim(H)=0$, hence f is a light map. Suppose that $\operatorname{int}\mathrm{f(H)}=0$. Using the fact that f is quasiregular on $D\setminus H$, it follows that f is differentiable on $D\setminus H$ and $J_f(x)\geq 0$ in $D\setminus H$, and since $\operatorname{int}\mathrm{f(H)}=\emptyset$, we apply Theorem 10 [6] to obtain that f is open and discrete on D. Using Theorem 6, we can find a constant $C(a,K)$ such that $d\big(f(aB_i)\big)^n/\mu_n\big(f(B_i)\big)\leq C(a,K)$ for every $i\in\mathbf{N}$. We apply now Theorem 7 to see that f is quasiregular on D. Since $\mu_n(H)=0$ and f is K-quasiregular on $D\setminus H$, it follows that f is K-quasiregular on D.

Another generalization of the removability result from [7] may be found in [8].

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