Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 155–163

REMARKS ON THE NONEXISTENCE OF DOUBLING MEASURES

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Abstract. We establish that there are bounded Jordan domains $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ that do not carry a (nontrivial) doubling measure with respect to the Euclidean distance. More generally, it is shown that every nonempty metric space (X, d) without isolated points has an open and dense subset A such that (A, d) does not carry a doubling measure.

1. Introduction

The fundamental result on the existence of doubling measures, due to Vol'berg and Konyagin [11], states that every doubling and compact metric space (X, d) carries a nontrivial doubling measure. This remains true also if instead of compactness one only requires that X is complete, see [4]. In particular, every closed subset of $\mathbf{R}^n$ carries a nontrivial doubling measure. The existence question gains special interest from the fact that, in order to extend classical results of analysis to a setting of a metric space X, one is often forced to postulate a doubling measure on X (see e.g. [9, Chapter I]).

In the present note we show that one cannot replace compactness by local compactness in the theorem of Vol'berg and Konyagin, thus answering negatively a question posed in [4]. In fact, counterexamples contain remarkably 'nice' spaces that appear in analysis: Theorem 3 yields a bounded Jordan domain $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ that carries no doubling measure with respect to the Euclidean distance. We also prove (Theorem 5) a more general result of nonexistence: for every nonempty metric space X which does not have isolated points there is an open and dense subset $\Omega \subset X$ such that Ω does not carry a doubling measure. Further observations are contained in Remarks 1 and 2 below.

We refer to [11], [3], [9, Chapter I, Section 8], [10], [12], [13], and [2] for additional results on doubling measures. In addition, the work of Staples and Ward [8] is directly related to Theorem 5, see Remark 3 below (we are grateful to J. Heinonen who noticed the connection between our results and [8]).

The research for this note was carried out during my visit to the University of Michigan, Ann Arbor. I thank F. Gehring and J. Heinonen for hospitality and for creating a stimulating mathematical atmosphere. Moreover, I thank J. Luukkainen for the careful reading of the manuscript and his valuable comments on it.

1991 Mathematics Subject Classification: Primary 28A12; Secondary 28A75.

Supported by the Academy of Finland.

2. Results

We first recall that if (X,d) is a metric space, then a Borel measure μ on X is doubling if there exists a constant $C \geq 1$ so that the inequality

$$0 < \mu\big(B(x,2r)\big) \leq C\mu\big(B(x,r)\big) < \infty \tag{1}$$

holds for all $x \in X$ and $r > 0$ (in this case we also say that μ is C-doubling). Here $B(x,r) = \{y \in X \mid d(x,y) < r\}$. If B is a ball in X with radius r, then for $k > 0$ the abbreviation kB denotes the ball with the same center and with radius kr. Moreover, we write $B_c(x,r) = \{y \in X \mid d(x,y) \leq r\}$.

A subset A' of a metric space A is ε-dense in A for $\varepsilon > 0$ if $A = \bigcup_{y\in A'} B(y,\varepsilon)$. A metric space (X,d) is doubling (in the metric sense) if there is a constant $K > 0$ such that for every set $A \subset B(x,2r)$ with $x \in X$, $r > 0$ that satisfies $d(y,z) \geq r$ for distinct $y,z \in A$, the number of points in A is bounded from above by K. If X carries a doubling measure, then X is known to be doubling. The family of Borel sets of a metric space X is denoted by $\mathscr{B}(X)$.

The following simple lemma is essentially known, but for completeness we include a proof.

Lemma 1. *Let A be a dense subset of a metric space X. Then every C-doubling measure μ on A extends to a C-doubling measure $\tilde\mu$ on X for which $\tilde\mu(S) = \mu(S \cap A)$ for $S \in \mathscr{B}(X)$.*

Proof. Note that A need not be a Borel subset of X, so that one needs to verify that $\tilde\mu$ is a well-defined measure. For that end it is enough to note that $\{S \in \mathscr{B}(X) \mid S \cap A \in \mathscr{B}(A)\} = \mathscr{B}(X)$, and this follows by observing that the left-hand side is a σ-field that contains all open subsets of X. Let then $x \in X$ and $r > 0$. Choose a sequence of points $x_k \in A$ so that $d(x_k,x) < \frac{1}{2}r$ and $d_k = d(x_k,x)$ decreases to 0 as $k \to \infty$. The inequality (1) follows by letting $k \to \infty$ in the estimate

$$\begin{aligned}\tilde\mu\big(B(x,2r-3d_k)\big) &\leq \mu\big(B(x_k,2r-2d_k)\cap A\big)\\ &\leq C\mu\big(B(x_k,r-d_k)\cap A\big) \leq C\tilde\mu\big(B(x,r)\big). \ \square\end{aligned}$$

Our second lemma is a geometrically refined variant of known estimates (see e.g. [12, Lemma 1] for doubling measures. In what follows, the word rectangle (or square) refers to the closure of a rectangular domain.

Lemma 2. *Let $S = [0,a]^2 \subset \mathbf{R}^2$ and $R = [0,b] \times [0,a]$, where $0 < b < \frac{1}{2}a$. Let μ be a C-doubling measure on a set $U \subset \mathbf{R}^2$ so that $U \cap S$ is $(b/16)$-dense in S. Then*

$$\frac{\mu(R\cap U)}{\mu(S\cap U)} \leq \big(1 - C^{-4}\big)^{\log_2(a/b)-2}.$$

Proof. The claim clearly holds for $b \in \left[\frac{1}{4}a, \frac{1}{2}a\right)$. By induction it is then enough to prove it for $b \in \left(0, \frac{1}{4}a\right)$ assuming that it is true for $2b$. Assume hence that $b \in \left(0, \frac{1}{4}a\right)$ and denote $R' = [0, 2b] \times [0, a]$. By the density assumption it is easily verified that there are disjoint balls $B_1, \dots, B_m$ with centers in U so that $\bigcup_{i=1}^m B_i \subset R' \setminus R$ and $R' \subset \bigcup_{i=1}^m 2^4 B_i$. The doubling property of μ implies

$$\mu(U \cap R') \le C^4 \mu\left(U \cap \left(\bigcup_{i=1}^m B_i\right)\right)$$

and hence

$$\mu(U \cap R) \le \mu(U \cap R') - \mu\left(U \cap \left(\bigcup_{i=1}^m B_i\right)\right) \le (1 - C^{-4})\mu(U \cap R').$$

The lemma now follows by applying the induction hypothesis on the rectangle R' since the needed density assumption is satisfied. □

Theorem 3. *For each $n \ge 2$ there is a bounded Jordan domain $\Omega \subset \mathbf{R}^n$ (even the image of B^n under a homeomorphism of $\mathbf{R}^n$) which does not carry a doubling measure.*

Proof. For reasons of clarity we first give a detailed proof in the case $n = 2$. Let us start by constructing a standard Cantor set $F \subset \mathbf{R}$ corresponding to a sequence $(\lambda_j)_{j=1}^\infty$ of scalars, such that $\lambda_j \in \left(0, \frac{1}{8}\right)$ and $\lim_{j\to\infty} \lambda_j = 0$. Thus $F = \bigcap_{j=0}^\infty F_j$, where $F_0 = [0, 1]$ and for each $j \ge 1$ the set F_j is obtained from F_{j-1} by dissecting a middle interval of length $\lambda_j |I|$ from each of the 2^{j-1} closed intervals I that comprise F_{j-1}. Finally we choose $\lambda_j = 2^{-5j}$ for $j \ge 1$ (however, we keep writing λ_j instead of 2^{-5j} until to the very end of the argument, since we will later make another choice for (λ_j) in Remark 2).

Set $K = F \times F \subset \mathbf{R}^2$. Since K is a Cantor set, it is well known that there is a planar Jordan curve J such that $K \subset J$, see e.g. [7, Theorems 12.8 and 13.7]. Let Ω be the bounded component of $\mathbf{R}^2 \setminus J$. We claim that Ω carries no doubling measure. Towards the proof of the claim denote by U the closure of Ω and assume that μ is a C-doubling measure on U. Lemma 1 implies that it is enough to show that μ cannot be supported on Ω, and this follows if we establish that

$$\mu(K) > 0. \tag{2}$$

We may write $K = \bigcap_{j=0}^\infty K_j$ with $K_j = F_j \times F_j$ and, moreover,

$$K_j = \bigcup_{i=1}^{4^j} S_j^i,$$

where each S_j^i $(i = 1, \dots, 4^j)$ is a square.

Choose $j_0 = j_0(C) \geq 1$ so that $b_j > \frac{1}{2}$ for $j \geq j_0$, where

$$b_j = 1 - C^4(1 - C^{-4})^{-3+\log_2((1-\lambda_j)/\lambda_j)}. \tag{3}$$

Let then $j \geq j_0$ and $i_0 \in \{1, \dots, 4^j\}$ be arbitrary and let $S_{j+1}^{i_r}$, $r = 1, 2, 3, 4$, be the four squares that comprise $S_j^{i_0} \cap K_{j+1}$. We show that

$$\sum_{r=1}^{4} \mu(S_{j+1}^{i_r} \cap U) \geq b_{j+1} \mu(S_j^{i_0} \cap U). \tag{4}$$

Before proving (4) we show how (2) follows from it. Note that (4) yields by induction the inequality $\mu(K_j \cap U) \geq \mu(K_{j_0} \cap U) \prod_{i=j_0+1}^{j} b_i$, and letting $j \to \infty$ we obtain

$$\mu(K) = \mu(K \cap U) \geq \Bigl(\prod_{j=j_0+1}^{\infty} b_j \Bigr) \mu(K_{j_0} \cap U).$$

We observe that evidently $\mu(K_{j_0} \cap U) > 0$. Moreover, the elementary inequality $b \geq e^{-2(1-b)}$ for $b \in \bigl(\frac{1}{2}, 1\bigr)$ implies that

$$\prod_{j=j_0+1}^{\infty} b_j \geq \exp\Bigl(-2C^4 \sum_{j=j_0+1}^{\infty} (1 - C^{-4})^{-3+\log_2((1-\lambda_j)/\lambda_j)} \Bigr) \geq \exp\Bigl(-C' \sum_{j=j_0+1}^{\infty} \lambda_j^d \Bigr),$$

where the positive constants d and C' depend only on C. Hence $\prod_{j=j_0+1}^{\infty} b_j > 0$ (independently of the value of C) provided that the sequence (λ_j) satisfies

$$\sum_{j=1}^{\infty} \lambda_j^{\delta} < \infty \quad \text{for all } \delta > 0. \tag{5}$$

We obtain (2) since (5) obviously holds for the choice $\lambda_j = 2^{-5j}$.

It remains to prove (4). Let $a_j = 2^{-j} \prod_{i=1}^{j} (1 - \lambda_i)$ be the sidelength of the square $S_j^{i_0}$, and write $S_j^{i_0}$ in an obvious manner as a union of nine sets:

$$S_j^{i_0} = S' \cup \Bigl(\bigcup_{r=1}^{4} R_r \Bigr) \cup \Bigl(\bigcup_{r=1}^{4} S_{j+1}^{i_r} \Bigr), \tag{6}$$

where each R_r $(r = 1, 2, 3, 4)$ is a rectangle with sides $\lambda_{j+1} a_j$ and $\frac{1}{2}(1 - \lambda_{j+1}) a_j$, and S' is a square with side $\lambda_{j+1} a_j$. Fix $r \in \{1, 2, 3, 4\}$. With a possible (and harmless) relabeling of the rectangles we may denote by R_r' the closed rectangle of the same size as R_r, having a common side with it, and contained in $S_{j+1}^{i_r}$. We now apply Lemma 2 and deduce that

$$\mu(R_r' \cap U) \leq (1 - C^{-4})^{-3+\log_2((1-\lambda_{j+1})/\lambda_{j+1})} \mu(S_{j+1}^{i_r} \cap U),$$

since the density hypothesis of Lemma 2 is satisfied owing to the construction of the Cantor set F and the fact that $\lambda_{j+2} = \lambda_{j+1}/32$. Because of the density we may also choose disjoint balls $B_1, \dots, B_m$ with centers in U so that $\bigcup_{i=1}^m B_i \subset R'_r$ and $S' \cup R_r \subset \bigcup_{i=1}^m 2^4 B_i$, which implies that

$$\mu\big((S' \cup R_r) \cap U\big) \le C^4 \mu(R'_r \cap U).$$

Combining these observations we deduce that

$$\mu\biggl(\biggl(S_j^{i_0} \setminus \biggl(\bigcup_{r=1}^4 S_{j+1}^{i_r}\biggr)\biggr) \cap U\biggr) \le C^4 (1 - C^{-4})^{-3+\log_2((1-\lambda_{j+1})/\lambda_{j+1})} \mu\biggl(\biggl(\bigcup_{r=1}^4 S_{j+1}^{i_r}\biggr) \cap U\biggr),$$

which yields (4).

Finally, in the case $n \ge 3$ we choose $\Omega = \Omega_2 \times (0,1)^{n-2}$, where Ω_2 is the two-dimensional domain constructed above. The proof of the theorem and a higher dimensional version of Lemma 2 remains the same almost verbatim if one replaces the two-dimensional sets A figuring in the proof by sets of the form $A \times [0,1]^{n-2}$. There are only minor differences: the power C^4 has to be replaced by a power C^k, where k depends on the dimension n, and the density condition of Lemma 2 together with the choice of λ_j must be adjusted accordingly. □

Remark 1. Under some additional conditions on the domain one can ensure the existence of a doubling measure on Ω with respect to the Euclidean distance. We give a simple example: every doubling measure on $\overline{\Omega}$ is supported on Ω if the domain Ω satisfies the following condition:

(A) *There is a constant $k > 1$ with the following property*: *For every $x \in \partial\Omega$ and for every $\varepsilon > 0$ there is $y \in \Omega$ so that $d(y, \partial\Omega) < \varepsilon$ and $x \in kB\big(y, d(y, \partial\Omega)\big)$.*

The existence of a doubling measure on Ω satisfying (A) follows from the Vol'berg–Konyagin theorem applied to $\overline{\Omega}$ (or from [4] if Ω is unbounded). Notice that (A) is satisfied e.g. by John domains, but also by some bounded domains that are not John (for the definition of a John domain, see [5, 2.1]). In order to prove our claim, we assume for simplicity that Ω is bounded. Denote $A_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) \le \varepsilon\}$. Given $\varepsilon > 0$, assumption (A) and an application of a standard covering theorem (see e.g. [6, Theorem 2.1]) to the ball family $\mathscr{B} = \big\{B\big(y, d(y, \partial\Omega)\big) \mid y \in A_\varepsilon\big\}$ yield a constant k_0 (that does not depend on ε) and disjoint balls $B_1, B_2, \dots$ from the family $\mathscr{B}$ so that $\bigcup_{i=1}^\infty B_i \subset A_{2\varepsilon}$ and $\partial\Omega \subset \bigcup_{i=1}^\infty 2^{k_0} B_i$. Let μ be C-doubling on $\overline{\Omega}$. It follows that $\mu(\partial\Omega) \le C^{k_0} \mu(A_{2\varepsilon})$, where the right-hand side can be made arbitrarily small since $\bigcap_{j=1}^\infty A_{2^{-j}} = \emptyset$.

Remark 2. There are domains that are 'nearly John' but which do not carry a doubling measure. More precisely, there is a bounded domain $\Omega \subset \mathbf{R}^2$ which does not carry a doubling measure but whose boundary is accessible in the following sense:

(B) *Let $\varepsilon > 0$. For every boundary point $x \in \partial\Omega$ there is a path γ: $[0,1] \to \overline{\Omega}$ with finite length, and such that $\gamma(0) = x$ and $\gamma(1) = x_0$, where $x_0 \in \Omega$ is fixed. Moreover, for all $t \in (0,1]$ it holds that*

$$B\bigl(\gamma(t), C_\varepsilon s^{1+\varepsilon}\bigr) \subset \Omega,$$

where s is the length of the subarc $\gamma([0,t])$.

(Compare with the previous remark.) In fact, we may choose $\Omega = B(0,3)\backslash K$, where K is the planar Cantor set constructed in the proof of Theorem 3 with the choice $\lambda_j = 2^{-j/\log(j+2)}$. The proof for our claim is practically equal to the proof of Theorem 3, now only the density conditions are automatically satisfied (independently of the choice of (λ_j)) so that it remains to verify (5), which is immediate. Finally, it is not difficult to verify that Ω fulfils (B) with the choice $x_0 = \bigl(\frac{1}{2}, \frac{1}{2}\bigr)$.

Question. Suppose that Ω is a bounded domain in $\mathbf{R}^n$ and $\partial\Omega$ is piecewise given by the graph of a continuous function. Does Ω support a doubling measure?

We next turn to a general observation about nonexistence.

Lemma 4. *Let X be a metric space without isolated points and assume that the set $I = \{C \geq 1 \mid$ there is a C-doubling measure on $X\}$ is nonempty (notice that I is a half-line). Let $y \in X$. Then there are functions f_1, f_2: $(0,1) \times I \to (0,\infty)$ with the following properties:*

(i) *f_1 is decreasing with respect to the second variable and f_2 is increasing with respect to both variables. Moreover, $\lim_{r\to 0^+} f_2(r,C) = 0$ for every $C \in I$.*

(ii) *Every C-doubling measure μ on X satisfies*

$$f_1(r,C)\mu\bigl(B(y,1)\bigr) \leq \mu\bigl(B(x,r)\bigr) \leq f_2(r,C)\mu\bigl(B(y,1)\bigr)$$

for $x \in B(y,1)$ and $r \in (0,1)$.

Proof. We first conclude that X is doubling, since $I \neq \emptyset$. Assume that μ is a C-doubling measure on X. The existence of the lower bound is obtained by a standard argument: given $r \in (0,1)$ and $x \in B(y,1)$, choose $k_0 = \bigl[\log_2(1/r)\bigr] + 2$ and notice that $\mu\bigl(B(y,1)\bigr) \leq \mu\bigl(B(x,2^{k_0}r)\bigr) \leq C^{k_0}\mu\bigl(B(x,r)\bigr)$. Hence an appropriate choice for f_1 is $f_1(r,C) = C^{-2}r^{\log_2 C}$, which is clearly decreasing which respect to C since $r < 1$.

Towards the upper bound, we first consider the case where X is complete. Then, since metric doubling clearly implies total boundedness for bounded sets, we see that every closed ball of X is compact. For $r \in (0,1)$ and $C \in I$ define

$$f_2(r,C) = \sup_{x\in B(y,1)} \bigl\{\mu\bigl(B(x,r)\bigr) \mid \mu \text{ is } C\text{-doubling on } X \text{ with } \mu\bigl(B(y,1)\bigr) = 1\bigr\}.$$

Clearly f_2 is well-defined since the supremum is bounded from above by C, and it is increasing with respect to both variables. We show that $\lim_{r\to 0^+} f_2(r,C)=0$, where $C\in I$ is fixed. Assuming the contrary we deduce the existence of a sequence (x_j) with $x_j\in B(y,1)$, a sequence (r_j) of positive radii with $r_j<1/j$, and a sequence of C-doubling measures μ_j on X with $\mu_j\bigl(B(y,1)\bigr)=1$, and such that $\mu_j\bigl(B(x_j,r_j)\bigr)\ge c_0>0$ for all $j\ge 1$. Note that $1\le \mu_j\bigl(B_c(y,3)\bigr)\le C^2$. We may extract a subsequence (j_k) such that $\mu_{j_k}\to\mu$ in the weak*-topology of Borel measures on the compact ball $B_c(y,3)$ and $x_{j_k}\to a$ as $k\to\infty$. A simple reasoning (compare the proof of [4, Theorem 1]) shows that μ is nonzero and satisfies the doubling property (1) (with possibly a larger doubling constant) for $x\in B_c(y,1)$ and $r\in(0,1)$. Next, let $\varepsilon\in(0,1)$. There are arbitrarily large k such that $B(x_{j_k},r_{j_k})\subset B\bigl(a,\frac12\varepsilon\bigr)$. It follows that $\mu\bigl(B(a,\varepsilon)\bigr)\ge c_0$ and hence $\mu(\{a\})>0$. However, this is clearly impossible since μ is doubling (for small radii) on a neighborhood of the nonisolated point a.

Consider finally the general case where X may possibly be noncomplete. Let Y be the completion of X. Then Y is doubling and perfect so that the above reasoning yields an appropriate f_2 for Y and $y\in X\subset Y$. If μ is a C-doubling measure on X, then Lemma 1 yields a C-doubling extension $\tilde\mu$ on Y. Then $\mu\bigl(B^X(x,r)\bigr)\le f_2(r,C)\mu\bigl(B^X(y,1)\bigr)$ for $x\in B^X(y,1)$ and $r\in(0,1)$, by the definition of the extension $\tilde\mu$. Hence f_2 (or a suitable restriction of it) will do the job for X. □

Theorem 5. *Let X be a nonempty metric space without isolated points. Then there is a dense open set $A\subset X$ such that A does not carry a doubling measure.*

Proof. We may assume that X carries a C_0-doubling measure for some $C_0\ge 1$ since otherwise we may choose $A=X$. Fix $y\in X$ and select functions f_1 and f_2 that satisfy the conditions stated in Lemma 4. Choose a dense sequence $(y_k)_{k=1}^\infty$ of points in X (because X is doubling it is separable) so that $y_k\ne y$ for $k\ge 1$. The set A will be constructed in the form $A=\bigcup_{k=1}^\infty B(x_k,r_k)$, where the positive radii $r_k\in\bigl(0,\frac18\bigr)$ and the points $x_k\in X$ are chosen inductively in such a way that the following conditions are satisfied for each $n\ge 1$:

(i) $x_n=y_{k_n}$, where $k_n=\inf\bigl\{k\mid y_k\notin\overline{\bigcup_{j=1}^{n-1}B(x_j,r_j)}\bigr\}$,

(ii) $d\bigl(x_n,\overline{\bigcup_{j=1}^{n-1}B(x_j,r_j)}\bigr)\ge 3r_n$ if $n>1$,

(iii) $y\notin\overline{\bigcup_{j=1}^{n}B(x_j,r_j)}$,

(iv) $f_2(r_n,n)\le\dfrac{1}{2n}\min\{f_1(r_{n-1},n-1),f_2(r_{n-1},n-1)\}$ if $n\ge C_0+1$,

(v) $\{z\in X\mid d(z,x_n)=2r_n\}\ne\emptyset$.

In order to start the induction set $x_1=y_1$, and since x_1 is not an isolated point we may choose $r_1>0$ satisfying (v) and with $r_1<\min\bigl\{\frac12 d(y_1,y),\frac18\bigr\}$. Assume then that $n\ge 2$ and that $x_1,\dots,x_{n-1}$ together with $r_1,\dots,r_{n-1}$ have been chosen so that conditions (i)–(v) hold for the respective indexes. We next choose x_n according to (i). This is possible according to the induction hypothesis since (iii) implies that $\{y_k | k \geq 1\} \setminus \overline{\bigcup_{j=1}^{n-1} B(x_j, r_j)} \neq \emptyset$. Then (ii)–(iv) hold once r_n is chosen small enough, and also (v) may be satisfied since x_n is not isolated in X. The induction argument is complete. Note that (iv) implies for $n \geq C_0$ the estimate

$$\text{(iv)} \qquad f_1(r_n, n) \geq 2n f_2(r_{n+1}, n+1) \geq n \sum_{k=n+1}^{\infty} f_2(r_k, k),$$

since the infinite series is bounded from above by a geometric majorant which is obtained from the observation

$$f_2(r_k, k) \leq f_2(r_{k-1}, k-1)/2k \leq f_2(r_{k-1}, k-1)/2.$$

Assume then that μ is a C-doubling measure on A. Note that (i) implies the density of A in X, and hence Lemma 1 extends μ to a C-doubling measure $\tilde{\mu}$ on X. We may assume that $\tilde{\mu}\big(B(y,1)\big) = 1$ and $C \geq C_0$. By the fact $\overline{A} = X$ and (iii) we may choose $n \geq 2C^2$ so that $x_n \in B\big(y, \frac{1}{2}\big)$. According to (v) there is $z \in X$ satisfying $d(x_n, z) = 2r_n$. Write $B_1 = B(x_n, r_n)$ and $B_2 = B(z, r_n)$. Then $B_1 \cap B_2 = \emptyset$ and (ii) implies that also $B_2 \cap B(x_k, r_k) = \emptyset$ for all $k < n$. Hence we may apply Lemma 4 and (vi) in order to deduce that

$$\begin{aligned} \tilde{\mu}(B_2) &\leq \mu\Bigg(\bigcup_{\{k \geq n+1 | x_k \in B(y,1)\}} B(x_k, r_k)\Bigg) \leq \sum_{\{k \geq n+1 | x_k \in B(y,1)\}} f_2(r_k, C) \\ &\leq \sum_{k=n+1}^{\infty} f_2(r_k, k) \leq \frac{1}{n} f_1(r_n, n) \leq \frac{1}{2C^2} f_1(r_n, C) \leq \frac{1}{2C^2} \tilde{\mu}(B_1). \end{aligned}$$

This contradicts the facts that $\tilde{\mu}$ is C-doubling and $B_1 \subset 4B_2$. Hence A carries no nontrivial doubling measure. □

Remark 3. In the case where $X = [0,1]$, the existence of a dense open subset of X without a doubling measure may in fact be deduced as a simple consequence of a result due to Staples and Ward [8, Theorem 1.2]. Namely, in [8] a subset $K \subset [0,1]$ is called *quasisymmetrically thick* if there is no quasisymmetric map ϕ from $[0,1]$ onto $[0,1]$ such that $|\phi(K)| = 0$ (for the definitions and properties of quasisymmetric maps on the real line we refer to [1]). Here $|\cdot|$ refers to the Lebesgue measure. Choose a closed and quasisymmetrically thick subset $K \subset [0,1]$ with dense complement K^c (such sets are provided by [8, Theorem 1.2] and, not surprisingly, our reasoning for Theorem 5 partly resembles their proof). Assume that K^c carries a normalized doubling measure μ. Let $\tilde{\mu}$ be the extension of μ onto $[0,1]$ provided by Lemma 1. For $x \in [0,1]$ define $\phi(x) = \tilde{\mu}([0,x])$. Then ϕ is a quasisymmetric map of $[0,1]$ onto itself such that $|\phi(K)| = 0$, which is impossible. Hence K^c carries no doubling measures.

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Received July 1997