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THE ALGEBRA PROPERTY OF THE INTEGRALS OF SOME ANALYTIC FUNCTIONS IN THE UNIT DISK

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Abstract. We consider the integrals of functions in the α-Bloch space $\mathscr{B}^\alpha$, in the Besov space B_p, in the Möbius invariant subspace Q_p of the weighted Dirichlet space, and we show that they form algebras with respect to the multiplication.

Introduction

Let X be a Banach space of analytic functions in the unit disk $D = \{z \in \mathbf{C}, |z| < 1\}$. For $f \in X$ we denote by F the function

$$F(z) = \int_0^z f(\zeta)\, d\zeta, \qquad z \in D.$$

The Banach spaces X we are considering in this paper are the following.

The α-Bloch space $\mathscr{B}^\alpha$, $\alpha > 0$, is defined to be the space of all functions f with

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

By $\mathscr{B}_0^\alpha$ we denote the space of all functions f with

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0$$

([10], [11], [13]). For $\alpha = 1$ we get the well-known Bloch space, denoted by $\mathscr{B}$, and the little Bloch space, denoted by $\mathscr{B}_0$ ([1], [2]).

The Besov space B_p, $1 < p < \infty$, consists of all functions f with

$$\|f\|_{B_p} = |f(0)| + \left\{\iint_D (1-|z|^2)^{p-2}|f'(z)|^p\, dx\, dy\right\}^{1/p} < \infty,$$

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$z = x + iy$. It is clear that $\mathscr{B} = B_\infty$. The space B_1 is defined as the space of all functions f with

$$\|f\|_{B_1} = |f(0)| + |f'(0)| + \iint_D |f''(z)|\,dx\,dy < \infty.$$

It is known that $B_1 \subset \mathscr{A}$, where $\mathscr{A}$ denotes the disk algebra of D, that is, all functions analytic in D and continuous on $\overline{D}$. Also $B_p \subset B_q \subset \mathscr{B}$ if $1 \le p < q$. Note that B_2 is the classical Dirichlet space $\mathscr{D}$ of the functions f with $\iint_D |f'(z)|^2\,dx\,dy < \infty$ (for the basic theory of B_p spaces see [3] or [12, p. 88–93]).

Finally we denote by Q_p, $0 < p < \infty$, the space of all functions f with

$$\|f\|_{Q_p} = |f(0)| + \left\{\sup_{\zeta\in D} \iint_D |f'(z)|^2 \log^p\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right|\,dx\,dy\right\}^{1/2} < \infty,$$

and by $Q_{p,0}$, $0 < p < \infty$, the space of all functions f with

$$\lim_{|\zeta|\to 1} \iint_D |f'(z)|^2 \log^p\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right|\,dx\,dy = 0.$$

It is known ([4]) that for $1 < p < \infty$ the spaces Q_p ($Q_{p,0}$) are all the same and equal to the Bloch space $\mathscr{B}$ ($\mathscr{B}_0$). We note that in [8] a boundary value criterion for functions in Q_p ($Q_{p,0}$) is given, and in [9] a Corona type theorem is proved for Q_p, $0 < p < 1$.

For $p = 1$ we have $Q_1 = \mathrm{BMOA}$, $Q_{1,0} = \mathrm{VMOA}$, where BMOA and VMOA are the classical spaces of analytic functions of bounded mean oscillation and vanishing mean oscillation ([12, p. 179]). For $0 < p < q$ it is known $Q_p \subset Q_q$ ([5]). The space $Q_{1,0} = \mathrm{VMOA}$ contains all B_p functions for $1 \le p < \infty$ ([12, p. 188]).

For the integrals F of Bloch and BMOA functions it is known that they form algebras with respect to the multiplication ([2], [6]).

1. The main result

In this paper we prove that the integrals F of functions in $\mathscr{B}^\alpha$ and in $\mathscr{B}^\alpha_0$ for $0 < \alpha < 2$, in B_p for $1 \le p < \infty$, in Q_p and in $Q_{p,0}$ for $0 < p < \infty$ form algebras with respect to the multiplication.

Our proposition is formulated as follows.

Theorem 1. *Let f_1, f_2 be two functions in X, where X is one of the following Banach spaces of analytic functions in the unit disk D: $\mathscr{B}^\alpha$, $\mathscr{B}^\alpha_0$ for $0 < \alpha < 2$, B_p for $1 \le p < \infty$, and Q_p, $Q_{p,0}$ for $0 < p < \infty$. If*

$$F_j(z) = \int_0^z f_j(\zeta)\,d\zeta, \qquad j = 1, 2,\ z \in D,$$

then

$$F_1(z)F_2(z) = \int_0^z h(\zeta)\,d\zeta, \qquad z \in D,$$

where $h \in X$ *and* $\|h\|_X \le C\|f_1\|_X\,\|f_2\|_X$, *where* C *depends only on* X.

Revising the proof of Theorem 1 we are able to prove a slightly different version.

Theorem 2. *Let* X *be one of the Banach spaces in Theorem* 1. *If*

$$\Phi_j(z) = \frac{1}{z}\int_0^z f_j(\zeta)\,d\zeta, \qquad j = 1, 2,\ z \in D,$$

then

$$\Phi_1(z)\Phi_2(z) = \frac{1}{z}\int_0^z h(\zeta)\,d\zeta, \qquad z \in D,$$

where $h \in X$. *Furthermore,* $\|h\|_X \le C\|f_1\|_X\,\|f_2\|_X$, *where* C *depends only on* X.

2. Proofs of the theorems

Proof of Theorem 1. First we show that the functions $F(z) = \int_0^z f(\zeta)\,d\zeta$, $z \in D$, belong to the disk algebra $\mathscr{A}$ if $f \in X$, where X is one of the Banach spaces in our theorem. If $X \subset \mathscr{B}$, we have (2.3) below which is a sufficient condition for F to be in $\mathscr{A}$ (cf. [7, Theorem 5.5.2]). This covers all cases except $\mathscr{B}^\alpha$, $1 < \alpha < 2$: here we have to use [7, Theorem 5.5.1] and estimate (2.2) below.

We show that if X is one of our Banach spaces, then

$$\|F\|_\infty \le C\|f\|_X. \tag{2.1}$$

For a function $f \in \mathscr{B}^\alpha$ we have $|f'(z)| \le \|f\|_{\mathscr{B}^\alpha}/(1-|z|^2)^\alpha$. By integration we get

$$\begin{aligned} |f(z)| &\le C\|f\|_{\mathscr{B}^\alpha}, && 0 < \alpha < 1,\\ |f(z)| &\le C\frac{\|f\|_{\mathscr{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, && 1 < \alpha < 2, \end{aligned} \tag{2.2}$$

and

$$|f(z)| \le \|f\|_{\mathscr{B}}\left(1 + \log\frac{1}{1-|z|}\right) \qquad \text{for } \alpha = 1. \tag{2.3}$$

Integrating again, we obtain (2.1) in the case $X = \mathscr{B}^\alpha$.

In the cases $X = B_p$, $1 \le p < \infty$, or $X = Q_p$, $0 < p < \infty$, we have $X \subset \mathscr{B}$ and we know that F must be in $\mathscr{A}$. Furthermore,

$$\|F\|_\infty \le C\|f\|_{\mathscr{B}} \le C\|f\|_X,$$

which is (2.1).

Next, we consider separately the various cases of our Banach spaces X.

(i) $X = \mathscr{B}^\alpha$, $0<\alpha<2$. First we note that $(F_1F_2)'(0)=0$. Further it follows from (2.2) and (2.3) that

$$
\begin{aligned}
|(F_1F_2)''| &= |F_1''F_2 + F_2''F_1 + 2F_1'F_2'| \\
&\le |f_1'||F_2| + |f_2'||F_1| + 2|f_1||f_2| \\
&\le C|f_1'|\,\|f_2\|_{\mathscr{B}^\alpha} + C|f_2'|\,\|f_1\|_{\mathscr{B}^\alpha} + C\|f_1\|_{\mathscr{B}^\alpha}\|f_2\|_{\mathscr{B}^\alpha}
\end{aligned}
\tag{2.4}
$$

for $0<\alpha<1$,

$$
|(F_1F_2)''| \le C|f_1'|\,\|f_2\|_{\mathscr{B}^\alpha} + C|f_2'|\,\|f_1\|_{\mathscr{B}^\alpha} + C\|f_1\|_{\mathscr{B}^\alpha}\|f_2\|_{\mathscr{B}^\alpha}\frac{1}{(1-|z|^2)^{2(\alpha-1)}}
\tag{2.5}
$$

for $1<\alpha<2$, and

$$
|(F_1F_2)''| \le C|f_1'|\,\|f_2\|_{\mathscr{B}} + C|f_2'|\,\|f_1\|_{\mathscr{B}} + C\|f_1\|_{\mathscr{B}}\|f_2\|_{\mathscr{B}}\left(1+\log\frac{1}{1-|z|}\right)^2
\tag{2.6}
$$

for $\alpha=1$.

By the above estimates and an easy calculation we get

$$
\|(F_1F_2)'\|_{\mathscr{B}^\alpha} \le C\|f_1\|_{\mathscr{B}^\alpha}\|f_2\|_{\mathscr{B}^\alpha}
$$

for $0<\alpha<2$.

(ii) $X = B_p$, $1\le p<\infty$. First we assume that $1<p<\infty$. We have

$$
\begin{aligned}
|(F_1F_2)''|^p &= |f_1'F_2 + f_2'F_1 + 2f_1f_2|^p \\
&\le C|f_1'|^p\|f_2\|_{B_p}^p + C|f_2'|^p\|f_1\|_{B_p}^p + C|f_1|^p|f_2|^p,
\end{aligned}
\tag{2.7}
$$

where C depends only on p. Since $f_j\in B_p$ implies $f_j\in\mathscr{B}$, we have

$$
\begin{aligned}
|f_j(z)| &\le \|f_j\|_{\mathscr{B}}\log\left(1+\frac{1}{1-|z|}\right) \\
&\le C\|f_j\|_{B_p}\log\left(1+\frac{1}{1-|z|}\right), \qquad z\in D,\ j=1,2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|(F_1F_2)'\|_{B_p}^p &= \iint_D (1-|z|^2)^{p-2}|(F_1F_2)''|^p\,dx\,dy \\
&\le C\|f_1\|_{B_p}^p\|f_2\|_{B_p}^p \\
&\quad + C\|f_1\|_{B_p}^p\|f_2\|_{B_p}^p\iint_D (1-|z|^2)^{p-2}\left(1+\log\frac{1}{1-|z|}\right)^{2p}dx\,dy \\
&\le C\|f_1\|_{B_p}^p\|f_2\|_{B_p}^p,
\end{aligned}
$$

which implies that

$$\|h\|_{B_p} \le C\|f_1\|_{B_p}\,\|f_2\|_{B_p}.$$

Keeping track of the constants, we can let $p \longrightarrow \infty$ to obtain the algebra property of integrals of functions in $\mathscr{B}$ (cf. [2, Section 3.5]).

We consider now the case $X = B_1$. By definition

$$\|(F_1F_2)'\|_{B_1} = |(F_1F_2)'(0)| + |(F_1F_2)''(0)| + \iint_D |(F_1F_2)'''|\,dx\,dy.$$

We observe that

$$(F_1F_2)'(0) = 0$$

and

$$|(F_1F_2)''(0)| = 2|f_1(0)|\,|f_2(0)| \le 2\|f_1\|_{B_1}\|f_2\|_{B_1}.$$

A simple calculation shows

$$|(F_1F_2)'''| \le C|F_1|\,|f_2''| + C|F_2|\,|f_1''| + C|f_1|\,|f_2'| + C|f_2|\,|f_1'|. \tag{2.8}$$

Now the functions f_j, $j = 1,2$, are in the disk algebra $\mathscr{A}$, and

$$\|f_j\|_\infty \le C\|f_j\|_{B_1}, \qquad j = 1,2. \tag{2.9}$$

Further

$$\|F_j\|_\infty \le C\|f_j\|_{B_1}, \tag{2.10}$$

and by [12, p. 58, Remark]

$$\iint_D |f_j'(z)|\,dx\,dy \le C|f_j'(0)| + C\iint_D |f_j''(z)|\,dx\,dy \le C\|f_j\|_{B_1}. \tag{2.11}$$

From (2.8), (2.9), (2.10) and (2.11) it follows immediately that

$$\|(F_1F_2)'\|_{B_1} \le C\|f_1\|_{B_1}\|f_2\|_{B_1}.$$

(iii) $X = Q_p$, $0 < p < \infty$. For a function $f \in Q_p$, $0 < p < \infty$, we have $f \in \mathscr{B}$, so that

$$|f(z)| \le \|f\|_{\mathscr{B}}\left(1 + \log\frac{1}{1-|z|}\right) \le C\|f\|_{Q_p}\left(1 + \log\frac{1}{1-|z|}\right), \qquad z \in D,$$

and $\|F\|_\infty \le C\|f\|_{Q_p}$, where C is a constant depending only on p.

By using the same notation as in (i) and (ii) we get again

$$(F_1F_2)'(0) = 0$$

and

$$|(F_1F_2)''|^2 \le C|f_1'|^2\|f_2\|_{Q_p}^2 + C|f_2'|\,\|f_1\|_{Q_p}^2 + C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2\left(1+\log\frac{1}{1-|z|}\right)^4.$$

We will use here the equivalent norm for functions in Q_p, which involves the Möbius transformation $\varphi_\zeta(z) = (\zeta - z)/(1-\overline{\zeta}z)$, $\zeta, z \in D$, and obtain

$$\|f\|_{Q_p} \sim |\!|\!|f|\!|\!|_{Q_p} = |f(0)| + \sup_{\zeta\in D}\left\{\iint_D |f'(z)|^2\big(1-|\varphi_\zeta(z)|^2\big)^p\,dx\,dy\right\}^{1/2}$$

(see [5, Proposition 1]).

We have now

$$\begin{aligned}\|(F_1F_2)'\|_{Q_p}^2 &\le C\sup_{\zeta\in D}\iint_D |(F_1F_2)''|^2\big(1-|\varphi_\zeta(z)|^2\big)^p\,dx\,dy\\ &\le C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2 + C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2\iint_D\left(1+\log\frac{1}{1-|z|}\right)^4dx\,dy\\ &\le C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2.\end{aligned}$$

(iv) $X = \mathscr{B}_0^\alpha$, $0<\alpha<2$ and $X = Q_{p,0}$, $0<p<\infty$. We consider first the case $X = \mathscr{B}_0^\alpha$, $0<\alpha<2$. It suffices to prove that if $f_1, f_2 \in \mathscr{B}_0^\alpha$, then $h \in \mathscr{B}_0^\alpha$.

By (2.4), (2.5) and (2.6) we have

$$\begin{aligned}(1-|z|^2)^\alpha|h'(z)| &= (1-|z|^2)^\alpha|F''(z)|\\ &\le C\|f_2\|_{B_\alpha}(1-|z|^2)^\alpha|f_1'(z)| + C\|f_1\|_{B_\alpha}(1-|z|^2)^\alpha|f_2'(z)|\\ &\quad + C\|f_1\|_{B_\alpha}\|f_2\|_{B_\alpha}(1-|z|^2)^\alpha\end{aligned}$$

for $z\in D$, $0<\alpha<1$,

$$\begin{aligned}(1-|z|^2)^\alpha|h'(z)| &\le C\|f_2\|_{B_\alpha}(1-|z|^2)^\alpha|f_1'(z)| + C\|f_1\|_{B_\alpha}(1-|z|^2)^\alpha|f_2'(z)|\\ &\quad + C\|f_1\|_{B_\alpha}\|f_2\|_{B_\alpha}(1-|z|^2)^{2-\alpha}\end{aligned}$$

for $z\in D$, $0<\alpha<2$, and

$$\begin{aligned}(1-|z|^2)|h'(z)| &\le C\|f_2\|_B(1-|z|^2)|f_1'(z)| + C\|f_1\|_B(1-|z|^2)|f_2'(z)|\\ &\quad + C\|f_1\|_B\|f_2\|_B(1-|z|^2)\left(1+\log\frac{1}{1-|z|}\right)^2\end{aligned}$$

for $z \in D$, $\alpha = 1$.

We see immediately that in all cases

$$(1-|z|^2)^\alpha |h'(z)| \to 0 \qquad \text{as } |z| \to 1.$$

(v) If $X = Q_{p,0}$, $0 < p < \infty$, we have to prove that

$$\iint_D |h'(z)|^2 \big(1 - |\varphi_\zeta(z)|^2\big)^p \, dx\, dy \to 0 \qquad \text{as } |\zeta| \to 1.$$

Inequality (2.7) for $p = 2$ and the definition of $Q_{p,0}$ imply that it suffices to show that

$$\iint_D |f_1|^2 |f_2|^2 B(\zeta, z)\, dx\, dy \to 0,$$

where $B(\zeta, z) = \big(1 - |\varphi_\zeta(z)|^2\big)^p \le 1$, $\zeta, z \in D$, and $f_1, f_2 \in Q_{p,0}$.

From the estimate

$$|f_j(z)| \le C \|f_j\|_{Q_p} \left(1 + \log \frac{1}{1-|z|}\right), \qquad z \in D,$$

(cf. the proof of case (iii)) it follows that

$$\begin{aligned} |f_1(z)|^2 |f_2(z)|^2 B(z,\zeta) &\le C M(z) \\ &= C \|f_1\|_{Q_p}^2 \|f_2\|_{Q_p}^2 \left(1 + \log \frac{1}{1-|z|}\right)^4, \qquad z, \zeta \in D. \end{aligned}$$

Obviously,

$$\iint_D M(z)\, dx\, dy < \infty$$

and $|f_1(z)|^2 |f_2(z)|^2 B(z,\zeta) \to 0$ for every $z \in D$ when $|\zeta| \to 1$.

Lebesgue's dominated convergence theorem implies that

$$\iint_D |f_1(z)|^2 |f_2(z)|^2 B(z,\zeta)\, dx\, dy \to 0 \qquad \text{as } |\zeta| \to 1.$$

The proof of our theorem for $X = Q_{p,0}$, $0 < p < \infty$, is now complete.

After this, by minor changes, we are able to prove Theorem 2.

Proof of Theorem 2. We have $\Phi(z) = \int_0^1 f(zt)\, dt$. We start with the case $X \subset \mathscr{B}$. If $L(z) = 1 + \log\big(1/(1-|z|)\big)$, then

$$|\Phi'(z)| \le \left| \int_0^1 t f'(tz)\, dt \right| \le C \|f\|_{\mathscr{B}} L(z) \le C \|f\|_X L(z), \qquad \|\Phi\|_\infty \le C \|f\|_X.$$

As in the proof of Theorem 1, we see that $\Phi \in \mathscr{A}$. It is clear that $h' = (z\Phi_1\Phi_2)''$ is a sum of terms of type $(z\Phi_1)''\Phi_2 = f_1'\Phi_2$, $z\Phi_1'\Phi_2'$ and $\Phi_1(z\Phi_2)'' = \Phi_1 f_2'$. The contributions to the estimate of $\|h\|_X$ from terms of the first and third type are of the form $C\|f_1\|_X\|f_2\|_X$.

A term of the second type is majorized by

$$C\|f_1\|_X\|f_2\|_X L(z)^2.$$

The same computation as in the proof of Theorem 1 will prove Theorem 2. There are some slight differences in the case $X = B_1$, which we leave to the reader.

If $X = \mathscr{B}^\alpha$, $1 < \alpha < 2$, we have

$$|\Phi'(z)| \le \|f\|_{\mathscr{B}^\alpha} \int_0^1 (1-|z|^2t^2)^{-\alpha}\,dt \le C\|f\|_X(1-|z|)^{1-\alpha}, \qquad \|\Phi\|_\infty \le C\|f\|_X,$$

omitting again the details.

Question. If we consider Möbius invariant function spaces, we always have $X \subset \mathscr{B}$. If $\|f\|_X$ is essentially $|\!|\!| f' |\!|\!|$ for some norm $|\!|\!| \cdot |\!|\!|$ such that $|\!|\!| L^2 |\!|\!| < \infty$, then the argument above works both for the F- and the Φ-transforms. Does it work for any other transforms?

3. Remark

If we consider the α-Bloch space $\mathscr{B}^\alpha$ with $\alpha \ge 2$, then we do not have the algebra property, since in this case $h \notin \mathscr{B}^\alpha$ in general.

To see this we take for example $f_1(z) = f_2(z) = (1-z)^{1-\alpha}$, $\alpha \ge 2$, $z \in D$. Then

$$F_1(z) = F_2(z) = \log\frac{1}{1-z} \qquad \text{if } \alpha = 2$$

and

$$F_1(z) = F_2(z) = \frac{1}{\alpha-2}\big((1-z)^{2-\alpha} - 1\big) \qquad \text{if } \alpha > 2.$$

It is easy to check that the function $h(z) = \big(F_1^2(z)\big)'$ does not belong to $\mathscr{B}^\alpha$.

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