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DISTANCE BETWEEN DOMAINS IN THE SENSE OF LEHTO IS NOT A METRIC

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Abstract. In this paper we consider the question whether the quotient of the set of domains conformally equivalent to a halfplane by the group of Möbius transformations with distance in the sense of Lehto is a metric space. The answer is shown to be negative in the general case. However, restricted to analytic domains the question has an affirmative answer.

1. Introduction

Let $\mathscr{D}$ denote the set of complex domains conformally equivalent to the upper half-plane $H = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Throughout this paper we shall denote elements of $\mathscr{D}$ by $D, \widetilde{D}, \widetilde{\widetilde{D}}$. The Poincaré density of the hyperbolic metric of D is defined by

$$\eta_D(z) = \frac{|\pi_D'(z)|}{\operatorname{Im} \pi_D(z)},$$

where $\pi_D\colon D \to H$ is a conformal mapping onto H.

The Schwarzian derivative, or Schwarzian, of a conformal mapping $f\colon D \to \widetilde{D}$ is defined in D as the holomorphic function

$$S(f,z) = S_f(z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2.$$

The following two properties of the Schwarzian derivative are well known:
(1) $S_f \equiv 0$ if and only if f is a Möbius transformation.
(2) Cayley's formula

$$S(g \circ f, z) = S\big(g, f(z)\big) f'(z)^2 + S(f,z)$$

holds for conformal mappings $f\colon D \to \widetilde{D}$ and $g\colon \widetilde{D} \to \widetilde{\widetilde{D}}$.

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Furthermore, if a weighted sup-norm is defined for functions φ holomorphic in D by

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)| \eta_D(z)^{-2},$$

we have the identities (see e.g. [3])

$$\|S_{g \circ f^{-1}}\|_{\widetilde{D}} = \|S_g - S_f\|_D, \qquad \|S_f\|_D = \|S_{f^{-1}}\|_{\widetilde{D}}. \tag{3}$$

Two domains D and $\widetilde{D}$ are said to be Möbius equivalent if there is a Möbius transformation of D onto $\widetilde{D}$. We denote by Ω the quotient set of the set $\mathscr{D}$ by the group of Möbius transformations. The Lehto distance between two domains D and $\widetilde{D}$ in $\mathscr{D}$ is defined by

$$\delta(D, \widetilde{D}) = \inf\{\|S_f\|_D : f\colon D \to \widetilde{D} \text{ conformal}\}.$$

The above properties of the Schwarzian derivative imply that

$$\delta(D, \widetilde{D}) = \delta(\widetilde{D}, D),$$
$$\delta(D, \widetilde{\widetilde{D}}) \le \delta(D, \widetilde{D}) + \delta(\widetilde{D}, \widetilde{\widetilde{D}})$$

for all domains $D, \widetilde{D}$ and $\widetilde{\widetilde{D}}$ in $\mathscr{D}$. Because $\delta(D, \widetilde{D}) = 0$ for Möbius equivalent domains D and $\widetilde{D}$, the Lehto distance δ defines a pseudometric in the quotient set Ω. We shall see in the sequel that for the subset of Ω consisting of domains with analytic boundaries the Lehto distance does, indeed, define a metric, but that for an arbitrary domain $D \in \mathscr{D}$, the equality $\delta(D, \widetilde{D}) = 0$ does not imply the Möbius equivalence of D and $\widetilde{D}$.

2. Reducing the problem to quadratic differentials

We can consider the space

$$Q = \{\varphi : \varphi \text{ holomorphic in } H,\ \|\varphi\|_H < \infty\}$$

as the space of quadratic differentials of bounded norm corresponding to the universal Teichmüller space $T(H)$; cf. [3]. If D is a domain in $\mathscr{D}$ and $f\colon H \to D$ a conformal mapping of the upper half-plane H onto D, the Kraus–Nehari theorem implies the norm inequality $\|\varphi\|_H \le \frac{3}{2}$. Thus every conformal mapping $f\colon H \to D$ of a domain $D \in \mathscr{D}$ determines a corresponding quadratic differential $\varphi = S_f \in Q$. For another domain $\widetilde{D} \in \mathscr{D}$ and conformal mapping $\tilde{f}\colon H \to \widetilde{D}$ we denote the respective quadratic differential by $\widetilde{\varphi} = S_{\tilde{f}} \in Q$. The vanishing $\delta(D, \widetilde{D}) = 0$ of the Lehto distance is equivalent to the existence of a sequence $g_n\colon D \to \widetilde{D}$ of conformal mappings so that $\|S_{g_n}\|_D \to 0$ when $n \to \infty$. For each g_n there is a Möbius transformation $A\colon H \to H$ such that $g_n = \tilde{f} \circ A_n{}^{-1} \circ f^{-1}$. Now by (3) we have the identity $\|S_{g_n}\|_D = \|S_{\tilde{f}} - S_{f \circ A_n}\|_H$, and thus we get the following

Lemma 1A. *The vanishing* $\delta(D,\widetilde{D}) = 0$ *of the Lehto distance between the domains* $D,\widetilde{D} \in \mathscr{D}$ *is equivalent to the existence of a sequence of Möbius transformations* $A_n\colon H \to H$ *such that* $\|\varphi - \varphi_n\|_H \to 0$ *when* $n \to \infty$, *where the sequence of quadratic differentials* $\varphi_n \in Q$ *is defined by* $\varphi_n(z) = \varphi(A_n z)A_n'(z)^2 = S_{f\circ A_n}$.

Similarly we have for the Möbius equivalence of domains

Lemma 1B. *Two domains* $D,\widetilde{D} \in \mathscr{D}$ *are Möbius equivalent, if and only if there exists a Möbius transformation* $A\colon H \to H$ *such that* $\widetilde{\varphi} = \varphi(Az)A'(z)^2$.

For any quadratic differential $\varphi \in Q$ we define the subset $N(\varphi) \subset Q$ by

$$N(\varphi) = \{(\varphi \circ A){A'}^2 : A\colon H \to H \text{ is a Möbius transformation}\}.$$

As an immediate consequence of Lemmas 1A and 1B we have

Lemma 1C. *Let* $f\colon H \to D$ *be a conformal mapping. The set of all* $\widetilde{D} \in \mathscr{D}$ *which satisfy* $\delta(D,\widetilde{D}) = 0$ *is equal to the set of domains Möbius equivalent to* D *if and only if* $N(S_f)$ *is closed in* Q.

Finally we prove

Lemma 1D. (Ω,δ) *is a metric space, if and only if* $N(\varphi)$ *is closed in* Q *for all* $\varphi \in Q$.

Proof. Should $N(\varphi)$ be closed in Q for each $\varphi \in Q$, then $\delta(D,\widetilde{D}) = 0$ if and only if D and $\widetilde{D}$ are Möbius equivalent by Lemma 1C, so that (Ω,δ) would be a metric space. Conversely any $\varphi \in Q$ with $\|\varphi\|_H, < \frac{1}{2}$ is the Schwarzian of a conformal mapping of the upper half-plane H by the Ahlfors–Weill theorem (cf. [2]). So should δ define a metric in Ω, then $N(\varphi)$ would by Lemma 1C be closed wherever $\|\varphi\|_H < \frac{1}{2}$, and thus actually for all $\varphi \in Q$. □

3. A metric for analytic domains

A domain $D \in \mathscr{D}$ is analytic, if the boundary of D is an analytic curve, i.e. the image of a circle K under a conformal mapping defined in a neighbourhood of K. We denote by $C_0(H)$ the subspace of $C(H)$ consisting of continuous functions f vanishing on the boundary of H, and define a subspace Q_0 of Q by

$$Q_0 = \{\varphi \in Q : \varphi\eta_H^{-2} \in C_0(H)\}.$$

Lemma 2. $S_f \in Q_0$ *when* $f\colon H \to D$ *is a conformal mapping onto an analytic domain* D.

Proof. Let $M\colon H \to U$ be a Möbius transformation of H onto the unit disc U, and $g = f \circ M^{-1}\colon U \to D$. Because D is an analytic domain, the mapping g extends to a conformal mapping defined in a neighbourhood of the closed unit disk $\overline{U}$. Having a holomorphic extension into a neighbourhood of $\overline{U}$, the Schwarzian S_g remains bounded on $\overline{U}$, so that $\lim_{|w|\to 1} S_g(w)\eta_U^{-2}(w) = 0$. Thus we also have $\lim_{z\to r} S_f(w)\eta_H^{-2}(z) = 0$ for every boundary point $r \in \overline{\mathbf{R}}$ of H, so that the Schwarzian S_f belongs to Q_0 when D is an analytic domain. □

Lemma 3. *$N(\varphi)$ is closed in Q for every $\varphi \in Q_0$.*

Proof. Let $\varphi \in Q_0$ and A_n a sequence of Möbius automorphisms of H such that $\varphi_n = (\varphi \circ A_n)A'^2_n \to \widetilde{\varphi}$ in Q when $n \to \infty$. By the compactness principle (e.g. [4]) we may suppose that the sequence A_n converges locally uniformly in H either to a Möbius automorphism $A\colon H \to H$, or to a constant $c \in \overline{\mathbf{R}}$. Should the sequence A_n converge to a constant function, we would have for all $z \in H$

$$\widetilde{\varphi}(z)\eta_H(z)^{-2} = \lim_{n\to\infty} \varphi(A_n z)\eta_H(A_n z)^{-2} = 0$$

because $\varphi \in Q_0$. Thus $\widetilde{\varphi}$ vanishes identically, and we have $\|\widetilde{\varphi}\|_H = 0$. But $\varphi_n \to \widetilde{\varphi}$ in Q, and as $\|\varphi_n\|_H = \|\varphi\|_H$ for all n, and we have $\varphi = \widetilde{\varphi} = 0 \in Q_0$, so that $\widetilde{\varphi} = \varphi \in N(\varphi)$. When the sequence A_n converges to a Möbius automorphism A, we have $\widetilde{\varphi} = (\varphi \circ A)A'^2$, which obviously belongs to $N(\varphi)$. Thus $N(\varphi)$ is a closed subset of Q for all $\varphi \in Q_0$. □

Denoting by Ω_A the quotient of the set of analytic domains by the group of Möbius transformations we get as an immediate consequence of Lemmas 2, 3 and 1D.

Theorem 1. *(Ω_A, δ) is a metric space.*

4. (Ω, δ) is not a metric space

We are going to give here three slightly different examples of quadratic differentials $\varphi \in Q$ for which $N(\varphi)$ is not closed in Q. To do this we construct a quadratic differential $\varphi \in Q$ and determine a sequence of Möbius transformations $A_n\colon H \to H$ so that the sequence $\varphi_n = (\varphi \circ A_n)A'^2_n$ converges in Q towards a quadratic differential $\widetilde{\varphi}$ not in $N(\varphi)$. Particularly, if we choose φ so that $\|\varphi\|_H < \frac{1}{2}$, then by the Ahlfors–Weill theorem there are conformal mappings $f\colon H \to D$ and $\tilde{f}\colon H \to \widetilde{D}$ with $S_f = \varphi$, $S_{\tilde{f}} = \widetilde{\varphi}$, so that we have $\delta(D, \widetilde{D}) = 0$ for the image domains D and $\widetilde{D}$, although D and $\widetilde{D}$ are not Möbius equivalent.

To begin with let us note that for all $a > 0$ the function e^{iaz} is in Q with the norm $\|e^{iaz}\|_H = (2/ae)^2$.

Example 1. Let $\varphi(z) = e^{2\pi i z} + e^{iz}$, $\widetilde{\varphi}(z) = e^{2\pi i z} - e^{iz}$ and $A_k(z) = z + n_k$, where $n_k = 2\pi m_k + \pi + o(1)$ when $k \to \infty$ ($n_k, m_k \in \mathbf{N}$). To establish the existence of such a sequence n_k we notice that the convergents $P_s/Q_s, P_s, Q_s \in \mathbf{N}$, of the continued fraction expansion of π satisfy

$$|\pi - P_s/Q_s| < \frac{1}{Q_s Q_{s+1}} < \frac{1}{Q_s^2}, \qquad P_s Q_{s-1} - Q_s P_{s-1} = (-1)^s$$

with a strictly increasing sequence of denominators Q_s (see e.g. [5]). Thus by choosing an appropriate subsequence we have $P_{s_k} = n_k$, $Q_{s_k} = 2m_k + 1$ with n_k, m_k satisfying the above conditions.

It is not difficult to see that $(\varphi \circ A_k) A_k'^2 \to \widetilde{\varphi}$ in Q. It remains to be shown that

$$\widetilde{\varphi} = (\varphi \circ A) A'^2 \tag{4}$$

holds for no Möbius transformation $A\colon H \to H$. Now φ has zeros at $z_k = (2k+1)\pi/(2\pi - 1)$ and $\widetilde{\varphi}$ at $\tilde{z}_k = 2k\pi/(2\pi - 1)$, $k \in \mathbf{Z}$. So should (4) hold for a Möbius transformation, then A would be a translation by an odd integral multiple of $\pi/2\pi - 1$. Now for any $c \in \mathbf{R}$ the coefficients of $e^{2\pi i z}$ and e^{iz} in the expansion

$$\varphi(z + c) = e^{2\pi i c} e^{2\pi i z} + e^{ic} e^{iz}$$

are uniquely determined. However, the equations $e^{2\pi i c} = 1$ and $e^{ic} = -1$ cannot be simultaneously satisfied for any c, so that $\widetilde{\varphi} \neq (\varphi \circ A) A^2$ holds for all Möbius automorphisms A of the upper half-plane H.

Example 2. Suppose that the sequence λ_k satisfies $\sum_{k=1}^{\infty} 3^{2k} e^{-2} \pi^{-2} |\lambda_k| < +\infty$ and that $\lambda_k \neq 0$ for infinitely many $k \in \mathbf{N}$. We define φ and $\widetilde{\varphi}$ by

$$\varphi(z) = \frac{e^{i\pi z}}{(1 - e^{i\pi z})^2} + \sum_{k=1}^{\infty} \lambda_k e^{2i\pi z/3^k},$$

$$\widetilde{\varphi}(z) = \frac{-e^{i\pi z}}{(1 + e^{i\pi z})^2} + \sum_{k=1}^{\infty} \lambda_k e^{2i\pi z/3^k}.$$

Since $\|e^{i\pi z}/(1 - e^{i\pi z})^2\|_H = 1/\pi^2$ and $\|e^{2i\pi z/3^k}\|_H = 3^{2k}/e^2\pi^2$, we conclude that $\varphi, \widetilde{\varphi} \in Q$. Setting $A_n(z) = z + 3^n$ it is easy to see that $(\varphi \circ A_n) A_n'^2 \to \widetilde{\varphi}$ in Q. It remains to be shown that $\widetilde{\varphi} \notin N(\varphi)$, i.e., that (4) cannot hold for any Möbius automorphism A of H. Now we notice that the limit

$$\lim_{z \to r} \varphi(z) \eta_H^{-2}(z)$$

is 0 for all $r \in \mathbf{R} \setminus 2\mathbf{Z}$ and does not exist if $r \in 2\mathbf{Z}$. Similarly, the expression $\widetilde{\varphi}(z)\eta_H^{-2}(z)$ equals 0 at all boundary points $r \in \mathbf{R}$ except for the set $2\mathbf{Z}+1$ of odd integers. Thus a Möbius transformation A satisfying (4) would be a translation by an odd integer. But in the expansion

$$\varphi(z+2l+1) - \widetilde{\varphi}(z) = \sum_{k=1}^{\infty} \lambda_k (e^{2i\pi(2l+1)/3^k} - 1) e^{2i\pi z/3^k}$$

the coefficients of $e^{i\pi z/3^k}$ are uniquely determined, so that $\varphi(z+2l+1) = \widetilde{\varphi}(z)$ cannot hold for any $2l+1 \in \mathbf{Z}$.

Example 3. We now define φ and $\widetilde{\varphi}$ by

$$\varphi(z) = (e^{2\pi iz} - e^{-\pi}) \Big(\sum_{k=1}^{\infty} \frac{1}{10^k} e^{\pi iz/2^k} \Big),$$

$$\widetilde{\varphi}(z) = (e^{2\pi iz} - e^{-\pi}) \Big(\sum_{k=1}^{\infty} \frac{1}{10^k} e^{\pi i/2^k (z + (4^k-1)/3)} \Big),$$

and set $A_n(z) = z + \frac{1}{3}(4^n - 1)$. Since $\|e^{\pi iz/2^k}\|_H = 4^{k+1}/e^2\pi^2$, the series

$$s(z) = \sum_{k=1}^{\infty} \frac{1}{10^k} e^{\pi iz/2^k}$$

converges in Q. Because $e^{2\pi iz} - e^{-\pi}$ is bounded in H, both functions φ and $\widetilde{\varphi}$ are in Q, and it is not difficult to see that $(\varphi \circ A_n)A_n'^2 \to \widetilde{\varphi}$ in Q.

For $\mathrm{Im}(z) \leq 1$ we have

$$\Big| \frac{1}{10^k} e^{\pi iz/2^k} \Big| > 2 \Big| \frac{1}{10^{k+1}} e^{\pi iz/2^{k+1}} \Big|,$$

so that the series $s(z)$ has no zeros with $\mathrm{Im}(z) < 1$. Since all zeros of $e^{2\pi iz} - e^{-\pi}$ are lying on the line $\mathrm{Im}(z) = \frac{1}{2}$, we see that the functions φ and $\widetilde{\varphi}$ do not have any zeros in the horizontal strip $\{z \in H : \mathrm{Im}(z) < \frac{1}{2}\}$. On the other hand, the functions φ and $\widetilde{\varphi}$ have zeros on the line $\mathrm{Im}(z) = \frac{1}{2}$ at the points $\frac{1}{2}i + l$ with $l \in \mathbf{Z}$. We conclude that a Möbius automorphism of H satisfying $\widetilde{\varphi} = (\varphi \circ A)A'^2$ would be a translation by an integer. Now for any $l \in \mathbf{Z}$ we have an expansion

$$\varphi(z+l) - \widetilde{\varphi}(z) = (e^{2\pi iz} - e^{-\pi}) \sum_{k=1}^{\infty} c_n e^{\pi iz/2^k}$$

with uniquely determined coefficients c_n. Hence, for $\varphi(z+l) - \widetilde{\varphi}(z)$ to vanish identically we should have

$$l \equiv \tfrac{1}{3}(4^k - 1) = 1 + 4 + \cdots + 4^{k-1} \pmod{2^{k+1}}$$

for every $k = 1, 2, \ldots$. This is obviously impossible for any fixed $l \in \mathbf{Z}$, so that $\widetilde{\varphi} \neq (\varphi \circ A)A'^2$ for all Möbius transformations A.

By Lemma 1D and the above three counterexamples we get thus the following theorem.

Theorem 2. (Ω, δ) *is not a metric space.*

5. Further results and discussion

Having shown in the previous section that $N(\varphi)$ is not closed for some $\varphi \in Q$ we now ask how large the set of such φ actually is.

Theorem 3. *The set of quadratic differentials $\varphi \in Q$ with $N(\varphi)$ not closed is nowhere dense in Q.*

Proof. We proved above that $N(\varphi)$ is not closed in Q if and only if there is a sequence of Möbius transformations A_n of H such that $(\varphi \circ A_n){A'}_n^2 \to \widetilde{\varphi}$ and $A_n \to c \in \overline{\mathbf{R}}$. We shall show that this is possible only for quadratic differentials φ in a nowhere dense subset of Q.

Let $\Psi_0 \in Q$ be an arbitrary quadratic differential. We shall show that within any ball $B \subset Q$ of radius $\varepsilon > 0$ centered at Ψ_0 there is a ball B' of radius $\frac{1}{4}\varepsilon$ such that $N(\varphi)$ is closed for every $\varphi \in B'$. First choose a quadratic differential $\Phi_0 \in Q_0$ with $\|\Phi_0\|_H = 1$ and a point $z_0 \in H$ such that $|\Psi_0(z_0)|\eta_H(z_0)^{-2} > \|\Psi_0\|_H - \frac{1}{4}\varepsilon$, and further a point $z_1 \in H$ with $|\Phi_0(z_1)|\eta_H(z_1)^{-2}$ sufficiently close to 1 and a Möbius transformation $A\colon H \to H$ mapping z_0 to z_1 so that

$$\left|\tfrac{3}{4}\varepsilon\Phi_0(Az_0)A'(z_0)^2 e^{i\theta} + \Psi_0(z_0)\right|\eta_H(z_0)^{-2} > \|\Psi_0\|_H + \tfrac{1}{2}\varepsilon.$$

Thus $\|\Psi_0 + \Phi\|_H > \|\Psi_0\|_H + \frac{1}{2}\varepsilon$ when Φ is defined by

$$\Phi = \tfrac{3}{4}\varepsilon(\Phi_0 \circ A){A'}^2 e^{i\theta}.$$

For any $\|\Phi_1\|_H < \frac{1}{4}\varepsilon$, the quadratic differential $\varphi = (\Psi_0 + \Phi) + \Phi_1$ is contained in the ball B of radius ε centered at Ψ_0 with a norm satisfying $\|\varphi\|_H > \|\Psi_0\|_H + \frac{1}{4}\varepsilon$. Because Φ, too, belongs to the subspace Q_0, we have $(\Phi \circ A_n){A'}_n^2 \to 0$ in H for any sequence A_n of Möbius automorphisms of H converging to a constant. Assuming that $(\varphi \circ A_n){A'}_n^2 \to \widetilde{\varphi} \in Q$ when $n \to +\infty$ we would thus also have $\bigl((\Psi_0 + \Phi_1) \circ A_n\bigr){A'}_n^2 \to \widetilde{\varphi}$ in H as $n \to +\infty$. But $\|\widetilde{\varphi}\|_H = \|\varphi\|_H > \|\Psi_0\|_H + \frac{1}{4}\varepsilon$, so that for some $z \in H$ and $n > n_0$ we would have

$$\left|(\Psi_0 + \Phi_1)(A_n z)A'_n(z)^2\right|\eta_H(z)^{-2} = \left|(\Psi_0 + \Phi_1)(A_n z)\right|\eta_H\bigl(A_n(z)\bigr)^{-2} > \|\Psi_0\|_H + \tfrac{1}{4}\varepsilon,$$

contradicting the inequality $\|\Psi_0 + \Phi_1\|_H < \|\Psi_0\|_H + \frac{1}{4}\varepsilon$. □

Let us finally discuss the background of our examples, especially Examples 3 and 2.

Let Γ_j be a decreasing sequence of Fuchsian groups and denote by $Q(\Gamma_j)$ the increasing sequence of subspaces of Q consisting of all quadratic differentials φ satisfying $\varphi = (\varphi \circ A){A'}^2$ for all $A \in \Gamma_j$. Let $P = \overline{\bigcup_{j=1}^{\infty} Q(\Gamma_j)}$. Consider now sequences A_n of Möbius transformations such that for every j, the transformations A_n are eventually in the same right coset of the group Γ_j. Thus for every j there is N_j such that $(\varphi \circ A_m){A'}_m^2 = (\varphi \circ A_n){A'}_n^2$ for all $\varphi \in Q(\Gamma_j)$ whenever $n, m > N_j$. Such a sequence A_n obviously induces a mapping $F\colon P \to Q$ when $F(\varphi)$ is defined by $F(\varphi) = \lim_{n\to\infty} (\varphi \circ A_n){A'}_n^2 \in Q$. Any fixed Möbius transformation $A\colon H \to H$ determines a mapping $F_A\colon P \to Q$, $\widetilde{\varphi}_A(\varphi) = (\varphi \circ A){A'}^2$ corresponding to the constant sequence $A_n = A$.

If $P \neq \bigcup_{j=1}^{\infty} Q(\Gamma_j)$, there can be mappings F which are not equal to F_A for any Möbius transformation A. This can be clearly seen from Example 3. There Γ_j is the group of translations by integral multiples of 2^j, and the mapping F corresponding to the sequence A_n acts as if it were a "translation" by the 2-adic number $1 + 4 + 4^2 + \cdots$. We could choose any other 2-adic number and "translate" any element of P, which is here the smallest closed subspace of Q containing all functions $e^{m\pi i z/2^n}$, $m, n \in \mathbf{N}$.

We end this paper by posing the natural problem of characterizing all $\varphi \in Q$ such that $N(\varphi)$ is closed in Q.

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