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PERIOD MATRICES OF ACCOLA–MACLACHLAN AND KULKARNI SURFACES

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Abstract. We compute the period matrices of the Riemann surfaces given by the equations $w^2 = z^{2g+2} - 1$ (Accola–Maclachlan surfaces) and $w^{2g+2} = z(z-1)^{g-1}(z+1)^{g+2}$ (Kulkarni surfaces). Furthermore, we obtain the triples of matrices associated to the real forms for these Riemann surfaces.

1. Introduction

A classical result of Torelli says that an isomorphism between the Jacobians of two compact Riemann surfaces comes from a conformal map, so that essentially the period matrix determines the surface. However, for $g \geq 2$ it is difficult to compute explicitly the period matrix, and results have been obtained for some famous surfaces of Klein, Fermat, Bring or Lefschetz. As a short historical background we may quote the papers by [B], [BT], [KN], [RL], [RR], [Sc1], [Sc2], [Sch], [W]. In [BT], the period matrix of Macbeath's curve of even genus is computed, whilst in [B] and [KN] the case of genus 2 surfaces is analyzed. The surfaces of genus 3 are studied in [RL] and some of the hyperelliptic ones are treated in [Sc1], [Sc2] and [Sch]. All of them use the method introduced by Rauch which involves the solution of a system of quadratic equations. The difficulty of this method increases when the genus becomes bigger. This is the reason why we need to use a different approach, which originally comes from Siegel [S] and Weyl [W] and was already used in [RR].

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We obtain here the period matrices and represent their automorphism groups for a large and remarkable family of surfaces. Namely, let X_g be the Riemann surface given by the algebraic equation $w^2 = z^{2g+2} - 1$; Accola [A] and Maclachlan [M] proved that its automorphism group has order $8g+8$ and that this is the largest possible order that can be constructed uniformly for every $g \geq 2$. Much later, Kulkarni [K] proved that if $g \neq 3 \mod 4$, such surface is unique, whereas for $g = 3 \mod 4$ there exists exactly one other surface Y_g of the equation $w^{2g+2} = z(z-1)^{g-1}(z+1)^{g+2}$ with an automorphism group of the same order. A problem still to be solved is to explain those results in terms of linear algebra data, namely, to find all principally polarized Abelian varieties with such groups of automorphisms.

Furthermore, those surfaces admit anti-conformal symmetries, and we proved in [BBCGG] that there are exactly three non-conjugated symmetries admitting fixed points. In other words, there are exactly three dianalytically non-equivalent bordered Klein surfaces whose double coverings are X_g or Y_g. We here compute its (real) Jacobians, and, to our knowledge, this is the first time such results have been obtained for Klein surfaces. To do so, in Section 2 we show that the period matrix of a Riemann surface is given by a pair of real matrices (C, R) and that the real Jacobian of a Klein surface is given by a triple (C, R, S), where S is the matrix of the anti-conformal involution. For real algebraic curves, the reader may consult Comesatti [C] from the beginning of this century, [GH], [Si], [SS] in modern language, and [R] for Klein's triples. As an example, we finally show the equivalence of the different points of view writing the matrices for the three real curves in $g = 3$.

2. Preliminaries

Let M be a compact Riemann surface of genus g and $B = [e_1, \ldots, e_{2g}]$ be an arbitrary basis of $H_1(M, \mathbf{Z})$. We define the intersection matrix C of the above basis with entries

$$c_{ij} = -e_i . e_j, \tag{1}$$

where the dot denotes the number of intersections, counting $+1$ at each intersection point where e_i crosses e_j as the x-axis crosses the y-axis, and -1 otherwise.

A basis $B^* = \{dw_1, \ldots, dw_{2g}\}$ of the *real* vector space $H^{1,0}(M, \mathbf{C})$ is said to be dual of B if

$$\operatorname{Re} \int_{e_i} dw_j = c_{ij}. \tag{2}$$

From Riemann bilinear relations it follows that the matrix S with entries

$$\operatorname{Im} \int_{e_i} dw_j = s_{ij} \tag{3}$$

is symmetric and positive definite.

The complex structure in $H^{1,0}(M,\mathbf{C})$ is given by a matrix R with respect to the basis B^* and satisfies $R^2=-I$. By equalities (2) and (3) we have the relation

$$CR = S. \tag{4}$$

After H. Weyl [W] and C.L. Siegel [S], the matrix R is called the period matrix of M with respect to the basis B.

Let a be a holomorphic automorphism of M, and $[a]$ denotes the matrix of its action on the homology and cohomology with respect to the above basis. Then,

$$[a]^{-1}R[a] = R, \qquad [a]'C[a] = C, \tag{5}$$

where the symbol $[a]'$ denotes the transpose of the matrix $[a]$.

On the other hand, if s is an antiholomorphic involution (a symmetry), then

$$[s]^{-1}R[s] = -R \qquad \text{and} \qquad [s]'C[s] = -C; \tag{6}$$

(see [R]).

If $\widetilde{C}$ is the intersection matrix of a canonical basis $\widetilde{B}=\{a_1,\dots,a_g\,,\,b_1,\dots,b_g\}$ of $H_1(M,\mathbf{Z})$, i.e.,

$$\widetilde{C} = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}, \tag{7}$$

we can obtain the ordinary Riemann $(g\times g)$-period matrix Z by the formula

$$Z = \widetilde{R}_{21}^{-1}\widetilde{R}_{22} - i\widetilde{R}_{21}^{-1}, \tag{8}$$

where

$$\widetilde{R} = \begin{pmatrix} \widetilde{R}_{11} & \widetilde{R}_{12} \\ \widetilde{R}_{21} & \widetilde{R}_{22} \end{pmatrix}$$

is our period matrix with respect to the dual basis in $H^{1,0}(M,\mathbf{C})$ of $\widetilde{B}$.

Notice that if a is a holomorphic automorphism of M whose matrix with respect to $\widetilde{B}$ is

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

then

$$PZ - ZS + ZRZ = Q. \tag{5'}$$

In our computations it will be essential to deal with the relation between automorphisms and period matrices. For these purposes, the formulae (5) are much better than the analogous $(5')$.

Recall that the pair (R,C) determines the analytic structure of a given Riemann surface in the following sense. Two such pairs, (R_1,C_1) and (R_2,C_2), determine the same structure if and only if there exists an integral matrix N whose determinant is 1 such that

$$N^{-1}R_1N = R_2; \qquad N'C_1N = C_2.$$

3. Period matrices

Accola and Maclachlan independently found for every genus $g \geq 2$ a surface X_g whose holomorphic automorphism group has order $8g+8$; precisely,

$$\mathrm{Aut}^+(X_g) = \langle a, d; a^{2(g+1)}, d^4, (ad)^2, ad^2a^{-1}d^2 \rangle. \tag{3.1}$$

The quotient of X_g under the action of the automorphism a has genus zero. This fact provides the following equation for X_g:

$$y^2 = x^{2g+2} - 1. \tag{3.2}$$

Later, Kulkarni proved that for large values of g, $g \not\equiv 3 \mod 4$, X_g is the only compact Riemann surface of genus g whose group of holomorphic automorphisms has order $8g+8$, whilst for $g \equiv 3 \mod 4$ there exists exactly one other surface Y_g whose holomorphic automorphism group has order $8g+8$. In fact, this group has the presentation

$$\mathrm{Aut}^+(Y_g) = \langle a, d; a^{2(g+1)}, d^4, (ad)^2, d^2ad^2a^g \rangle. \tag{3.3}$$

Also in this case, the quotient of Y_g under the action of the automorphism a is the sphere, and from [BBCGG] its equation is

$$y^{2g+2} = x(x-1)^{g-1}(x+1)^{g+2}. \tag{3.4}$$

In order to compute the period matrices of X_g and Y_g, we uniformize both surfaces by Fuchsian surface groups.

To simplify the notations in what follows, let $t = g+1$.

The coverings $X_g \to X_g/\langle a \rangle$ and $Y_g \to Y_g/\langle a \rangle$ have three branched points of orders $2t$, $2t$ and t. Therefore, we consider the triangular group

$$N_t = \langle a, b, c; a^{2t}, b^{2t}, c^t, abc \rangle, \tag{3.5}$$

whose fundamental region is the union of two hyperbolic isosceles triangles of angles $\pi/2t$, $\pi/2t$, π/t and vertices O, A, B, C (see Figure 1).

Figure 1.

We denote by a the rotation at O with angle π/t, by b the rotation at A with angle π/t, and by c the rotation at C with angle $2\pi/t$. The monodromy epimorphisms of the coverings $X_g \to X_g/\langle a\rangle$ and $Y_g \to Y_g/\langle a\rangle$ are $\theta_t\colon N_t \to C_{2t} = \langle \alpha; \alpha^{2t} = 1\rangle$ and $\eta_t\colon N_t \to C_{2t}$, respectively, and defined by $\theta_t(a) = \alpha = \theta_t(b)$, $\theta_t(c) = \alpha^{-2}$ and $\eta_t(a) = \alpha$, $\eta_t(b) = \alpha^{t+1}$, $\eta_t(c) = \alpha^{t-2}$. The kernels of θ_t and η_t are Fuchsian groups uniformizing X_g and Y_g, respectively, and have as the fundamental region a hyperbolic polygon of $4t$ sides constructed with $2t$ quadrilaterals centered at O.

Let us now obtain the identification rules of the sides of the above polygon. The sides are labelled by $1, 2, \dots, 4t$ counter-clockwise. To obtain the identification of the sides, observe that $a^2c \in \ker\theta_t$ and $a^{t+2}c \in \ker\eta_t$. Hence for X_g we have:

$$\text{if } k \text{ is odd: } k \xrightarrow{c} k-1 \xrightarrow{a^2} k+3 \bmod(4t), \tag{3.6}$$

and for Y_g we obtain:

$$\text{if } k \text{ is odd: } k \xrightarrow{c} k-1 \xrightarrow{a^{t+2}} 2t+k+3 \bmod(4t). \tag{3.7}$$

After the identifications of sides, three types of vertices appear: A, B and C.

We illustrate the case $t = 4$ (the first in which both X_g and Y_g occur). In Figure 2 we have a regular 16-gon whose sides have been labelled with integer numbers 1 to $16 = 4t$ and the vertices with the letters A, B and C. The pairwise identifications of sides for X_g and Y_g appear in two columns in Figure 2.

Let us define the following elements in $H_1(M, \mathbf{Z})$, where M denotes either X_g or Y_g:

$$e_1 = (4t) + (1)$$

and rotate via a to build

$$\begin{aligned} e_2 &= a(e_1) = (2) + (3),\\ e_3 &= a^2(e_1) = (4) + (5),\\ &\ \vdots\\ e_{2t-2} &= a^{2t-3}(e_1) = (4t-6) + (4t-5), \end{aligned} \tag{3.8}$$

where (k) denotes the side with label k.

Using the identification rules (3.6) and (3.7), we obtain in both cases:

$$\begin{aligned} e_{2t-1} &= a^{2t-2}(e_1) = -e_1 - e_3 - \cdots - e_{2t-3},\\ e_{2t} &= a^{2t-1}(e_1) = -e_2 - e_4 - \cdots - e_{2t-2}. \end{aligned} \tag{3.9}$$

Figure 2. X_g: 1–4, 3–6, 5–8, 7–10, 9–12, 11–14, 13–16, 15–2; Y_g: 1–12, 3–14, 5–16, 7–2, 9–4, 11–6, 13–8, 15–10

Lemma 3.1. *With the same notation as above, $B = [e_1, \dots, e_{2t-2}]$ is a basis of $H_1(M, \mathbf{Z})$ for M equal to X_g or Y_g. The matrix of the automorphism a with respect to B is*

$$[a] = (a_{ij}) \qquad a_{ij} = \begin{cases} -1 & \text{if } i \text{ is odd and } j = 2t-2, \\ 1 & \text{if } i = j+1,\ 1 \le j \le 2t-3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let Γ_t be the kernel of θ_t or η_t. The group Γ_t is generated by the hyperbolic elements that provide the side identifications. In the case $M = X_g$, the axis of each of these hyperbolic elements is homotopically equivalent to one of the e_i, $i = 1, \dots, 2t$. By (3.9), this shows that $\{e_1, \dots, e_{2t-2}\}$ generate Γ_t, and since $rkH_1(M, \mathbf{Z}) = 2g = 2t-2$, then B is a basis. In the case $M = Y_g$, we repeat the argument using the fact that the axis of each hyperbolic generator is homotopically equivalent to a sum of some of the e_i, $i = 1, \dots, 2t$.

Applying (3.8) and (3.9), we easily obtain the matrix $[a]$.

In the next lemma we shall compute the intersection matrices for the basis above.

Lemma 3.2. *Let $C_X = (x_{ij})$ and $C_Y = (y_{ij})$ be the intersection matrices* *of X_g and Y_g, respectively, with respect to the basis given in Lemma* 3.1. *Then,*

$$\begin{aligned} x_{ii+1} &= -1, && i = 1, \dots, 2t-3, \\ x_{ij} &= 0, && \textit{if } i < j \textit{ and } j \neq i+1, \\ x_{ij} &= -x_{ji}, && \textit{for all } i, j, \end{aligned}$$

and

$$y_{11} = y_{12} = y_{1t+1} = 0,$$

$$y_{1k} = \begin{cases} (-1)^k & \textit{if } 3 \le k < t+1, \\ (-1)^{k+1} & \textit{if } t+1 < k \le 2t-2, \end{cases}$$

$$\begin{aligned} y_{ij} &= y_{1j+1-i} && \textit{if } i < j, \\ y_{ij} &= -y_{ji} && \textit{for all } i, j. \end{aligned}$$

Proof. By their very definition, both C_X and C_Y are antisymmetric; so we compute only x_{ij} and y_{ij} with $i < j$.

Notice that the matrix $[a]$ of a with respect to the basis B is the same for both surfaces X_g and Y_g.

Since $[a]' C_X [a] = C_X$ and $[a]' C_Y [a] = C_Y$, then

$$x_{i+1,j+1} = x_{i,j}, \qquad y_{i+1,j+1} = y_{i,j}$$

for $i < j < 2t-2$ and so

$$x_{ij} = x_{1,j-i+1}, \qquad y_{ij} = y_{1,j-i+1}$$

for all $i < j$. Hence we need only to compute the first row of the matrices C_X and C_Y. Let j be an integer such that $1 < j \le 2t-2$. The cycle e_j intersects e_1 just at the point A, so we need to analyze the situation in this vertex.

Figure 3.

Figure 4.

Figure 5.

We begin with the surface X_g. In Figure 3 we represent the images of the polygon of Figure 2 under the action of the group $\ker\theta_t$ uniformizing X_g in a neighborhood of A.

The reader is referred to Figures 4 and 5, where we represent the cycles e_1 and e_j.

It follows that e_1 intersects e_j non-trivially only in the case $j = 2$. Then $x_{ij} = 0$ if $j \geq 3$ and $x_{12} = -(e_1.e_2) = -1$ (see Figure 6).

Now we deal with the surfaces Y_g. As before, e_1 intersects e_j just at the point A, and we look at a full neighborhood of A (see Figure 7).

Notice that we have labelled $l_0, l_2, \cdots, l_{2t}$ the pairs of identified sides arriving at A. In fact, l_k consists of the sides labelled as $(k-1)2t+2k-1$ and $k(2t+2)+2$. Let us draw the cycle e_1 in Figure 8.

Observe that $e_1 = 1 + 4t$ and the label l_{t-1} corresponds to the side $4t$. This explains the figure.

Since a is a conformal automorphism and $e_j = a^{j-1}(e_1)$, see 3.8, we can represent the cycle e_j as in Figure 9, where $k \equiv 1 + (j-1)(t+1) \mod 2t$, because $a(l_k) = l_{k+(t+1) \bmod 2t}$.

Figure 6.

Figure 7 and Figure 8.

Figure 9 and Figure 10.

In Figure 10 we have represented with distinct drawings the different e_j's having intersection number $+1$ with e_1.

Using the automorphism a, it is easy to compute the values of j such that e_j is one of the above, i.e., $k \equiv 1 + (j-1)(t+1) \bmod 2t$, where $k = 2, \ldots, t-1$. The solutions are the odd integers j with $3 \le j \le t-1$, and the even integers j with $t+2 \le j \le 2t-2$.

Figure 11.

In Figure 11 we have represented the drawings of the e_j's having intersection 0. This implies that y_{11}, y_{12} and y_{1t+1} are the only zeros in the first row of C_Y, and using the formula (1) we obtain the matrices C_X and C_Y as stated in Lemma 3.2.

Our next step is to compute the period matrix R. First we impose the condition $[a]^{-1}R[a] = R$, and we remark that this *almost* determines the matrix R. More precisely:

Lemma 3.3. *Let $\Delta = (\Delta_{ij})$ be the $(2t-2)$ x $(2t-2)$ matrix with entries*

$$\Delta_{ij} = \begin{cases} \sin\bigl(k(j-1)\pi/t\bigr) & \text{if } i = 2k, \\ \cos\bigl(k(j-1)\pi/t\bigr) & \text{if } i = 2k-1. \end{cases}$$

Let $[\hat{a}]$ and $\widehat{R}$ be the following matrices:

$$[\hat{a}] = \bigoplus_{k=1}^{t} \begin{pmatrix} \cos(k\pi/t) & -\sin(k\pi/t) \\ \sin(k\pi/t) & \cos(k\pi/t) \end{pmatrix}, \qquad \widehat{R} = \bigoplus_{k=1}^{t} \varepsilon_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\varepsilon_k \in \{-1, 1\}$. Then

$$[a] = \Delta^{-1}[\hat{a}]\Delta \tag{3.10}$$

and

$$R = \Delta^{-1}\hat{R}\Delta. \tag{3.11}$$

Proof. It is straightforward linear algebra.

Now we must decide among the 2^t possible choices of the ε_k's, which makes CR positive definite, or, equivalently, $\widehat{C}\widehat{R}$ positive definite, where

$$C = \Delta' \widehat{C} \Delta. \tag{3.12}$$

This is the content of the next lemma, which is also elementary:

Lemma 3.4. *The matrix $\widehat{R}$ has the form*

$$\bigoplus_{k=1}^{t-1} \varepsilon_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with $\varepsilon_k = -1$ for all k for the surface X_g and

$$\varepsilon_k = \begin{cases} -1 & \text{if } k \text{ is odd and } k > \frac{1}{2}t, \\ 1 & \text{otherwise,} \end{cases}$$

for the surface Y_g.

Remark 3.5. In order to compute $\widehat{R}$, we have only used the monodromy epimorphisms θ_t and η_t on the cyclic groups. Since the resulting period matrix $\widehat{R}$ is unique, then we can conclude that the monodromy completely determines these surfaces. This result has been previously obtained using combinatorial methods by Conder and the first author in [BC].

From a direct change of basis we get the following description of R.

Theorem 3.6. *The period matrix R of the Accola–Maclachlan surface X_g and the Kulkarni surface Y_g with respect to the basis $[e_1, \ldots, e_{2t-2}]$ is*

$$R = \Delta^{-1} \widehat{R} \Delta,$$

where Δ and $\widehat{R}$ are described in 3.3 *and* 3.4*, respectively.*

To finish this section, we shall illustrate the claim announced at the beginning of the paper: we compute from R the Riemann matrix Z for the Accola–Maclachlan surface of genus 2.

Example 3.7. It is easily seen that the basis $\widetilde{B} = [a_1 = e_1, a_2 = e_4, b_1 = e_2, b_2 = -e_1 - e_3]$ is canonical. Let

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

be the matrix of the change of basis. Following Theorem 3.6, we obtain the matrix

$$R = \begin{pmatrix} 0 & \frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} \\ -\frac{2}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} & 0 \\ 0 & -\frac{1}{3}\sqrt{3} & 0 & \frac{2}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{3} & 0 \end{pmatrix}.$$

With respect to $\widetilde{B}$, the period matrix is

$$\widetilde{R} = P^{-1}RP = \begin{pmatrix} 0 & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} & 0 \\ -\frac{1}{3}\sqrt{3} & 0 & 0 & \frac{2}{3}\sqrt{3} \\ -\frac{2}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{3} \\ 0 & -\frac{2}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & 0 \end{pmatrix}.$$

This provides us with the matrices

$$\widetilde{R}_{21} = \begin{pmatrix} -\frac{2}{3}\sqrt{3} & 0 \\ 0 & -\frac{2}{3}\sqrt{3} \end{pmatrix} \qquad \text{and} \qquad \widetilde{R}_{22} = \begin{pmatrix} 0 & \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} & 0 \end{pmatrix}$$

and by the formula (8):

$$Z = \begin{pmatrix} \frac{1}{2}i\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}i\sqrt{3} \end{pmatrix}.$$

This matrix coincides with the matrix obtained by Kuusalo and Näätänen in [KN, Case F, p. 411]).

4. Klein surfaces

Klein surfaces are the quotients of Riemann surfaces under the action of antiholomorphic involutions (symmetries). The Accola–Maclachlan and Kulkarni surfaces admit exactly three anticonformal involutions with fixed points producing dianalytically non-equivalent bordered Klein surfaces (i.e., three nonbirationally equivalent real curves whose complexification is the given Riemann surface; see [BBCGG]).

Let M/s be a Klein surface defined as the quotient of the Riemann surface M under the action of the symmetry s. In [R], the fourth author associated to M/s a triple $(R, C, [s])$, where (R, C) has the meaning of the preceding sections and $[s]$ is the matrix in homology of the symmetry s, and he proved that two such triples $(R_1, C_1, [s_1])$ and $(R_2, C_2, [s_2])$ are associated to dianalytically equivalent Klein surfaces if and only if there exists an integer matrix N of determinant ± 1 such that

$$N^{-1}R_1N = R_2; \qquad N'C_1N = C_2; \qquad N^{-1}[s_1]N = [s_2].$$

In this section we shall compute the matrices of the three symmetries, s_1, s_2 and s_3, with respect to the basis $[e_1, \dots, e_{2t-2}]$ introduced in (3.8).

Let M denote either the Accola–Maclachlan or the Kulkarni surface. Let $\mathrm{Aut}^{\pm}(M)$ be the group of automorphisms of M including the anticonformal ones. In [BBCGG], we proved that the quotient $M/\mathrm{Aut}^{\pm}(M)$ can be uniformized by a hyperbolic Coxeter group $(2, 4, 2t)$, i.e., generated by three reflections whose axes form a hyperbolic triangle with angles $\frac{1}{2}\pi, \frac{1}{4}\pi, \pi/2t$.

The reflections with axes on the lines OA and OC in Figure 1 are symmetries of the polygon in Figure 2. Even more, they preserve the pairwise identifications (3.6) and (3.7) corresponding to the two types of surfaces.

Figure 12.

Let us construct a hyperbolic triangle ODE (see Figure 12) with angles $\frac{1}{2}\pi, \frac{1}{4}\pi, \pi/2t$, with the vertex E on the line OA and D on OC.

The reflections with axes on DE and OE generate a dihedral group D_4. The action of D_4 on the triangle ODE produces a triangulation of the quadrilateral $OCAB$ of Figure 1. The action of the rotation of order $2t$ with center O provides us with a triangulation with hyperbolic triangles of type $\frac{1}{2}\pi, \frac{1}{4}\pi, \pi/2t$. The above triangulation corresponds to the lifting of the triangle uniformizing $M/\mathrm{Aut}^{\pm}(M)$ by the covering $M \to M/\mathrm{Aut}^{\pm}(M)$ (see Figure 13 in the case $g = 3$).

Let us choose as s_1 the lifting of the reflection with axis OA such that s_1 has as the fixed point set the line joining in the polygon of Figure 1 the center O with the vertex A where the sides labelled $2t-1$ and $2t$ intersect.

It follows directly from Figure 2 that $[s_1]$ acts on the basis given in (3.8) by the following formulae:

$$[s_1](e_1) = \sum_{k=1}^{t-1} e_{2k}, \quad [s_1](e_2) = \sum_{k=1}^{t-1} e_{2k-1}, \quad [s_1](e_k) = -e_{2t-k+1}, \quad 3 \le k \le 2t-2.$$

Analogously, let us choose as s_2 the symmetry whose fixed point set is the line joining O with the vertex B, where the sides $2t$ and 1 intersect. From Figure 2

Figure 13.

we have:

$$[s_2](e_1) = -e_1, \qquad [s_2](e_2) = \sum_{k=1}^{t-1} e_{2k}, \qquad [s_2](e_3) = \sum_{k=1}^{t-1} e_{2k-1},$$

$$[s_2](e_k) = -e_{2t-k+2}, \qquad 4 \le k \le 2t-2.$$

To analyze the third symmetry we must separately lift the reflection with axis DE to the Accola–Maclachlan and the Kulkarni surfaces. Let us begin with the first case. Since the fixed point set of s_3 is the union of the orthogonal lines joining k with $k+3 \bmod (4t)$ for all $k \equiv 3 \bmod 4$, it follows that

$$[s_3](e_i) = (-1)^{i+1} e_i \qquad \text{for all } i.$$

To finish with, we study the action of $[s_3]$ for the surface Y_g. In this case, the fixed point set of s_3 consists of the orthogonal lines joining k with $k+3 \bmod (4t)$ for all $k \equiv 7 \bmod 8$.

Let us compute $[s_3](e_{2l})$, for odd l. By 3.8, $e_{2l} = 2(2l-1) + (4l-1)$, and $4l-2$ is identified with $2t+4l-5$; see 3.7. Since, again by 3.8, $e_{t+2l-2} = (2t+4l-6) + (2t+4l-5)$, we conclude that

$$[s_3](e_{2l}) = e_{t+2l-2}.$$

On the other hand, if l is even, let $k = \frac{1}{2}(2l+2-t)$, which is odd. Hence $[s_3](e_{2k}) = e_{t+2k-2} = e_{2l}$, and since s_3 is an involution,

$$[s_3](e_{2l}) = e_{2l+2-t}.$$

Now we compute $[s_3](e_k)$ for odd k, and we begin with the case $k = 4l+1$. Note that $e_{4l+1} = (8l) + (8l+1)$, and so

$$[s_3](e_{4l+1}) = (8l-1) + e_{4l+1} + (8l+2).$$

But using the identifications in (3.1), we get

$$(8l-1) + (8l+2) = \sum_{j=0}^{(t/4)-1} (e_{2+4l+4j} + e_{4l-4j}),$$

and so

$$[s_3](e_{4l+1}) = e_{4l+1} + \sum_{j=0}^{(t/4)-1} (e_{2+4l+4j} + e_{4l-4j}).$$

In a similar way we obtain

$$[s_3](e_{4l+3}) = e_{4l+3+2t} + \sum_{j=0}^{(t/4)-1} [e_{2+4l+4j+2t} + e_{4l-4j+2t}].$$

In all computations above, whenever either e_{2t-1} or e_{2t} appears, the reader should change them by their values given in (3.9). For example,

$$[s_3](e_{(t/2)+1}) = e_{2t} = -(e_2 + e_4 + \cdots + e_{2t-2}).$$

This finishes the computation of the matrices of the actions in homology of the symmetries s_1, s_2 and s_3 in both surfaces X_g and Y_g.

This way, the triples $(R, C, [s_i])$, $i = 1, 2, 3$, where R is given in (3.11), C was calculated in Lemma 3.2 and the matrices $[s_i]$ are the ones just obtained, are the representants of the three non-dianalytically equivalent Klein surfaces having X_g and Y_g as double covers.

Comessatti [C], Gross–Harris [GH] and Shimura [S] proved that for every compact Riemann surface M of genus $g \geq 2$, and every symmetry τ on M, there exists a symplectic basis $[a_1, \dots, a_g, b_1, \dots, b_g]$ of $H_1(M, \mathbf{Z})$ such that the Riemann period matrix Z of M with respect to this basis has the form

$$Z = \tfrac{1}{2}H + iY$$

with $H \in M_g(\mathbf{Z})$, $Y \in M_g(\mathbf{R})$, and the matrix $[\tau]$ with respect to this basis is

$$[\tau] = \begin{pmatrix} I_g & H \\ 0 & -I_g \end{pmatrix}.$$

It is proved in [GH] that the Klein surfaces corresponding to (Z_1, τ_1) and (Z_2, τ_2) are dianalytically distinct if and only if the matrices Z_1 and Z_2 are linearly different. This classification agrees with the one given at the beginning of this section; see [R]. We now illustrate how to obtain the period Riemann matrices Z_i, once we know the triples $(R, C, [s_i])$ for X_g and Y_g in the case $g = 3$. For $g = 3$, the surface Y_3, as it was proved by Watson [Wa], has only two non-conjugated symmetries with fixed points, s_1 and s_2.

Example. We begin with the surface X_3 and its symmetry s_1. The matrix of the change of basis, whose columns express the vectors $[a_1, a_2, a_3, b_1, b_2, b_3]$ in terms of our previous basis $[e_1, e_2, e_3, e_4, e_5, e_6]$, is

$$P_1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

With respect to this new basis,

$$[s_1] = \begin{pmatrix} I_3 & H_1 \\ 0 & -I_3 \end{pmatrix} \qquad \text{and} \qquad H_1 = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Following the same procedure as in Example 3.7, one gets

$$Z_1 = \begin{pmatrix} -\frac{1}{2} + \frac{1}{4}(1+\sqrt{2}\,)i & -\frac{1}{2} - \frac{1}{4}i & -\frac{1}{4}i \\ -\frac{1}{2} - \frac{1}{4}i & \frac{1}{4}(1+\sqrt{2}\,)i & -\frac{1}{2} + \frac{1}{4}(1-\sqrt{2}\,)i \\ -\frac{1}{4}i & -\frac{1}{2} + \frac{1}{4}(1-\sqrt{2}\,)i & \frac{1}{4}(1+\sqrt{2}\,)i \end{pmatrix}.$$

Note that $2\,\mathrm{Re}\,Z_1 = H_1$.

For the other symmetries on X_3 with similar changes of basis, we have:

$$Z_2 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{2}\,)i & -1 & -\frac{1}{2}i \\ -1 & \frac{1}{4}\sqrt{2}\,i & -\frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} & \frac{1}{2}i \end{pmatrix},$$

$$Z_3 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{2}\,)i & -\frac{1}{2}\sqrt{2}\,i & \frac{1}{2}(1-\sqrt{2}\,)i \\ -\frac{1}{2}\sqrt{2}\,i & \sqrt{2}\,i & \frac{1}{2}\sqrt{2}\,i \\ \frac{1}{2}(1-\sqrt{2}\,)i & \frac{1}{2}\sqrt{2}\,i & \frac{1}{2}(1+\sqrt{2}\,)i \end{pmatrix}.$$

There are only two non-conjugated symmetries, s_1 and s_2, with fixed points (see [Wa]), and the matrices are:

$$Z_1 = \begin{pmatrix} -\frac{1}{2} + \frac{3}{2}i & -\frac{1}{2} & \frac{1}{2} - i \\ -\frac{1}{2} & -\frac{3}{2} + i & 1 - \frac{1}{2}i \\ \frac{1}{2} - i & 1 - \frac{1}{2}i & -\frac{1}{2} + i \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} -2 + \frac{3}{2}i & -1 + \frac{1}{2}i & -1 - \frac{1}{2}i \\ -1 + \frac{1}{2}i & -\frac{3}{2} + \frac{1}{2}i & -\frac{1}{2} \\ -1 - \frac{1}{2}i & -\frac{1}{2} & \frac{1}{2}i \end{pmatrix}.$$

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