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(BB) PROPERTIES ON FRÉCHET SPACES

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Abstract. In this paper we point out some relations concerning properties $(BB)_n$ and $(BB)_{n,s}$ related to the "Problème des topologies" of Grothendieck and give a first example of a Fréchet space with the $(BB)_2$ property but without the $(BB)_3$ property.

1. Introduction

After Taskinen's counterexample ([T1]) to the "Problème des topologies" of Grothendieck ([Gr]), the study of the property (BB) for Fréchet spaces has got the attention of several authors (see, for instance, [T2], [T3], [BoDi], [GGM], [Din2], [DiMe], [DeM], [Din3], [P], [Bl2], [DeP]). We recall that a couple of locally convex spaces (E,F) has the (BB) property if for every bounded subset B in the completion of the projective tensor product $E\hat{\otimes}_\pi F$ there are bounded subsets $C\subset E$ and $D\subset F$ such that B is contained in the closed convex hull of $C\otimes D=\{x\otimes y : x\in C,\ y\in D\}$. For the definition and properties of tensor products see [DeF].

Here we are interested in a generalization of property (BB) due to Dineen ([Din3]), the so called $(BB)_n$ property, in both cases, the full and the symmetric. In this last case we will denote it by $(BB)_{n,s}$. These properties are defined as follows: a locally convex space E has property $(BB)_n$ (respectively $(BB)_{n,s}$), for a given natural number $n\geq 2$, if for every bounded subset B in the completed projective tensor product $\hat{\otimes}^n_\pi E = E\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi E$ (respectively in the completed symmetric projective tensor product $\hat{\otimes}^n_{s,\pi}E = E\hat{\otimes}_{s,\pi}\cdots\hat{\otimes}_{s,\pi}E$) there is a bounded subset C in E such that B is contained in the closed convex hull of $\otimes^n C = \{x_1\otimes\cdots\otimes x_n : x_1,\dots,x_n\in C\}$ (respectively $\otimes^n_s C=\{\otimes^n x = x\otimes\cdots\otimes x : x\in C\}$). For $n=1$ we use the convention $\otimes^n E=\otimes^n_s E = E$. Note that a locally convex space E has property $(BB)_2$ if and only if the pair (E,E) has property (BB).

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All Banach spaces have property $(BB)_n$ for all n, but there are Fréchet spaces which are even Montel spaces without the $(BB)_{2,s}$ property ([AnT]).

It is known that for a given $n \geq 3$, property $(BB)_{n,s}$ implies property $(BB)_{n-1,s}$ ([Bl2]), and also property $(BB)_n$ implies property $(BB)_{n-1}$ (see below). Here we give a first example of a Fréchet space E which has property $(BB)_2$ but not property $(BB)_{3,s}$. Having in mind that property $(BB)_n$ implies property $(BB)_{n,s}$ (see below) we have that E has property $(BB)_{2,s}$ but not property $(BB)_{3,s}$ and that E has property $(BB)_2$ but not property $(BB)_3$. Note that all known examples of Fréchet spaces with property $(BB)_2$ have property $(BB)_n$ for all n (see for instance [GGM], [DeM], [Din3]). Our example gives the answer to a question studied in recent years by different authors (see in particular Problem 31 by Peris in the list of problems collected during the meeting on Polynomials and Holomorphic Functions on Infinite Dimensional Spaces held in Dublin in September 1994).

2. Stability properties of $(BB)_n$ and $(BB)_{n,s}$

In this section we state some stability properties of $(BB)_n$ and $(BB)_{n,s}$ that we are going to use to obtain the announced example. Other related properties can be seen in [Bl1].

Proposition 2.1. *Let E be a locally convex space with property $(BB)_n$. Then E has property $(BB)_{n,s}$.*

Proof. Let B be a bounded subset in $\hat{\otimes}^n_{s,\pi}E$. Since $\hat{\otimes}^n_{s,\pi}E$ is a subspace of $\hat{\otimes}^n_{\pi}E$, B is a bounded subset of $\hat{\otimes}^n_{\pi}E$. As E has the $(BB)_n$ property, there is an absolutely convex bounded set $C \subset E$ such that $B \subset \overline{\Gamma}(\otimes^n C)$, where $\overline{\Gamma}$ denotes absolutely convex closed hull. Hence

$$B = \sigma(B) \subset \sigma\big(\overline{\Gamma}(\otimes^n C)\big) = \overline{\Gamma}\big(\sigma(\otimes^n C)\big) \subset \overline{\Gamma}\bigg(\otimes^n_s \frac{n}{(n!)^{1/n}} C\bigg)$$

(see [Din4, (1.16)]), where σ denotes the symmetrization map given by

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)}$$

and S_n stands for the group of permutations of $\{1, \ldots, n\}$. □

Remark 2.2. There are not known examples of spaces with the $(BB)_{n,s}$ property and without the $(BB)_n$ property. Does $(BB)_{n,s}$ imply $(BB)_n$? This question has already been formulated by Peris in the collection of problems mentioned in the introduction.

Proposition 2.3. *Let E be a locally convex space and let F be a complemented subspace of E. If E has the $(BB)_n$ (respectively $(BB)_{n,s}$) property then F also has it.*

Proof. The $(BB)_n$ case is essentially obtained in [T1]. The symmetric case can be obtained as follows.

Let B be a bounded subset in $\hat{\otimes}^n_{s,\pi}F$. Then B is a bounded subset in $\hat{\otimes}^n_{s,\pi}E$, so there is a bounded subset C in E such that $B \subset \overline{\Gamma}(\otimes^n_s C)$. Let Π_F be a continuous projection from E onto F. Then $\Pi_F(C)$ is a bounded subset in F and

$$B = (\otimes^n \Pi_F)(B) \subset \overline{\Gamma}\big(\otimes^n_s \Pi_F(C)\big). \ \square$$

On the other hand, complementation properties lead to the following result, which will be used later on.

Proposition 2.4. (a) *If given $n \in \mathbf{N}$, $n \geq 2$, the locally convex space E has the $(BB)_n$ property, then E has the $(BB)_m$ property for each positive integer m, $2 \leq m \leq n$.*

(b) *If given $n \in \mathbf{N}$, $n \geq 2$, the locally convex space E has the $(BB)_{n,s}$ property, then E has the $(BB)_{m,s}$ property for each positive integer m, $2 \leq m \leq n$.*

Proof. (a) There is no loss of generality in assuming that $m = n-1$. Let $e \in E$, $e \neq 0$ and $F = [e]$, F is canonically complemented in E. Denote by J_F and Π_F the injection and projection, respectively, which give that complementation. The mappings $J\colon \hat{\otimes}^{n-1}E \to \hat{\otimes}^n E$ and $\Pi\colon \hat{\otimes}^n E \to \hat{\otimes}^{n-1}E$ defined by

$$J(x_1 \otimes \cdots \otimes x_{n-1}) = x_1 \otimes \cdots \otimes x_{n-1} \otimes e$$

and

$$\Pi(x_1 \otimes \cdots \otimes x_n) = \lambda_n x_1 \otimes \cdots \otimes x_{n-1},$$

with λ_n such that $\Pi_F(x_n) = \lambda_n e$ (and extended in an obvious way to their respective domains), are continuous, linear and $\Pi \circ J = \mathrm{Id}$.

Let B be a bounded subset in $\hat{\otimes}^{n-1}_\pi E$, then $J(B)$ is bounded in $\hat{\otimes}^n_\pi E$ and there is a bounded subset C in E such that

$$B = \Pi\big(J(B)\big) \subset \Pi\big(\overline{\Gamma}(\otimes^n C)\big) = \overline{\Gamma}\big(\Pi(\otimes^n C)\big) \subset \overline{\Gamma}(k \otimes^{n-1} C) = \overline{\Gamma}(\otimes^{n-1} k^{1/(n-1)} C),$$

where $k = \sup\{|\lambda| : \lambda e \in \Pi_F(C)\}$.

The proof of (b) is much more technical, it can be seen in [Bl2, Corollary 8]. $\square$

The following result gives a useful representation of the symmetric projective tensor product of a finite cartesian product (or direct sum) of locally convex spaces. It will be used in the proof of Proposition 2.6 below and could also be used to get a proof of Proposition 2.3 above.

Proposition 2.5 ([AnF, 2.2 and 3.4]). *If $F_1, \dots, F_m$ are locally convex spaces then*

$$\hat{\otimes}^n_{s,\pi}\Big(\prod_{j=1}^m F_j\Big) \simeq \prod_{\substack{l_1+\cdots+l_m=n\\ l_j\in\{0,\dots,n\}}} [\hat{\otimes}^{l_1}_{s,\pi}F_1]\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi[\hat{\otimes}^{l_m}_{s,\pi}F_m],$$

with the obvious meaning for $\hat{\otimes}^{l_j}_{s,\pi}F_j$ when $l_j = 0$.

One of the keys to get the example we are looking for is the following proposition.

Proposition 2.6. *Let E and F be locally convex spaces. Then $E \times F$ has property $(BB)_{n,s}$ if and only if the following two properties are satisfied:*
(a) *E and F have the $(BB)_{n,s}$ property.*
(b) *The pair $(\hat{\otimes}^k_{s,\pi}E, \hat{\otimes}^{n-k}_{s,\pi}F)$ has the (BB) property for each $k \in \{1, \dots, n-1\}$.*

Proof. Assume that $E \times F$ has property $(BB)_{n,s}$. Since E and F are complemented in $E \times F$, (a) follows from Proposition 2.3.

Let us see now how (b) is also verified. The topological isomorphism in Proposition 2.5 between $\hat{\otimes}^n_{s,\pi}(E \times F)$ and $\prod_{k=0}^n([\hat{\otimes}^k_{s,\pi}E]\hat{\otimes}_\pi[\hat{\otimes}^{n-k}_{s,\pi}F])$, is defined by $Q = (Q_0, \dots, Q_n)$, where for $k = 0, \dots, n$,

$$Q_k\big(\otimes^n(x, y)\big) = [\otimes^k x] \otimes [\otimes^{n-k} y].$$

This map induces a continuous injection J_k: $[\hat{\otimes}^k_{s,\pi}E]\hat{\otimes}_\pi[\hat{\otimes}^{n-k}_{s,\pi}F] \to \hat{\otimes}^n_{s,\pi}(E \times F)$ given by

$$J_k(z) = Q^{-1}(0, \dots, 0, \overset{k}{\smile}{z}, 0, \dots, 0)$$

for each $k = 1, \dots, n-1$. We have that $Q_k \circ J_k = \mathrm{Id}_{[\hat{\otimes}^k_{s,\pi}E]\hat{\otimes}_\pi[\hat{\otimes}^{n-k}_{s,\pi}F]}$.

Let B be a bounded subset in $[\hat{\otimes}^k_{s,\pi}E]\hat{\otimes}_\pi[\hat{\otimes}^{n-k}_{s,\pi}F]$. Then $J_k(B)$ is bounded in $\hat{\otimes}^n_{s,\pi}(E \times F)$. Since $E \times F$ has the $(BB)_{n,s}$ property, there exists a bounded subset $C = C_1 \times C_2 \subset E \times F$, where C_1 and C_2 are bounded subsets in E and F respectively, such that $J_k(B) \subset \overline{\Gamma}\big(\otimes^n_s(C_1 \times C_2)\big)$. Hence

$$B = Q_k(J_k(B)) \subset \overline{\Gamma}\big(Q_k\big(\otimes^n_s(C_1 \times C_2)\big)\big) = \overline{\Gamma}\big([\otimes^k_s C_1] \otimes [\otimes^{n-k}_s C_2]\big),$$

and since that $\otimes^k_s C_1$ is a bounded subset in $\hat{\otimes}^k_{s,\pi}E$ and $\otimes^{n-k}_s C_2$ is a bounded subset in $\hat{\otimes}^{n-k}_{s,\pi}F$, the pair $(\hat{\otimes}^k_{s,\pi}E, \hat{\otimes}^{n-k}_{s,\pi}F)$ has property (BB).

On the other hand, each bounded subset B in the symmetric tensor product $\hat{\otimes}^n_{s,\pi}(E \times F)$ is contained in a set $Q^{-1}\big(\prod_{k=0}^n B_k\big)$, where B_o (respectively B_n) is a bounded subset in $\hat{\otimes}^n_{s,\pi}F$ (respectively $\hat{\otimes}^n_{s,\pi}E$) and for each $k = 1, \dots, n-1$,

B_k is a bounded subset in $[\hat{\otimes}^k_{s,\pi}E]\hat{\otimes}_\pi[\hat{\otimes}^{n-k}_{s,\pi}F]$. By (b), $B_k \subset \overline{\Gamma}(C_k \otimes D_k)$, for bounded subsets C_k and D_k in $\hat{\otimes}^k_{s,\pi}E$ and $\hat{\otimes}^{n-k}_{s,\pi}F$ respectively. Our hypothesis (a) and Proposition 2.4(b) imply that E and F have property $(BB)_{m,s}$ for every $m \in \{2,\ldots,n\}$. In particular, for $k = 1,\ldots,n-1$, there are absolutely convex bounded subsets $C'_k \subset E$ and $D'_k \subset F$ such that $C_k \subset \overline{\Gamma}(\otimes^k_s C'_k)$ and $D_k \subset \overline{\Gamma}(\otimes^{n-k}_s D'_k)$. Hence

$$B_k \subset \overline{\Gamma}\big(\overline{\Gamma}(\otimes^k_s C'_k) \otimes \overline{\Gamma}(\otimes^{n-k}_s D'_k)\big) = \overline{\Gamma}\big((\otimes^k_s C'_k) \otimes (\otimes^{n-k}_s D'_k)\big).$$

For $k = 0$ (respectively $k = n$) let D'_o (respectively C'_n) such that $B_o \subset \overline{\Gamma}(\otimes^n_s D'_o)$ (respectively $B_n \subset \overline{\Gamma}(\otimes^n_s C'_n)$ and let C (respectively D) an absolutely convex bounded subset in E (respectively F) which contains $\bigcup_{k=0}^n C'_k$ (respectively $\bigcup_{k=0}^n D'_k$). Thus

$$\begin{aligned}
B &\subset Q^{-1}\Big(\prod_{k=0}^n B_k\Big) \subset Q^{-1}\Big(\prod_{k=0}^n \overline{\Gamma}\big((\otimes^k_s C'_k) \otimes (\otimes^{n-k}_s D'_k)\big)\Big)\\
&\subset Q^{-1}\Big((n+1)\overline{\Gamma}\Big(\prod_{k=0}^n \big((\otimes^k_s C'_k) \otimes (\otimes^{n-k}_s D'_k)\big)\Big)\Big)\\
&= (n+1)\overline{\Gamma}\Big(Q^{-1}\Big(\prod_{k=0}^n (\otimes^k_s C'_k) \otimes (\otimes^{n-k}_s D'_k)\Big)\Big)\\
&\subset (n+1)\overline{\Gamma}\Big(Q^{-1}\Big(\prod_{k=0}^n \big((\otimes^k_s C) \otimes (\otimes^{n-k}_s D)\big)\Big)\Big)\\
&\subset (n+1)\overline{\Gamma}\big((n+1)n! \otimes^n (C \times D)\big).
\end{aligned}$$

Since B consists of symmetric tensors, we have

$$\begin{aligned}
B = \sigma(B) &\subset \sigma\big((n+1)\overline{\Gamma}\big((n+1)n! \otimes^n (C \times D)\big)\big)\\
&\subset \overline{\Gamma}\Big((n+1)^2 n! \frac{n^n}{n!} \otimes^n_s (C \times D)\Big)\\
&= \overline{\Gamma}\big(\otimes^n_s \big(n(n+1)^{2/n}(C \times D)\big)\big),
\end{aligned}$$

as we desired to prove. □

3. An example concerning $(BB)_{2,s}$ and $(BB)_{3,s}$

Our example will be built on the Fréchet space

$$l_{p+} = \bigcap_{q>p} l_q = \bigcap_k l_{p_k},$$

$(p_k)_k$ being a strictly decreasing sequence of real numbers convergent to p. Its topology is given by the norms

$$\|(x_j)_j\|_{p_k} = \left(\sum_{j=1}^{\infty} |x_j|^{p_k}\right)^{1/p_k}.$$

The space l_{p+} is a Fréchet non Montel space which has the Heinrich density condition among other properties (see [MM]). Spaces l_{p+} have been already used to obtain examples and counterexamples to different questions in functional analysis (see [Di], [P], [DeP], [ABP] for instance).

For p, with $2 \leq p < \infty$, the spaces l_{p+} have property $(BB)_2$ ([DeP, Example 5.5(2')]).

Apart from the space l_{p+} to get our example we need the particular Banach space obtained in the following known result:

Proposition 3.1 ([DeP, Example 5.3].) *For $1 < p < 2$ there is a Banach space X of cotype 2 such that (l_{p+}, X') does not have property (BB).*

The last ingredient we need to get the announced example is a result we are going to prove about the complementation of l_{p+} in $\hat{\otimes}^n_{s,\pi} l_{np+}$.

Proposition 3.2. *Suppose $1 \leq p < \infty$ and $n \in \mathbf{N}$. Then the space l_{p+} is isomorphic to a complemented subspace of $\hat{\otimes}^n_{s,\pi} l_{np+}$.*

Proof. Let (e_k) be the canonical basis of l_{np+}. The mapping $J\colon l_{p+} \to \hat{\otimes}^n_{s,\pi} l_{np+}$ defined by

$$\begin{array}{rccc} J: & l_{p+} & \to & \hat{\otimes}^n_{s,\pi} l_{np+} \\ & (x_j)_j & \mapsto & \sum_{j=1}^{\infty} x_j \otimes^n e_j \end{array}$$

gives an injection from l_{p+} into $\hat{\otimes}^n_{s,\pi} l_{np+}$.

Let us see how $J\big((x_j)_j\big) \in \hat{\otimes}^n_{s,\pi} l_{np+}$. For $q > p$ and $k, m \in \mathbf{N}$, $k \leq m$, $\sum_{j=k}^{m} x_j \otimes^n e_j \in \otimes^n_{s,\pi} l_{nq}$ and, using s-projective norms instead of the equivalent projective norms which we have used until now (see [F]), we have,

$$\otimes^n_s \| \cdot \|_{nq} \left(\sum_{j=k}^{m} x_j \otimes^n e_j\right) = \sup\left\{\left|\sum_{j=k}^{m} x_j P(e_j)\right| : P \in \mathscr{P}(^n l_{nq}),\ \|P\| = 1\right\},$$

where $\mathscr{P}(^n l_{np})$ stands for the space of all n homogeneous continuous polynomials on l_{np}.

Since $(x_j)_j \in l_q$ and $\big(P(e_j)\big)_j \in l_{nq/(nq-n)} = l_{q'}$ (see [Z]), it follows from Hölder's inequality that

$$\left|\sum_{j=k}^{m} x_j P(e_j)\right| \leq \left\|\sum_{j=k}^{m} x_j e_j\right\|_q \|\big(P(e_j)\big)_j\|_{q'} \leq \left\|\sum_{j=k}^{m} x_j e_j\right\|_q$$

for all $P \in \mathscr{P}({}^n l_{nq})$, $\|P\| = 1$. This implies that $\left(\sum_{j=1}^m x_j \otimes^n e_j\right)_m$ is a Cauchy sequence in the space $\otimes_{s,\pi}^n l_{np+}$ and then $\sum_{j=1}^\infty x_j \otimes^n e_j \in \hat{\otimes}_{s,\pi}^n l_{np+}$. Moreover we have got that

$$\otimes_s^n \| \cdot \|_{nq} \left(\sum_{j=1}^\infty x_j \otimes^n e_j \right) \le \|(x_j)_j\|_q$$

which gives the continuity of J.

Now define

$$\begin{array}{rccc} \Pi\colon & \otimes_{s,\pi}^n l_{np+} & \to & l_{p+} \\ & \otimes^n (x_j)_j & \mapsto & (x_j^n)_j \end{array}$$

extending it to the whole $\otimes_{s,\pi}^n l_{np+}$ by linearity.

It is clear that Π is well defined and for $q > p$,

$$\|\Pi\big(\otimes^n (x_j)_j\big)\|_q^q = \sum_{j=1}^\infty |x_j^n|^q = \|(x_j)_j\|_{nq}^{nq},$$

and then

$$\|\Pi\big(\otimes^n (x_j)_j\big)\|_q = \otimes_s^n \| \cdot \|_{nq}\big(\otimes^n (x_j)_j\big).$$

For $z = \sum_{r=1}^N \lambda_r \otimes^n (x_{r,j})_j$ we have that

$$\|\Pi(z)\|_q \le \sum_{r=1}^N |\lambda_r| \, \|\Pi\big(\otimes^n (x_{r,j})_j\big)\|_q = \sum_{r=1}^N |\lambda_r| \, \|(x_{r,j})_j\|_{nq}^n.$$

Taking the infimum over all representations of z we have

$$\|\Pi(z)\|_q \le \inf \left\{ \sum_{r=1}^N |\lambda_r| \, \|(x_{r,j})_j\|_{nq}^n : z = \sum_{r=1}^N \lambda_r \otimes^n (x_{r,j})_j \right\} = \otimes_s^n \| \cdot \|_{nq}(z)$$

(see [F], [Din4]). So Π is continuous and then also its extension to $\hat{\otimes}_{s,\pi}^n l_{np+}$. Finally it is easy to see that $\Pi \circ J = \mathrm{Id}$. □

We note that another proof of this proposition can be obtained using results either from [ArFa] or from [O].

Our main result can be established as follows:

Theorem 3.3. *Suppose $1 < p < 2$, X as in Proposition 3.1 and $E = l_{2p+} \times X'$. Then E has property $(BB)_2$ and does not have property $(BB)_{3,s}$.*

Proof. First of all we prove that E verifies the $(BB)_2$ property. It is easy to see that

$$E\hat{\otimes}_\pi E = (l_{2p+} \times X')\hat{\otimes}_\pi(l_{2p+} \times X')$$
$$\cong (l_{2p+}\hat{\otimes}_\pi l_{2p+}) \times (l_{2p+}\hat{\otimes}_\pi X') \times (X'\hat{\otimes}_\pi l_{2p+}) \times (X'\hat{\otimes}_\pi X').$$

Let us see how the bounded subsets in any of the factors on the right side splits. As we have already mentioned, l_{2p+} has property $(BB)_2$ and then (l_{2p+}, l_{2p+}) has property (BB) (note that $2 < 2p < \infty$). On the other hand since X has cotype 2 its bidual also has cotype 2 (see [DJT, Corollary 11.9]), so the dual of X' has cotype 2 and this, by [DeP, Proposition 3(2′)], implies that (l_{2p+}, X') has property (BB). Of course (X', l_{2p+}) also has property (BB) and finally (X', X') has property (BB) because all Banach spaces have property $(BB)_n$ for all $n \geq 2$ and X is Banach. A similar argument to the one used to prove the only if part in Proposition 2.6, using the above isomorphism, gives that E has property $(BB)_2$ (see [T1, Corollary 3.6]).

Now assume E has property $(BB)_{3,s}$, then by Proposition 2.6 the pair $(\hat{\otimes}^2_{s,\pi} l_{2p+}, X')$ has the (BB) property and, since by Proposition 3.2, l_{p+} is a complemented subspace of $\hat{\otimes}^2_{s,\pi} l_{2p+}$, we get that (l_{p+}, X') has the (BB) property also (see [T1, 3.4]). This is a contradiction with Proposition 3.1 (note that $1 < p < 2$) and finishes the proof. □

Corollary 3.4. *Let $E = l_{2p+} \times X'$ as in Theorem 3.3. Then*
(a) *E has property $(BB)_{2,s}$ but not property $(BB)_{n,s}$ for any $n \geq 3$.*
(b) *E has property $(BB)_2$ but not property $(BB)_n$ for any $n \geq 3$.*

Proof. From the above theorem and Proposition 2.1 it follows that E has property $(BB)_{2,s}$ and that E does not have property $(BB)_{3,s}$. Then, by Proposition 2.4(b) it does not have any of the properties $(BB)_{n,s}$ for $n \geq 3$. This gives part (a), part (b) follows in a similar way. □

Corollary 3.5. *If we denote by $\mathscr{P}(^nE)$ the space of all n homogeneous continuous polynomials on E, which turns to be the dual of $\hat{\otimes}^n_{s,\pi} E$, then the topology τ_b of uniform convergence on the bounded subsets of E and the strong topology β as a dual space agree on $\mathscr{P}(^2E)$ but they are different on $\mathscr{P}(^nE)$ for all $n \geq 3$ when $E = l_{2p+} \times X'$.*

Proof. Having in mind that the supremum of the modulus of a continuous polynomial on a bounded subset B in E is the same that the supremum of the modulus of its linearization on $\overline{\Gamma}(\otimes^n_s B)$, it follows that $\tau_b = \beta$ on $\mathscr{P}(^nE)$ if and only if E has the $(BB)_{n,s}$ property (see [Din3]). Then we get from Corollary 3.4(a) that $\tau_b = \beta$ on $\mathscr{P}(^2E)$ and $\tau_b \neq \beta$ on $\mathscr{P}(^nE)$ for all $n \geq 3$. □

Let us denote by τ_ω the Nachbin ported topology on $\mathscr{P}(^nE)$ defined as the inductive limit of the normed spaces $(\mathscr{P}(^nE_V), \|\cdot\|)$ when V ranges over the family of all absolutely convex open neigborhoods of 0 in E. It happens that $\tau_b \le \beta \le \tau_\omega$ and for metrizable spaces E, τ_ω is the barrelled topology associated with τ_b on $\mathscr{P}(^nE)$ (see [Din1, Proposition 3.41]).

Corollary 3.6. *The τ_b and τ_ω topologies agree on $\mathscr{P}(^n E)$ for $n = 1, 2$, with $E = l_{2p+} \times X'$, but they are different on $\mathscr{P}(^nE)$ for all $n \ge 3$.*

Proof. Since by Theorem 3.3 this particular Fréchet space E has the $(BB)_2$ property and the density condition because l_{2p+} and X' have it, the space $E\hat{\otimes}_\pi E$ also has the density condition [BiBo]. Then it is distinguished which gives that its strong dual $(\mathscr{P}(^2E), \beta)$ is barrelled and so $\beta = \tau_\omega$ on $\mathscr{P}(^2E)$. Since $\tau_b = \beta$ on $\mathscr{P}(^2E)$ because E has the $(BB)_{2,s}$ property, we get that $\tau_b = \tau_\omega$ on $\mathscr{P}(^2E)$ and then on $\mathscr{P}(^1E)$. On the other hand since E does not have the $(BB)_{n,s}$ property for $n \ge 3$ (Corollary 3.4(a)) we have $\tau_b \ne \tau_\omega$ on $\mathscr{P}(^nE)$ for all $n \ge 3$. □

Remark 3.7. An example of a Fréchet (even Montel) space E such that $\tau_b = \tau_\omega$ on $\mathscr{P}(^1E)$ and $\tau_b \ne \tau_\omega$ on $\mathscr{P}(^2E)$ is already known (see [AnT]).

References

[ABP] Ansemil, J.M., F. Blasco, and S. Ponte: On the "three-space problem" for spaces of polynomials. - Proceedings of the II International Workshop on Functional Analysis at Trier University, Trier Univ. 17, 1997, 189–195.

[AnF] Ansemil, J.M., and K. Floret: The symmetric tensor product of a direct sum of locally convex spaces. - Studia Math. 129, 1998, 285–295.

[AnT] Ansemil, J.M., and J. Taskinen: On a problem of topologies in infinite dimensional holomorphy. - Arch. Math. (Basel) 54, 1990, 61–64.

[ArFa] Arias, A., and J.D. Farmer: On the structure of tensor products of l_p-spaces. - Pacific J. Math. 175, 1996, 13–37.

[BiBo] Bierstedt, K., and J. Bonet: Density conditions in Fréchet and (DF)-spaces. - Rev. Mat. Univ. Complut. Madrid 2, 1989, 59–75.

[Bl1] Blasco, F.: Complementación, casinormabilidad y tonelación en espacios de polinomios. - Ph. D. Thesis, Universidad Complutense de Madrid, 1996.

[Bl2] Blasco, F.: Complementation in spaces of symmetric tensor products and polynomials. - Studia Math. 123, 1997, 165–173.

[BoDi] Bonet, J., and J.C. Díaz: The problem of topologies of Grothendieck and the class of Fréchet T-spaces. - Math. Nachr. 150, 1991, 109–118.

[DeF] Defant, A., and K. Floret: Tensor Norms and Operator Ideals. - North-Holland Math. Stud. 176, 1993.

[DeM] Defant, A., and M. Maestre: Property (BB) and holomorphic functions on Fréchet–Montel spaces. - Math. Proc. Cambridge Philos. Soc. 115, 1993, 305–313.

[DeP] Defant, A., and A. Peris: Maurey's extension theorem and Grothendieck's "problème des topologies". - J. London Math. Soc. (2) 58, 1998, 679–696.

[Di] Díaz, J.C.: An example of a Fréchet space, non Montel, without infinite dimensional normable subspaces. - Proc. Amer. Math. Soc. 96, 1986, 721.

[DiMe] Díaz, J.C., and G. Metafune: The problem of topologies of Grothendieck for quojections. - Results Math. 21, 1992, 299–311.

[DJT] Diestel, J., H. Jarchow, and A. Tonge: Absolutely Summing Operators. - Cambridge Stud. Adv. Math. 43, 1995.

[Din1] Dineen, S.: Complex Analysis in Locally Convex Spaces. - North-Holland Math. Stud. 57, 1981.

[Din2] Dineen, S.: Holomorphic functions on Fréchet–Montel spaces. - J. Math. Anal. Appl. 163, 1992, 581–587.

[Din3] Dineen, S.: Holomorphic functions and the (BB) property. - Math. Scand. 74, 1994, 215–236.

[Din4] Dineen, S.: Complex Analysis on Infinite Dimensional Spaces. - Springer-Verlag Monograph Series, 1999.

[F] Floret, K.: Natural norms on symmetric tensor products of normed spaces. - Proceedings of the II International Workshop on Functional Analysis at Trier University, Trier Univ., 17, 1997, 153–188.

[GGM] Galindo, P., D. García, and M. Maestre: The coincidence of τ_0 and τ_ω for spaces of holomorphic functions on some Fréchet–Montel spaces. - Proc. Roy. Irish Acad. Sect. A 91, 1991, 137–143.

[Gr] Grothendieck, A.: Produits tensoriels topologiques et espaces nucleaires. - Mem. Amer. Math. Soc. 16, 1955.

[MM] Metafune, G., and V.B. Moscatelli: On the space $l^{p+} = \bigcap_{q>p} l^q$. - Math. Nachr. 147, 1990, 7–12.

[O] Oja, E.: Sur la réflexivité des produits tensoriels et les sous-spaces des produits tensoriels projectifs. - Math. Scand. 51, 1982, 75–288.

[P] Peris, A.: Quasinormable spaces and the problem of topologies of Grothendieck. - Ann. Acad. Sci. Fenn. Math. 19, 1994, 167–203.

[T1] Taskinen, J.: Counterexamples to “Problème des Topologies” of Grothendieck. - Ann. Acad. Sci. Fenn. Math. Diss. 63, 1986.

[T2] Taskinen, J.: The projective tensor product of Fréchet–Montel spaces. - Studia Math. 91, 1988, 17–30.

[T3] Taskinen, J.: Examples of non-distinguished Fréchet spaces. - Ann. Acad. Sci. Fenn. Math. 14, 1989, 75–88.

[Z] Zalduendo, I.: An estimate for multilinear forms on l^p spaces. - Proc. Roy. Irish. Acad. Sect. A 93, 1993, 137–142.

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