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ESTIMATES FOR ASYMPTOTICALLY CONFORMAL MAPPINGS

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Abstract. Let f be a univalent function in the unit disc $\mathbf{D}$, which admits a quasiconformal extension to the whole plane with a complex dilatation μ, $|\mu| \le \varkappa < 1$. The following main estimate is established:

$$\left|\frac{f''(z)}{f'(z)}\right| \le C(1-|z|)^{-\varkappa} \int_{1-|z|}^{\infty} \frac{\omega(z,t)}{t^{2-\varkappa}}\, dt, \qquad |z| < 1,$$

where

$$\omega(z,t) = \left(\frac{1}{\pi t^2} \int_{|\zeta - z| \le t} |\mu(\zeta)|^2\right)^{1/2}$$

is the square mean value of μ.

The main estimate is sharp and implies many new and old results, concerning the boundary behaviour of asymptotically conformal mappings. Among these are a new condition for the rectifiability of the image $f(\mathbf{T})$ in terms of the Littlewood–Paley quadratic function of μ, and an alternative proof of Astala–Zinsmeister's result on the embedding $\log f' \in \mathrm{BMOA}$.

Introduction

Let Φ be a quasiconformal mapping of the plane with the complex dilatation $\mu(z) = \Phi_{\bar z}/\Phi_z$. If μ vanishes in a neighborhood of some closed subset E of the plane then Φ is analytic on E. If μ tends to zero when $z \to E$, then one expects some close-to-analytic behaviour of Φ on E. Anyway, the restriction $\Phi|_E$ ought to be more or less smooth.

Two settings of such a problem are well known.

In the first, one considers a univalent function f in the unit disc $\mathbf{D}$ admitting a quasiconformal extension to the whole plane, with $f(\infty) = \infty$, such that its complex dilatation $\mu(z) \to 0$ as $|z| \to 1+0$. Such functions are called *asymptotically conformal* [21]. The problem is: what is the relation between the decay of μ near the unit circle $\mathbf{T}$ and the boundary smoothness of f? For example [21], *the asymptotic conformality itself is equivalent to the condition*

$$\beta(z) = (1-|z|)\left|\frac{f''(z)}{f'(z)}\right| \to 0, \qquad |z| \to 1-0.$$

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The other setting of the problem is due to Carleson [10]. Let Φ be a self-mapping of $\mathbf{C}\setminus\mathbf{D}$, $\Phi(\infty)=\infty$, with a complex dilatation μ. How does the decay of μ near $\mathbf{T}$ influence the smoothness of $\Phi|_{\mathbf{T}}$?

Let K be the monotonic majorant of μ, that is

$$K(t)=\operatorname*{ess\,sup}_{1<|z|<1+t}|\mu(z)|.$$

Carleson [10] proved that *if*

$$\int_0^1 \frac{K(t)}{t}\,dt<+\infty$$

then Φ is a C^1-diffeomorphism of $\mathbf{T}$, and if

$$\int_0^1 \frac{K(t)^2}{t}\,dt<+\infty$$

then Φ is absolutely continuous on the circle and its derivative belongs to $L^2(\mathbf{T})$.

Becker [7] obtained, using Lehto's [18] maximum principle, the following estimate of β in terms of K for univalent functions with asymptotically conformal extensions:

$$\sup_{1-t<|z|<1}\beta(z)\le 4\big[K(t^{1-\varepsilon})+t^{\varepsilon/(1-\varepsilon)}\big],\qquad 0<t<1.$$

It follows that *f is C^1-smooth up to $\mathbf{T}$ if*

$$\int_0^1 \frac{K(t)}{t}\,dt<+\infty,$$

and $\Gamma=f(\mathbf{T})$ is rectifiable if

$$\int_0^1 \frac{K(t)^2}{t}\,dt<+\infty.$$

Furthermore, Becker [7] has observed that there is a relation between the two settings. For any quasiconformal self-mapping of $\mathbf{C}\setminus\mathbf{D}$ there exists a conformal welding [19]

$$\Phi=g\circ f,$$

where f is a univalent function in $\mathbf{D}$ admitting a quasiconformal extension with the same μ outside $\mathbf{D}$, and g is a Riemann mapping of $\mathbf{C}\setminus f(\mathbf{D})$ onto $\mathbf{C}\setminus\mathbf{D}$.

Having some estimates for the univalent functions f and g, one can obtain the desired information on Φ.

This program was carried out by Anderson, Becker and Lesley [2], who obtained and refined Carleson's results by this approach.

Anderson and Hinkkanen [3] obtained further results on higher boundary smoothness of f or Φ in the case where μ decays faster. In particular, they proved that if $K(t) = O(t^{\alpha})$ for some $\alpha > 0$ then f and Φ belong to the Hölder class Λ^{β} for any $\beta < \alpha + 1$.

By a certain estimate of the Schwarzian derivative of f, Astala and Zinsmeister [5] obtained the following result: *If μ satisfies the Carleson condition*

$$\frac{1}{|I|} \int_{\square(I)} \frac{|\mu(z)|^2}{|z|-1}\, dx\, dy \le A$$

for any arc $I \subset \mathbf{T}$, then $\log f' \in \mathrm{BMOA}(\mathbf{D})$. Here

$$\square(I) = \{z = re^{i\theta} : e^{i\theta} \in I,\ 1 < r < 1 + |I|\}.$$

Formerly, Semmes [22] showed that *this condition, with A small enough, implies that* $\log \Phi' \in \mathrm{BMO}(\mathbf{T})$ *in the Carleson problem.* If one replaces the $O(1)$ estimate in the Carleson condition by the $o(1)$ then the corresponding functions belong to VMO instead of BMO.

This paper presents a new universal approach to asymptotically conformal mappings. It is based on a sharp estimate of the logarithmic derivative f''/f' of a univalent function f with a quasiconformal extension in terms of its complex dilatation μ. The approach covers all known estimates and provides new results on boundary smoothness and rectifiability of the boundary curve.

In Section 0 necessary preliminary information on quasiconformal mappings, smooth functions and BMO space is recalled.

In particular, in the Carleson problem framework, the following improvement of Becker's approach is presented. Starting with the decomposition $\Phi = g \circ f$, we construct a new quasiconformal extension of Φ from $\mathbf{T}$ to a neighborhood of $\mathbf{T}$. This extension has a complex dilatation of order $\beta(z)$ (or $\beta(1/\bar{z})$), and its derivatives are completely controllable. There are three steps in the construction. First, we extend f to a neighborhood of $\mathbf{T}$ by Becker's method [8]. Then, following [13], we construct a special quasiconformal symmetry with respect to $\Gamma = f(\mathbf{T})$. Finally, following [13], this symmetry is used to extend g to $G = f(\mathbf{D})$.

This extension provides a unified approach to various questions on the smoothness of $\Phi|_{\mathbf{T}}$ in terms of μ.

The main theorem, Theorem 1, is proved in Section 1. This theorem gives a sharp estimate of β in terms of μ. It asserts that *for a univalent function f in* $\mathbf{D}$ *with $\varkappa$-quasiconformal extension one has*

$$\beta(z) \le \mathrm{const} \cdot (1-|z|)^{1-\varkappa}\left[1 + \int_{1-|z|}^{1} \frac{\omega(z,t)}{t^{2-\varkappa}}\, dt\right], \qquad |z| < 1,$$

where

$$\omega(z,t) = \left(\frac{1}{\pi t^2}\int_{|\zeta - z|\le t} |\mu(\zeta)|^2\right)^{1/2}$$

is the square mean value of μ.

A similar estimate holds for higher order derivatives of f as well.

This inequality is sharp in terms of the monotonic majorant K. It strengthens the above Becker estimate essentially.

The square mean can be replaced by the mean of power p, with some $p < 2$ in the general case, and any $p > 1$ in the asymptotically conformal case (when $\mu(z) \to 0$ for $|z| \to 1 + 0$), but it is not known whether one can take $p = 1$.

Remark. It will be clear from the discussion in Section 0.6 that one can replace the exponent $\varkappa$, which appears in Theorem 1, by $\operatorname{ess\,sup}_{1<|z|<R_0} |\mu(z)|$ for any given $R_0 > 1$. In particular, if $\lim_{|z|\to 1} \mu(z) = 0$ then $\varkappa$ in Theorem 1 can be made arbitrarily small.

In Section 2 we obtain various results on the boundary smoothness of f and Φ.

We prove that $\log f'$ is continuous in the closed disc if the integral $\int_0^1 \omega(z,t)\,\frac{dt}{t}$ converges uniformly in $z \in \mathbf{T}$.

A similar condition for the C^1-smoothness of $\Phi|_{\mathbf{T}}$ in the Carleson problem is proved as well.

Having the C^1-smoothness condition proved, one can obtain various other smoothness conditions using the estimates of higher order derivatives from Section 1 and the re-extension of Φ from Section 0.4.

In particular, for the case $K(t) = O(t^\alpha)$, $\alpha > 0$, the main estimate leads to the sharp result $f \in A^{\alpha+1}(\mathbf{D})$ and $\Phi|_{\mathbf{T}} \in \Lambda^{\alpha+1}(\mathbf{T})$. Here A^s and Λ^s are the Hölder–Zygmund classes of order s (see Section 0 for precise definition).

The result is sharp. Indeed, if f is univalent, $f \in A^{\alpha+1}(\mathbf{D})$ and $n > \alpha$, then the formula

$$f(z) = \sum_0^n f^{(k)}(z^*)(z - z^*)^k, \qquad z^* = 1/\bar{z},\ |z| > 1,$$

defines a quasiconformal extension of f to a neighborhood of the closed unit disc with

$$\mu(z) = O\big((|z| - 1)^\alpha\big).$$

Section 3 is devoted to the Luzin function

$$S(z) = S_\alpha(z) = \left(\int_{\Gamma_\alpha(z)} \left|\frac{f''(\zeta)}{f'(\zeta)}\right|^2 d\xi\, d\eta\right)^{1/2}, \qquad z \in \mathbf{T},$$

where

$$\Gamma(z) = \Gamma_\alpha(z) = \big\{\zeta \in \mathbf{D} : |\zeta - z| \le (1+\alpha)(1-|\zeta|)\big\}$$

is the Luzin cone of aperture α with vertex z.

The main estimate enables us to bound S by the well-known Littlewood–Paley quadratic function of μ:

$$Q(\mu)(z) = \left(\int_{1<|\zeta|<2} \frac{|\mu(\zeta)|^2}{|\zeta - z|^2}\, d\xi\, d\eta\right)^{1/2}.$$

It turns out that

$$S(z) \le \mathrm{const}\, Q(\mu)(z)$$

at any point z of $\mathbf{T}$.

In Section 3 we apply this bound to obtain the following results:

1. *If $Q(\mu) \in L^1(\mathbf{T})$ then $G = f(\mathbf{D})$ is a Smirnov domain.* (It means that the derivative f' is an outer function in $\mathbf{D}$ [16], [21].)

2. *There exists $a > 0$ such that if $\exp aQ(\mu)^2 \in L^1(\mathbf{T})$ then the curve $\Gamma = f(\mathbf{T})$ is rectifiable.*

This result on rectifiability is stronger than all known before. For example, according to the Astala–Zinsmeister theorem, the rectifiability of Γ follows from the Carleson condition above with small A. It turns out, however, that under the Carleson condition one has $Q(\mu)^2 \in \mathrm{BMO}(\mathbf{T})$, and so $\exp aQ(\mu)^2 \in L^1$ due to the John–Nirenberg theorem.

In Section 4 we obtain the cited Astala–Zinsmeister theorem on BMOA from our main estimate without using the Schwarzian derivative of f.

At the end of the section we present an example of μ such that the rectifiability condition from Section 3 holds for this μ, while the Carleson condition fails.

It is worth noting that Semmes' result on $\log \Phi' \in \mathrm{BMO}$, cited above, can be obtained from the main estimate as well.

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0. Preliminaries

0.1. Notation. Complex variables are denoted by $z = x + iy$, $\zeta = \xi + i\eta$ and $s = \sigma + i\tau$.

$\mathbf{D} = \{z : |z| < 1\}$ is the unit disc, $\mathbf{T} = \partial\mathbf{D}$ is the unit circle.

C and c with or without indices are various positive constants, not necessarily the same throughout a formula.

$\rho(z, E)$ is the distance from the point z to the set E.

For an arc $I \subset \mathbf{T}$: $z_I \in \mathbf{T}$ is its central point, $|I|$ is its length, kI, $k > 1$, is the arc with the same central point, but of the length $k|I|$.

The interior and exterior upbuildings over I are

$$\square_i(I) = \{z = re^{i\theta} : e^{i\theta} \in I,\ 1 - |I| \le r \le 1\}$$

and

$$\square_e(I) = \{z = re^{i\theta} : e^{i\theta} \in I,\ 1 \le r \le 1 + |I|\}.$$

$f \asymp g$ if

$$c\,|g| \le |f| \le C\,|g|$$

for some positive constants c and C.

0.2. Quasiconformal mappings. A quasiconformal self-mapping f of the plane is called $\varkappa$-quasiconformal if its complex dilatation $\mu(z) = f_{\bar z}/f_z$ is bounded by $\varkappa$: $|\mu(z)| \le \varkappa < 1$ almost everywhere.

Given a disc $\Delta = \{z : |z - z_0| < r\}$, then $w_0 = f(z_0)$ is referred to as the center of the quasidisc $B = f(\Delta)$ and

$$R = \inf_{w\in\partial B} |w - w_0|$$

as its radius. It is known [1], [19] that

$$\sup_{w\in\partial B} |w - w_0| \le c_1 R,$$

where c_1 depends on $\varkappa$ only.

If $r_1 < r_2$ then the corresponding numbers R_1 and R_2 satisfy the estimate

$$\frac{R_2}{R_1} \le c_2\Big(\frac{r_2}{r_1}\Big)^{\gamma}, \tag{0.1}$$

where the exponent $\gamma = \gamma_f > 0$ depends on f. In general, $\gamma \le (1+\varkappa)/(1-\varkappa)$. However, the value of γ for special mappings may be much smaller. In the next section we discuss the possible values of γ for the univalent functions in $\mathbf{D}$ with a quasiconformal extension and for their inverse mappings.

Let $\Gamma = f(\mathbf{T})$. One can transfer the standard symmetry $\zeta \mapsto 1/\bar\zeta$ with respect to $\mathbf{T}$ to a neighborhood of Γ by the formula

$$z^* = f(1/\bar\zeta), \qquad z = f(\zeta).$$

This is a quasiconformal symmetry with respect to Γ and, according to the well-known distortion theorems for conformal and quasiconformal mappings [16], [19], [20],

$$\rho(z^*, \Gamma) \asymp |z^* - z| \asymp \rho(z, \Gamma).$$

0.3. Univalent functions with quasiconformal extension. Let f be a univalent function in $\mathbf{D}$, $f(0)=0$, $f'(0)=1$, with a $\varkappa$-quasiconformal extension to the whole plane such that $f(\infty)=\infty$, denoted by f also. Due to the normalization, for any z, $|z|\le 2$,

$$|f(z)|\le C, \tag{0.2}$$

where C depends only on $\varkappa$.

Let $z_0\in\mathbf{T}$. The following estimate is a corollary of the well-known Goluzin–Kühnau inequality [20]:

$$c_1\Big(\frac{r}{s}\Big)^{\varkappa}\le\Big|\frac{f'((1-r)z_0)}{f'((1-s)z_0)}\Big|\le c_2\Big(\frac{s}{r}\Big)^{\varkappa},\qquad 0<r<s<1. \tag{0.3}$$

In particular

$$c_1(1-|z|)^{\varkappa}\le|f'(z)|\le c_2(1-|z|)^{-\varkappa}. \tag{0.4}$$

Furthermore, consider once again the quasidisc $B=f(\{z:|z-z_0|<r\})$, $0<r<1$, and its radius R. Due to the Koebe distortion theorem

$$R\asymp|f(z_0)-f((1-r)z_0)|\asymp r\big|f'\big((1-r)z_0\big)\big|.$$

Together with (0.3) this means that the estimate (0.1) holds for our f and any $z_0\in\mathbf{T}$ with $\gamma=\gamma_f=1+\varkappa$.

Throughout the article we use the notation

$$\beta(z)=(1-|z|)\Big|\frac{f''(z)}{f'(z)}\Big|,\qquad z\in\mathbf{D}. \tag{0.5}$$

One can try to extend f outside of $\mathbf{T}$ by the formula

$$f(z)=f(1/\bar z)+f'(1/\bar z)(z-1/\bar z),\qquad |z|>1. \tag{0.6}$$

For this extension

$$f_z=f'(1/\bar z)\qquad\text{and}\qquad f_{\bar z}=-\frac{1}{\bar z^2}(z-1/\bar z)f''(1/\bar z),$$

so the corresponding complex dilatation

$$\Big|\frac{f_{\bar z}}{f_z}\Big|\asymp\frac{1}{|z|^2}\beta(1/\bar z). \tag{0.7}$$

In general, the mapping (0.6) is not homeomorphic, but Becker and Pommerenke [8] have proved that (0.6) is indeed a quasiconformal extension of f to a neighborhood of $\mathbf{T}$, if

$$\overline{\lim_{|z|\to 1-0}}\, \beta(z) < 1. \tag{0.8}$$

Unlike the initial quasiconformal extension of f, this new extension has a completely controllable gradient.

The image $G = f(\mathbf{D})$ is a quasidisc with boundary $\Gamma = f(\mathbf{T})$, $0 \in G$. The inverse mapping $g = f^{-1}$ is a univalent function in G, $g(0) = 0$, $g'(0) = 1$, with a $\varkappa$-quasiconformal extension onto the whole plane, $g(\infty) = \infty$. In general, a possible exponent γ in the inequality (0.1) for g may be big. However, suppose that

$$\overline{\lim_{|\zeta|\to 1-0}}\, \beta(\zeta) < \varepsilon < \tfrac{1}{2}. \tag{0.9}$$

Then, an analogue of the inequality (0.3) holds for f with the exponent ε instead of $\varkappa$ if r and s are small enough.

Due to the Koebe distortion theorem this means that the inequality (0.1) holds for g, $z_0 \in \Gamma$ and r small enough, with the exponent $\gamma = \gamma_g = 1 + 2\varepsilon$.

0.4. Self-mappings of $\mathbf{C} \setminus \mathbf{D}$. Let Φ be a $\varkappa$-quasiconformal self-mapping of $\mathbf{C} \setminus \mathbf{D}$, $\Phi(\infty) = \infty$, with the complex dilatation μ.

We need the well-known decomposition $\Phi = g \circ f$, where f and g are two conformal mappings of complementary domains [19]. Apparently, Becker [7] was the first who applied it to the Carleson problem.

We may extend Φ to the whole plane by the formula

$$\Phi(z) = \frac{1}{\overline{\Phi(1/\bar z)}}, \qquad z \in \mathbf{D}.$$

Now, μ is defined as the complex dilatation of Φ on the whole plane too.

Let f be a homeomorphic solution of the Beltrami equation [1], [19]

$$\frac{f_{\bar z}}{f_z} = \mu(z), \quad |z| > 1, \qquad \frac{f_{\bar z}}{f_z} = 0, \quad |z| < 1,$$

normalized by the conditions $f(0) = 0$, $f'(0) = 1$, $f(\infty) = \infty$. This map f is a conformal mapping of the unit disc with a $\varkappa$-quasiconformal extension. It maps $\mathbf{D}$ onto a quasidisc G.

Now, consider a homeomorphic solution g of the Beltrami equation

$$\frac{g_{\bar z}}{g_z} = \mu(\zeta)\frac{f'(\zeta)}{\overline{f'(\zeta)}}, \quad \zeta = f^{-1}(z), \quad z \in G, \qquad \frac{g_{\bar z}}{g_z} = 0, \quad z \notin G,$$

with $g(0) = 0$ and $g(\infty) = \infty$. The composition $g \circ f$ is a quasiconformal mapping of the plane, and its complex dilatation coincides with μ almost everywhere. Therefore [19],

$$\Phi = F[g \circ f]$$

for some entire function F. But this F must be one-to-one in the whole plane, satisfy $F(0) = 0$ and $F(\infty) = \infty$. So, $F(z) \equiv az$ for some constant a. Including this a in the definition of g one can suppose that $a = 1$.

Thus, one obtains the decomposition

$$\Phi = g \circ f.$$

In particular, g maps conformally $\mathbf{C} \setminus G$ onto $\mathbf{C} \setminus \mathbf{D}$, which is an alternative definition of g.

Suppose now that f satisfies the condition (0.8) and consider its extension (0.6) to a neighborhood of the disc. As in Section 0.2, we can define a quasiconformal symmetry $z \mapsto z^*$ with respect to $\Gamma = f(\mathbf{T})$ (in a neighborhood of Γ) by the rule

$$z^* = f(1/\bar{\zeta}) = f(\zeta) + f'(\zeta)(1/\bar{\zeta} - \zeta) \in \mathbf{C} \setminus G, \quad \text{for} \quad z = f(\zeta), \quad \zeta \in \mathbf{D}.$$

A straightforward calculation shows that in this case

$$\left|\frac{\partial z^*}{\partial \bar{z}}\right| \asymp 1, \qquad \left|\frac{\partial z^*}{\partial z}\right| \asymp \beta(\zeta).$$

Using the symmetry we can define a new extension of g to some neighborhood of Γ in G (see [13]):

$$g(z^*) = \frac{1}{\overline{g(z)}}, \qquad z \in G.$$

For this extension, we have near Γ

$$\left|\frac{\partial g}{\partial z}\right| \asymp |g'(z^*)|, \qquad \left|\frac{\partial g}{\partial \bar{z}}\right| \asymp |g'(z^*)|\beta(\zeta), \qquad z = f(\zeta),\ \zeta \in \mathbf{D}. \tag{0.10}$$

In particular, this extension of g has a completely controllable gradient.

Combining these extensions of f and g we obtain a new quasiconformal extension of $\Phi|_{\mathbf{T}}$ by the formula $\Phi(\zeta) = g\,[f(\zeta)]$, defined in some neighborhood of $\mathbf{T}$. This re-extension does not coincide with the initial mapping, but it is completely controllable and its complex dilatation is equivalent to β.

The re-extension constructed above is our main tool in the investigation of the properties of $\Phi|_{\mathbf{T}}$ in terms of μ.

0.5. Asymptotically conformal curves. Let G be a quasidisc with boundary Γ, and f be the Riemann (conformal) mapping of the unit disc onto G.

The quasicircle Γ is called *asymptotically conformal* [21] if

$$\max_{z\in\Gamma(z_1,z_2)} \frac{|z-z_1|+|z-z_2|}{|z_1-z_2|} \to 1, \qquad |z_1-z_2|\to 0,$$

where $\Gamma(z_1,z_2)$ stands for the shorter arc of Γ between two points $z_1, z_2 \in \Gamma$. It is well known that the boundedness of this expression means exactly that the curve is a quasicircle.

Γ is asymptotically conformal if and only if [21]

$$\beta(z)\to 0, \qquad |z|\to 1-0,$$

and if and only if f admits a quasiconformal extension to the whole plane such that

$$\mu_f(z)\to 0, \qquad |z|\to 1+0.$$

0.6. Localization of μ. The purpose of this article is to investigate the boundary behaviour of a univalent function with a quasiconformal extension f or of a quasiconformal self-mapping Φ of $\mathbf{C}\setminus\mathbf{D}$ in terms of the corresponding μ, provided μ vanishes at the boundary in a sense. We are interested in such properties as smoothness of $f|_{\mathbf{T}}$ or $\Phi|_{\mathbf{T}}$, inclusion $\log f' \in \mathrm{BMO}(\mathbf{T})$ and so on.

In what follows we may suppose that $\mu(z) = 0$ outside the annulus $\{z : 1\le |z| \le 2\}$. Indeed, let f be a univalent function in $\mathbf{D}$ with a quasiconformal extension normalized as in Section 0.3. Consider the solution $\tilde f$ of the Beltrami equation

$$\frac{\tilde f_{\bar z}}{\tilde f_z} = \mu(z), \quad 1<|z|<2, \qquad \frac{\tilde f_{\bar z}}{\tilde f_z} = 0 \quad \text{otherwise},$$

such that $\tilde f(0)=0$, $\tilde f'(0)=1$, $\tilde f(\infty)=\infty$.

Then, $f = F\circ\tilde f$, where F is a univalent function in the quasidisc $\tilde f(\{z : |z|<2\})$, $F(0)=0$, and $F'(0)=1$. It is clear that all interesting properties of f and of $\tilde f$ near $\mathbf{T}$ are the same. We can replace f by $\tilde f$ in the whole problem without loss of generality.

In the case of Φ, extend Φ by symmetry in the whole plane as in the beginning of Section 0.4 and consider a solution $\tilde\Phi$ of the Beltrami equation

$$\frac{\tilde\Phi_{\bar z}}{\tilde\Phi_z} = \mu_\Phi(z), \quad \tfrac12<|z|<2, \qquad \frac{\tilde\Phi_{\bar z}}{\tilde\Phi_z} = 0 \quad \text{otherwise},$$

such that $\tilde\Phi(0)=0$, $\tilde\Phi(1)=1$ and $\tilde\Phi(\infty)=\infty$. Due to the assumed symmetry of μ_Φ, $\tilde\Phi$ maps $\mathbf{D}$ onto itself.

As before, $\Phi = F\circ\tilde\Phi$, where F is a univalent function in the quasiannulus $\tilde\Phi(\{z : \frac12 < |z| < 2\})$, and all interesting properties of $\Phi|_{\mathbf{T}}$ and $\tilde\Phi|_{\mathbf{T}}$ are the same.

Remark. One can choose any fixed radius $R_0 > 1$ instead of 2 in the above construction and assume that $\mu = 0$ outside the annulus $\{z : 1 < |z| < R_0\}$.

If $\mu(z) \to 0$ uniformly as $|z| \to 1$, that is, if our mapping is asymptotically conformal in the simplest sense, this means that one can assume the parameter $\varkappa = \operatorname{ess\,sup}_{1<|z|<R_0} |\mu(z)|$ in question to be arbitrarily small.

This observation plays a crucial role in the work [3] by Anderson and Hinkkanen, because their approach depends heavily on the assumption that $\varkappa$ can be made arbitrarily small. In our approach we do not need such an assumption.

0.7. Hölder–Zygmund classes. We need in this article smooth function spaces on $\mathbf{T}$ of two different kinds—Hölder–Zygmund and Carleman classes.

Let $s > 0$ and $n = [s]$ be its integral part. If s is not an integer, the Hölder–Zygmund class $\Lambda^s(\mathbf{T})$ consists of all functions $\varphi(e^{i\theta})$ on $\mathbf{T}$ with continuous n^{th} derivative

$$\varphi^{(n)}(e^{i\theta}) = \Big(\frac{d}{d\theta}\Big)^n \varphi(e^{i\theta}),$$

satisfying the Hölder condition of the order $s - n$:

$$|\varphi^{(n)}(e^{i(\theta+h)}) - \varphi^{(n)}(e^{i\theta})| \le Ch^{s-n}, \qquad h > 0.$$

If $s = n$ is an integer, $\Lambda^s(\mathbf{T})$ consists of all functions with continuous $(n-1)^{\mathrm{th}}$ derivative, satisfying the Zygmund condition

$$|\varphi^{(n-1)}(e^{i(\theta+h)}) - 2\varphi^{(n-1)}(e^{i\theta}) + \varphi^{(n-1)}(e^{i(\theta-h)})| \le Ch, \qquad h > 0.$$

The classes Λ^s have the following description in terms of the pseudoanalytic extension [12]: a continuous function φ on $\mathbf{T}$ belongs to Λ^s if and only if it admits a continuous extension to the whole plane, C^∞-smooth outside $\mathbf{T}$ and such that

$$\frac{\partial\varphi}{\partial\bar z} = O[\,\rho(z, \mathbf{T})^{s-1}\,].$$

The "analytic" Hölder–Zygmund class A^s consists of all functions f, analytic in $\mathbf{D}$, such that $f|_{\mathbf{T}} \in \Lambda^s$.

The well-known Hardy–Littlewood criterion for an analytic in $\mathbf{D}$ function f to belong to A^s asserts: $f \in A^s$ *if and only if there is an integer* $n > s$ *such that*

$$|f^{(n)}(z)| = O\bigg(\frac{1}{(1-|z|)^{n-s}}\bigg), \qquad z \in \mathbf{D}.$$

Remark. The standard formulation of the criterion (see, e.g., [11, p. 74]) involves only the least possible value $n = [s] + 1$. However, it is easy to check that if the estimate above holds for one $n > s$ then it holds for all such n. So, one can use any convenient value of $n > s$.

0.8. Carleman classes. *A regular majorant* is a strictly positive increasing function h on $[0,1]$, $h(+0)=0$, with logarithm $\psi=\log 1/h$, such that $\psi(e^{-t})$ is a convex function of $t\geq 0$ and $h(r)/r\leq h(Q_0 r)$, $0<r<1$, for some constant Q_0.

Define a positive sequence $\{M_n\}$ by the formula

$$M_n = n!\sup_{0<r<1}\frac{h(r)}{r^n}.$$

The Carleman class $C\{M_n\}$ on $\mathbf{T}$ consists of all functions $\varphi\in C^\infty(\mathbf{T})$ such that for some C and Q

$$\Big|\Big(\frac{\partial}{\partial\theta}\Big)^n\varphi(e^{i\theta})\Big|\leq CQ^nM_n, \qquad e^{i\theta}\in\mathbf{T},\ n=0,1,\dots .$$

The Carleman classes have the following description in terms of the pseudo-analytic extension [12]: $\varphi\in C\{M_n\}$ if and only if φ admits a C^∞-extension to the whole plane such that

$$\frac{\partial\varphi}{\partial\bar z} = O\big[h\big(Q\rho(z,\mathbf{T})\big)\big]$$

for some $Q>0$.

For example, the well-known Gevrey class $C\{(n!)^{1+\alpha}\}$ corresponds to

$$h(r)=\exp(-1/r^{1/\alpha}).$$

As before, $A\{M_n\}$ is the "analytic" subclass of $C\{M_n\}$. It consists of all functions f, analytic in $\mathbf{D}$, such that $f|_{\mathbf{T}}$ belongs to $C\{M_n\}$.

0.9. Carleson condition and BMO**.** Let ν be a positive measure in $\mathbf{D}$. The Carleson condition on ν is the following estimate:

$$\frac{\nu\big(\square_i(I)\big)}{|I|}\leq A \tag{0.11}$$

for any arc $I\subset\mathbf{T}$.

The least A in (0.11) is called *the Carleson constant* of ν. The measure itself is called a *Carleson measure.*

Furthermore, define *the limit Carleson constant of* ν:

$$N_0(\nu)=\lim_{\delta\to 0}\sup_{|I|<\delta}\frac{\nu\big(\square_i(I)\big)}{|I|}.$$

If the left-hand side of (0.11) tends to zero for $|I|\to 0$, i.e. $N_0(\nu)=0$, one says that the measure satisfies the $o(1)$-Carleson condition.

One can also formulate a similar condition in $\mathbf{C}\setminus\mathbf{D}$ with $\square_e(I)$ instead of $\square_i(I)$.

A function $\varphi\in L^1(\mathbf{T})$ belongs to the space $\mathrm{BMO}(\mathbf{T})$ if there exists a constant $A>0$ such that for any arc $I\subset\mathbf{T}$

$$\frac{1}{|I|}\int_I|\varphi-c_I|\le A$$

for some constant c_I. The least possible A in this inequality is called the BMO norm $\|\varphi\|_{\mathrm{BMO}}$ of φ.

The subspace of all $\varphi\in\mathrm{BMO}$ with

$$\lim_{|I|\to 0}\frac{1}{|I|}\int_I|\varphi-c_I|=0$$

is called $\mathrm{VMO}(\mathbf{T})$.

Analytic functions $f\in H^1(\mathbf{D})$, whose boundary values belong to $\mathrm{BMO}(\mathbf{T})$ or $\mathrm{VMO}(\mathbf{T})$, form the spaces BMOA and VMOA respectively.

The following criterion [14] is used to check the inclusion of a function to BMO.

Let φ be an analytic function in $\mathbf{D}$. If the measure

$$d\nu=(1-|z|)|\varphi'(z)|^2\,dx\,dy$$

satisfies the Carleson condition (0.11) in $\mathbf{D}$ with some constant A, then $\varphi\in$ BMOA.

Furthermore,

$$\overline{\lim_{r\to 1-0}}\,\|\varphi(e^{i\theta})-\varphi(re^{i\theta})\|_{\mathrm{BMO}}\le CN_0(\nu)^{1/2}.$$

Therefore, if ν satisfies the $o(1)$-Carleson condition then $\varphi(re^{i\theta})$ converge in BMO norm and the boundary function belongs to VMO.

1. Estimate for logarithmic derivative

1.1. The main theorem. The following estimate is the main technical tool of this work.

Theorem 1. *Let f be a univalent function in the unit disc, such that $f(0)=0$ and $f'(0)=1$. Suppose that f admits a $\varkappa$-quasiconformal extension to the whole plane with a complex dilatation μ. Then*

$$(1-|z|)\left|\frac{f''(z)}{f'(z)}\right|\le C(1-|z|)^{1-\varkappa}\left[1+\int_{1-|z|}^{1}\frac{\omega(z,t)}{t^{2-\varkappa}}\,dt\right],\qquad |z|<1,\tag{1.1}$$

where

$$\omega(z,t)=\left(\frac{1}{\pi t^2}\int_{|\zeta-z|<t}|\mu(\zeta)|^2\,d\xi\,d\eta\right)^{1/2}\tag{1.2}$$

is the local square mean of $|\mu|$. The constant C depends on $\varkappa$ only.

Proof. Due to (0.4) it suffices to prove the estimate

$$|f''(z)| \le C_1|f'(z)|\frac{1}{(1-|z|)^{\varkappa}}\int_{1-|z|}^{1}\frac{\omega(z,t)}{t^{2-\varkappa}}\,dt + C_2. \tag{1.3}$$

Let $r = 1-|z|$ be small enough. By the Cauchy–Green formula and in view of (0.2)

$$|f''(z)| \le \left|\frac{2}{\pi}\int_{1<|\zeta|<2}\frac{\partial f}{\partial\bar\zeta}\frac{d\xi\,d\eta}{(\zeta-z)^3}\right| + C_2.$$

The contribution of the annulus

$$2^k r < |\zeta - z| < 2^{k+1}r, \qquad k = 0,1,\ldots,$$

to this integral does not exceed

$$\begin{aligned}\frac{1}{(2^k r)^3}\int_{|\zeta-z|<2^{k+1}r}\left|\frac{\partial f}{\partial\bar\zeta}\right|d\xi\,d\eta &\le \frac{1}{(2^k r)^3}\left(\int_{|\zeta-z|<2^{k+1}r}|\mu(\zeta)|^2\right)^{1/2}\\ &\qquad\times\left(\int_{|\zeta-z|<2^{k+1}r}\left|\frac{\partial f}{\partial\zeta}\right|^2\right)^{1/2}\\ &\le \frac{2\sqrt{\pi}}{(2^k r)^2}\,\omega(z,2^{k+1}r)\left(\int_{|\zeta-z|<2^{k+1}r}\left|\frac{\partial f}{\partial\zeta}\right|^2\right)^{1/2}.\end{aligned}$$

However

$$\int_{|\zeta-z|<2^{k+1}r}\left|\frac{\partial f}{\partial\zeta}\right|^2 \le \frac{1}{1-\varkappa^2}\int J(\zeta) \le \frac{1}{1-\varkappa^2}\operatorname{mes} f\{|\zeta - z_0| < 2^{k+2}r\},$$

where $J(z)$ is the Jacobian of the mapping f and $z_0 = z/|z|$.

According to Section 0.3, the inequality (0.1) holds for f and z_0 with the exponent $\gamma = 1+\varkappa$. Therefore, due to the Koebe theorem,

$$\operatorname{mes} f\{|\zeta - z_0| < 2^{k+2}r\} \le c\cdot(2^k)^{2\gamma}\operatorname{mes} f\{|\zeta - z_0| < r\} \le c\cdot(2^k)^{2+2\varkappa}\cdot|f'(z)|^2\cdot r^2.$$

Collecting all the contributions we obtain

$$\begin{aligned}|f''(z)| &\le C_1|f'(z)|\sum_{2^k r\le 3}\frac{1}{(2^k r)^2}\omega(z,2^k r)\cdot 2^{k\varkappa}2^k r + C_2\\ &\le C_1\,\frac{1}{r}|f'(z)|\sum\frac{\omega(z,2^k r)}{2^{k(1-\varkappa)}} + C_2,\end{aligned}$$

which implies (1.3). □

1.2. Higher derivatives. In the same way, one can prove estimates for higher derivatives of f: *under the assumptions of Theorem* 1,

$$(1.4)\qquad (1-|z|)^{n-1}\left|\frac{f^{(n)}(z)}{f'(z)}\right| \le CQ^n n!\,(1-|z|)^{n-1-\varkappa}\left[1+\int_{1-|z|}^{1}\frac{\omega(z,t)}{t^{n-\varkappa}}\,dt\right]$$

for any n. *Here the constants* C *and* Q *do not depend on* n.

1.3. Remarks. 1) We used in the proof the Schwarz inequality and the square mean of μ. According to F. Gehring [15], the Jacobian J satisfies the reverse Hölder inequality: there exists $q>1$, depending on $\varkappa$, such that

$$\left(\frac{1}{|\Delta|}\int_{\Delta} J^q\right)^{1/q} \le C\frac{1}{|\Delta|}\int_{\Delta} J$$

for any disc Δ. Using this result and applying Hölder's inequality one can obtain an exact analog of (1.1) for

$$(1.5)\qquad \omega_p(z,t)=\left(\frac{1}{\pi t^2}\int_{|\zeta-z|<t}|\mu(\zeta)|^p\,d\xi\,d\eta\right)^{1/p}$$

with some $p<2$ instead of (1.2). This p depends on $\varkappa$ only.

Suppose now that our mapping is asymptotically conformal, i.e. $\mu(z)\to 0$ for $|z|\to 1+0$. Then, according to Astala's recent result [4], the reverse Hölder inequality for J holds for any $q>1$ if the disc Δ lies in a sufficiently small neighborhood of $\mathbf{D}$ (depending on q). Applying this instead of Gehring's result one can obtain (1.1) in the asymptotically conformal case with ω_p for any $p>1$. It is unknown whether this is true for $p=1$, i.e. whether (1.1) holds with ω_1 instead of $\omega=\omega_2$ even for an asymptotically conformal mapping.

2) Let us assume that

$$\frac{1}{t^2}\int_{|\zeta-z|<t}|\mu(\zeta)|\,d\xi\,d\eta\to 0,\qquad t\to 0,$$

uniformly in $z\in\mathbf{T}$. Then $\omega(z,t)\to 0$ uniformly as well, and by Theorem 1

$$\beta(z)=(1-|z|)\frac{f''(z)}{f'(z)}\to 0,\qquad |z|\to 1-0.$$

Therefore our mapping f is asymptotically conformal [21] (cf. [17]).

3) Clearly, the right hand side of (1.1) is a sort of averaging of ω, that is of μ. In the simplest special case consider the monotonic majorant of μ,

$$K(t) = \operatorname{ess\,sup}\{|\mu(z)| : 1 < |z| < 1+t\}, \qquad t > 0. \tag{1.6}$$

Of course, $\omega(z,t) \le K(t)$ for any t when $z \in \mathbf{D}$.

Now (1.1) becomes

$$\beta(z) = (1-|z|)\left|\frac{f''(z)}{f'(z)}\right| \le C(1-|z|)^{1-\varkappa}\left[1 + \int_{1-|z|}^{1} \frac{K(t)}{t^{2-\varkappa}}\,dt\right], \qquad |z| < 1. \tag{1.7}$$

In the asymptotically conformal case $K(+0) = 0$. Assuming for example that $K(t)/t^{\varepsilon}$ is nonincreasing for some $\varepsilon < 1-\varkappa$, one obtains from (1.7)

$$(1-|z|)\left|\frac{f''(z)}{f'(z)}\right| \le C\,K(1-|z|).$$

Conversely, let the right hand side of the last inequality tend to 0 as $|z| \to 1$. Then by the Becker–Pommerenke theorem [8], the extension (0.6) of f is quasiconformal in some neighborhood of the disc. However, for such an extension

$$|\mu(1/\bar{z})| \asymp |z|^2(1-|z|)\left|\frac{f''(z)}{f'(z)}\right|, \qquad |z| < 1,$$

and therefore our estimate is sharp.

A weaker estimate

$$(1-|z|)\left|\frac{f''(z)}{f'(z)}\right| \le 4\big[K(\delta^{1-\varepsilon}) + \delta^{\varepsilon/(1-\varepsilon)}\big], \qquad \delta = 1-|z|,$$

where $\varepsilon > 0$ is arbitrary but unavoidable, was obtained by Becker [7] with the aid of Lehto's maximum principle [18].

4) $\varkappa$ in the formulation of Theorem 1 is a global parameter: $\varkappa = \|\mu\|_{L^\infty(\mathbf{C})}$. However, according to Section 0.6, one can choose any $R_0 > 0$ and factorize $f = F \circ \tilde{f}$, where $\mu_{\tilde{f}}(z) = \mu_f(z)$ for $1 < |z| < R_0$, $\mu_{\tilde{f}} = 0$ otherwise, and F is conformal in $\tilde{f}(\{z : |z| < R_0\})$. Evidently, $\beta_f \asymp \beta_{\tilde{f}}$ in $\mathbf{D}$ up to a constant, depending on $\varkappa$ only. Thus, the main estimate (1.1) holds with the localized value $\widetilde{\varkappa} = \operatorname{ess\,sup}_{1<|z|<R_0} |\mu(z)|$ instead of global one. Particularly, in the asymptotically conformal case $\lim_{|z|\to 1} \mu(z) = 0$, $\widetilde{\varkappa}$ can be made arbitrarily small. The constant C in (1.1) depends both on R_0 and on the initial $\varkappa$.

1.4. Estimate for the inverse mapping. Let f satisfy the conditions of Theorem 1 and g be its inverse. Then g is a $\varkappa$-quasiconformal mapping of the plane, conformal in the quasidisc $G = f(\mathbf{D})$, $g(0) = 0$, $g'(0) = 1$, and $g(\infty) = \infty$. As before, let $\Gamma = f(\mathbf{T})$ be the boundary of G. The mapping g has its own complex dilatation μ_g and its own exponent γ_g in the inequality (0.1).

Theorem 1′. *If* $\gamma_g < 2$ *then*

$$(1.8)\quad \rho(z,\Gamma)\Big|\frac{g''(z)}{g'(z)}\Big| \le C\rho(z,\Gamma)^{2-\gamma_g}\left[1+\int_{\rho(z,\Gamma)}^{1}\frac{\tilde{\omega}(z,t)}{t^{3-\gamma_g}}\,dt\right],\ z\in G,\ \rho(z,\Gamma)<1,$$

where

$$(1.9)\qquad \widetilde{\omega}(z,t) = \left(\frac{1}{\pi t^2}\int_{|\zeta-z|<t}|\mu_g(\zeta)|^2\,d\xi\,d\eta\right)^{1/2}.$$

The constant C *depends on* $\varkappa$ *and* γ_g.

Proof. One can repeat the whole proof of Theorem 1, because the Koebe theorem

$$1-|g(z)| \asymp \rho(z,\Gamma)|g'(z)|, \qquad z\in G,$$

is valid for g as well. ◻

In particular, if f satisfies the condition

$$(0.9)\qquad \varlimsup_{|\zeta|\to 1-0}\beta(\zeta) < \varepsilon < \tfrac{1}{2}$$

from Section 0.3, then one can take $\gamma_g = 1+2\varepsilon$ and the estimate (1.8) holds.

2. Smoothness up to the boundary

2.1. C^1**-smoothness in the univalent function case.** Suppose that f is a univalent function in $\mathbf{D}$ and f satisfies the assumptions of Theorem 1.

Theorem 2. *If the integral*

$$(2.1)\qquad \int_0^1 \omega(z,t)\,\frac{dt}{t}$$

converges uniformly in $z\in\mathbf{T}$, *then* $\log f'$ (*and thereby* f' *and* $1/f'$) *is continuous in the closed disc.*

Remarks. 1) If one replaces $\omega=\omega_2$ by ω_1 in (2.1) (see (1.5) for the definition), then it turns to the very well-known expression

$$\int_{|\zeta-z|<1}\frac{|\mu(\zeta)|}{|\zeta-z|^2}\,d\xi\,d\eta.$$

The classical result of Teichmüller, Wittich and Belinskij [19] asserts that the convergence of the last integral implies differentiability of the mapping f at the point z. It is unknown whether its uniform convergence for $z\in\mathbf{T}$ implies C^1-smoothness of f in the closed disc. No sharp condition of such smoothness in terms of μ is known at all.

2) Consider the monotonic majorant $K(t)$ of μ defined in (1.6). Then Theorem 2 gives in terms of K the following (cf. [7]):

The function f is C^1-smooth up to the boundary if

$$\int_0^1 \frac{K(t)}{t}\,dt < +\infty.$$

3) Suppose that the integral (2.1) converges for one fixed $z \in \mathbf{T}$ only. Then the proof of Theorem 2 below shows that the non-tangential limit of f' exists at the point z. In other words, the function f has at the point z an angular derivative.

Proof of Theorem 2. In view of $f''/f' = (\log f)'$, it is enough to prove that the integral

$$\int_0^1 \beta\big((1-r)z\big)\frac{dr}{r}$$

converges uniformly in $z \in \mathbf{T}$. Due to (1.1),

$$\int_0^1 \beta\big((1-r)z\big)\,\frac{dr}{r} \le C\int_0^1 \frac{dr}{r^{\varkappa}}\left[1 + \int_r^1 \frac{\omega\big((1-r)z,t\big)}{t^{2-\varkappa}}\,dt\right].$$

Furthermore, for $t > r$ and $z \in \mathbf{T}$

$$\omega\big((1-r)z,t\big) \le c\,\omega(z,2t),$$

so the integral does not exceed

$$C\left[\int_0^1 \frac{dr}{r^{\varkappa}} + \int_0^1 \frac{\omega(z,t)}{t^{2-\varkappa}}\,dt\int_0^t \frac{dr}{r^{\varkappa}}\right] \le C\left[\int_0^1 \frac{dr}{r^{\varkappa}} + \frac{1}{1-\varkappa}\int_0^1 \frac{\omega(z,t)}{t}\,dt\right].$$

The last integral converges uniformly in z by assumption. ▫

2.2. C^1-smoothness in the Carleson problem. Let Φ be a $\varkappa$-quasiconformal self-mapping of $\mathbf{C}\setminus\mathbf{D}$, $\Phi(\infty)=\infty$, with complex dilatation μ. In order to investigate the C^1-smoothness of $\Phi|_{\mathbf{T}}$ we use the factorization

$$\Phi = g \circ f$$

of Section 0.4, where f is conformal in $\mathbf{D}$ (with the standard normalization), $\mu_f = \mu$ outside $\mathbf{D}$, and g is a Riemann mapping of the complement of $G = f(\mathbf{D})$ onto the complement of $\mathbf{D}$.

Suppose that the integral (2.1) converges uniformly in $z \in \mathbf{T}$. Then, according to Theorem 2, f is C^1-smooth in the closed unit disc. In order to prove the boundary C^1-smoothness of g too, we need a slightly stronger condition.

For $z \in \mathbf{T}$, set

$$\Omega(z,t) = \left\{ \frac{1}{t^2} \int_{|\zeta - z|<2t} |\mu(\zeta)|^2 \log \frac{3t}{|\zeta| - 1} \, d\xi \, d\eta + t^{1-\varkappa} \int_{2t<|\zeta-z|<1} \frac{|\mu(\zeta)|^2}{|\zeta - z|^{3-\varkappa}} \, d\xi \, d\eta \right\}^{1/2}.$$

Lemma. *For $z \in \mathbf{T}$ and $t > 0$*

$$\left(\frac{1}{t^2} \int_{|\zeta - z|<t} |\beta(\zeta)|^2 \, d\xi \, d\eta \right)^{1/2} \le C_1 \Omega(z,t) + C_2 t^{1-\varkappa}.$$

Proof. In view of (1.1)

$$\begin{aligned} \frac{1}{t^2} \int_{|\zeta - z|<t} |\beta(\zeta)|^2 \, d\xi \, d\eta &\le C \frac{1}{t^2} \int_{|\zeta - z|<t} \left[(1 - |\zeta|)^{1-\varkappa} \left(1 + \int_{1-|\zeta|}^1 \frac{\omega(\zeta, r)}{r^{2-\varkappa}} \, dr \right) \right]^2 d\xi \, d\eta \\ &\le C_1 \frac{1}{t^2} \int |\mu(s)|^2 \, d\sigma \, d\tau \int_{H(s)} \frac{d\xi \, d\eta \, dr}{(1 - |\zeta|)^{\varkappa - 1} r^{4-\varkappa}} + C_2 t^{2-2\varkappa}, \end{aligned}$$

where the domain of integration $H(s)$ is given by

$$H(s) = \{ (\zeta, r) : \zeta \in \mathbf{D}, \ r > 0, \ |\zeta - s| < r, \ |\zeta - z| < t \}.$$

The integration in r shows that the inner integral is less than

$$C \int_{|\zeta - z|<t} \frac{(1 - |\zeta|)^{1-\varkappa}}{|\zeta - s|^{3-\varkappa}} \, d\xi \, d\eta.$$

Therefore, if $|s - z| < 2t$ the inner integral does not exceed

$$C \int_{|s|-1<|\zeta - s|<3t} \frac{d\xi \, d\eta}{|\zeta - s|^2} = C \log \frac{3t}{|s| - 1},$$

because $|\zeta - s| \ge (1 - |\zeta|) + (|s| - 1)$.

If $|s - z| > 2t$ then $|\zeta - s| \asymp |z - s|$ and the inner integral does not exceed

$$C \frac{t^{3-\varkappa}}{|s - z|^{3-\varkappa}}. \quad \square$$

Theorem 3. *If the integral*

$$\int_0^1 \Omega(z,t) \, \frac{dt}{t}$$

converges uniformly in $z \in \mathbf{T}$ then Φ is C^1-smooth on $\mathbf{T}$.

Proof. Evidently $\omega(z,t) \le \Omega(z,t)$. Therefore, (2.1) holds and f is C^1-smooth. We have to prove that g is C^1-smooth up to Γ as well.

The extension (0.6) of f is a diffeomorphism of a neighborhood of $\bar{\mathbf{D}}$ onto a neighborhood of $\bar{G}$. In Section 0.4 we constructed a quasiconformal symmetry, which was related to f, and a quasiconformal extension of g to a neighborhood of Γ. For this extension, according to (0.10),

$$|\mu_g(z)| \asymp \beta(\zeta), \qquad z = f(\zeta) \in G.$$

Furthermore, $\lim_{|\zeta|\to 1} \beta(\zeta) = 0$ in our case. Therefore, Γ is asymptotically conformal (See Section 0.5), and for g, the condition (0.9) holds with any $\varepsilon > 0$. Thus, $\gamma_g < 2$, hence one can apply Theorem 1′ and repeat the whole proof of Theorem 3 for g. The uniform convergence of the analogue of the integral (2.1), which appears in the proof, follows from lemma. ▫

Consider once again the monotonic majorant $K(t)$ of μ defined in (1.6). Then Theorem 3 leads to the well-known condition for C^1-smoothness of Φ in terms of K [10]. Evidently

$$\Omega(z,t) \le C\left(t^{1-\varkappa} \int_t^2 \frac{K(\tau)^2}{\tau^{2-\varkappa}}\, d\tau\right)^{1/2}, \qquad 0 < t < 1. \tag{2.2}$$

Corollary 1. *If*

$$\int_0^1 \frac{K(t)}{t}\, dt < +\infty \tag{2.3}$$

then Φ is C^1-smooth on $\mathbf{T}$.

Proof. We have to check the assumptions of Theorem 3 in this case. Set $a_k = K(2^{-k})$, $k = 0, 1, \dots$. Then (2.3) yields

$$\sum a_k < +\infty.$$

Now, (2.2) gives, for any $n > 0$,

$$\Omega(z, 2^{-n})^2 \le C 2^{-n(1-\varkappa)}\left[1 + \sum_0^n a_k^2 2^{k(1-\varkappa)}\right] \le C 2^{-n(1-\varkappa)}\left[1 + \left(\sum_0^n a_k 2^{k(1-\varkappa)/2}\right)^2\right].$$

Therefore

$$\Omega(z, 2^{-n}) \le C 2^{-n(1-\varkappa)/2} + C \sum_0^n a_k 2^{(k-n)(1-\varkappa)/2},$$

and

$$\sum_0^\infty \Omega(z, 2^{-n}) \le C \sum_0^\infty 2^{-n(1-\varkappa)/2} + C \sum_{k=0}^\infty a_k \sum_{n=k}^\infty 2^{(k-n)(1-\varkappa)/2} \le C + C \sum_0^\infty a_k < +\infty.$$

This estimate does not depend on z, so the integral $\int_0^1 \Omega(z,t)\, dt/t$ converges uniformly. ▫

2.3. Smoothness of higher order. Let f be a univalent function in $\mathbf{D}$, and Φ a quasiconformal self-mapping of $\mathbf{C}\setminus\mathbf{D}$. Suppose that f and Φ satisfy the conditions of Section 2.1 and 2.2, respectively.

For simplicity, we consider here only the monotonic majorant (1.6) of μ.

Fix $\alpha > 0$ and a regular majorant h (see Section 0.8). Consider the corresponding Hölder classes $\Lambda^{\alpha+1}$ and $A^{\alpha+1}$ (Section 0.7), and the corresponding Carleman classes $C\{M_n\}$ and $A\{M_n\}$ (Section 0.8).

Theorem 4. (i) *If $K(t) = O(t^\alpha)$, $t\to 0$, then $f\in A^{\alpha+1}$.*
(ii) *If $K(t) = O[h(t)]$, $t\to 0$, then $f\in A\{M_n\}$.*

Proof. (i) For any $n > \alpha+2$, (1.4) gives

$$|f^{(n)}(z)| \le C\frac{|f'(z)|}{(1-|z|)^{\varkappa}}\int_{1-|z|}^{\infty}\frac{t^\alpha}{t^{n-\varkappa}}\,dt \le C|f'(z)|(1-|z|)^{n-(\alpha+1)}.$$

However, f' is uniformly bounded due to Theorem 2, and our claim on $A^{\alpha+1}$ follows from the Hardy–Littlewood criterion (Section 0.7).

(ii) In the case of the Carleman class, it follows again from (1.4) and from the boundedness of f' that

$$\begin{aligned}|f^{(n+1)}(z)| &\le CQ^{n+1}\,(n+1)!\left\{1+\frac{1}{(1-|z|)^{\varkappa}}\int_{1-|z|}^{1}\frac{h(t)}{t^{n+1-\varkappa}}\,dt\right\}\\ &\le CQ^{n+1}\frac{(n+1)!}{(1-|z|)^{\varkappa}}\left\{1+\sup_{0<t<1}\frac{h(t)}{t^n}\right\}\end{aligned}$$

and the desired result follows by integration. □

Theorem 5. (i) *If $K(t) = O(t^\alpha)$, $t\to 0$, then $\Phi|_{\mathbf{T}}\in\Lambda^{\alpha+1}$.*
(ii) *If $K(t) = O[h(t)]$, $t\to 0$, then $\Phi|_{\mathbf{T}}\in C\{M_n\}$.*

Proof. We will use the factorization $\Phi = g\circ f$ discussed in Section 2.2.

(i) It is known from Theorem 4 that in our case $f\in A^{\alpha+1}(\mathbf{D})$ and f' does not vanish on the circle. Therefore ([12]) f admits a C^1-extension to some neighborhood of $\mathbf{D}$ with the estimate

$$\left|\frac{\partial f}{\partial\bar\zeta}\right| = O[\rho(\zeta,\mathbf{T})^\alpha].$$

The extension is a diffeomorphism of the neighborhood.

According to [13], this means that one can construct a quasiconformal symmetry $z\mapsto z^*$ in a neighborhood of $\Gamma = f(\mathbf{T})$ satisfying the estimate

$$\left|\frac{\partial z^*}{\partial z}\right| = O[\rho(z,\Gamma)^\alpha].$$

Following [13], the existence of such symmetry implies that g, in its turn, admits an extension to a neighborhood of Γ such that

$$\left|\frac{\partial g}{\partial \bar z}\right| = O[\rho(z,\Gamma)^\alpha].$$

Now $g\circ f$ is a new extension of Φ from $\mathbf{T}$ to a neighborhood of $\mathbf{T}$ satisfying the estimate

$$\left|\frac{\partial \Phi}{\partial \bar\zeta}\right| = O[\rho(\zeta,\mathbf{T})^\alpha].$$

Therefore $\Phi|_{\mathbf{T}} \in \Lambda^{\alpha+1}$ by [12].

(ii) We know from Theorem 4 that in the present case $f \in C^\infty(\bar{\mathbf{D}})$ and $f|_{\mathbf{T}} \in C\{M_n\}$. Therefore, by [12], f admits a C^1-extension to a neighborhood of $\mathbf{D}$ with the estimate

$$\left|\frac{\partial f}{\partial \bar\zeta}\right| = O\big[h\big(Q\rho(\zeta,\mathbf{T})\big)\big]$$

for some Q. According to [13], one can construct a quasiconformal symmetry $z \mapsto z^*$ in a neighborhood of $\Gamma = f(\mathbf{T})$ satisfying the estimate

$$\left|\frac{\partial z^*}{\partial z}\right| = O\big[h\big(Q\rho(z,\Gamma)\big)\big].$$

Following [13], the existence of such symmetry implies that g admits an extension to a neighborhood of Γ such that

$$\left|\frac{\partial g}{\partial \bar z}\right| = O\big[h\big(Q\rho(z,\Gamma)\big)\big].$$

Now $g\circ f$ is a new extension of Φ from $\mathbf{T}$ to a neighborhood of $\mathbf{T}$ satisfying the estimate

$$\left|\frac{\partial \Phi}{\partial \bar\zeta}\right| = O\big[h\big(Q\rho(\zeta,\mathbf{T})\big)\big].$$

Therefore $\Phi|_{\mathbf{T}} \in C\{M_n\}$ according to [12]. □

Remarks. 1) Anderson and Hinkkanen [3] have proved in another way that $f \in A\{M_n\}$ and $\Phi|_{\mathbf{T}} \in C\{M_n\}$ if $K(t) = O[h(t)]$. They have proved also that if $K(t) = O(t^\alpha)$ then $f \in A^\beta$ and $\Phi|_{\mathbf{T}} \in \Lambda^\beta$ for any $\beta < \alpha + 1$. Apparently, their approach cannot give the sharp result with $A^{\alpha+1}$. Moreover, they conjectured in [3] that this sharp inclusion is not true in general. However, Theorems 4 and 5 show that the conjecture fails and the sharp inclusion holds.

2) Evidently, one can replace $K(t)$ in Theorems 4 and 5 by the square mean value of μ: if

$$\sup_{z\in\mathbf{T}} \omega(z,t) = O(t^\alpha), \qquad t \to +0,$$

then $f \in A^{\alpha+1}$ and so on. For example, one can treat this way the case of a uniform but not monotonic estimate on μ.

3. Estimate for the Luzin function

Assume again that f is a univalent function in the unit disc admitting a $\varkappa$-quasiconformal extension with the complex dilatation μ. According to Section 0.6, we suppose in what follows that μ vanishes outside the annulus $\{z : 1 < |z| < 2\}$.

3.1. The estimate. Let $\alpha > 0$ and $z \in \mathbf{T}$. Consider the Luzin function for $\log f'$

$$S(z) = S_\alpha(z) = \left(\int_{\Gamma_\alpha(z)} \left|\frac{f''(\zeta)}{f'(\zeta)}\right|^2 d\xi\, d\eta\right)^{1/2}, \qquad z \in \mathbf{T},$$

where

$$\Gamma(z) = \Gamma_\alpha(z) = \{\zeta \in \mathbf{D} : |\zeta - z| \le (1+\alpha)(1 - |\zeta|)\}$$

is the Luzin cone of aperture α with vertex z.

Furthermore, consider the following well-known quadratic function for μ:

$$Q(\mu)(z) = \left(\int_{1<|\zeta|<2} \frac{|\mu(\zeta)|^2}{|\zeta - z|^2}\, d\xi\, d\eta\right)^{1/2}.$$

Theorem 6. *For any* $z \in \mathbf{T}$

$$S(z) \le C_1 + C_2\, Q(\mu)(z).$$

Here the constants C_1 and C_2 depend on $\varkappa$ and on α in the definition of S.

Proof. It suffices to estimate $S(1)$. According to (1.1)

$$\begin{aligned} S(1)^2 &\le C \int_{\Gamma(1)} \frac{dx\, dy}{(1-|z|)^{2\varkappa}} \left[1 + \int_{1-|z|}^1 \frac{\omega(z,t)}{t^{2-\varkappa}}\, dt\right]^2 \\ &\le C_1 + C_2 \int_{\Gamma(1)} \frac{dx\, dy}{(1-|z|)^{1+\varkappa}} \int_{1-|z|}^1 \frac{\omega(z,t)^2}{t^{2-\varkappa}}\, dt \\ &\le C_1 + C_2 \int_{1<|\zeta|<2} |\mu(\zeta)|^2\, d\xi\, d\eta \int_{\Omega(\zeta)} \frac{dx\, dy\, dt}{(1-|z|)^{1+\varkappa} t^{4-\varkappa}}, \end{aligned}$$

where the domain of integration is

$$\Omega(\zeta) = \{(z,t) : z \in \Gamma(1),\ |\zeta - z| < t\}.$$

The integration in t reduces the inner integral to

$$\int_{\Gamma(1)} \frac{dx\, dy}{(1-|z|)^{1+\varkappa} |z-\zeta|^{3-\varkappa}}.$$

If $1-|z|<\frac{1}{2}|\zeta-1|$ then $|z-\zeta|\asymp|\zeta-1|$ and the corresponding contribution to the integral is less than

$$\frac{C}{|\zeta-1|^{3-\varkappa}}\int_{z\in\Gamma(1),1-|z|<|\zeta-1|}\frac{dx\,dy}{(1-|z|)^{1+\varkappa}}\le\frac{C}{|\zeta-1|^{2}}.$$

If, on the contrary, $1-|z|>\frac{1}{2}|\zeta-1|$ then $|z-\zeta|>1-|z|$ and the corresponding contribution is less than

$$C\int_{z\in\Gamma(1),1-|z|>1/2\,|\zeta-1|}\frac{dx\,dy}{(1-|z|)^{4}}\le\frac{C}{|\zeta-1|^{2}}.$$

Therefore, the whole inner integral does not exceed $C/|\zeta-1|^2$ which gives immediately $S(1)\le C_1+C_2\,Q(\mu)(1)$. □

3.2. Smirnov domains. Recall ([16], [21]) that the domain $G=f(\mathbf{D})$ is a *Smirnov domain* if f' is an outer function in the disc in the Beurling sense.

Corollary 2. *If $Q(\mu)\in L^1(\mathbf{T})$ then $G=f(\mathbf{D})$ is a Smirnov domain.*

Proof. According to Theorem 6, $S\in L^1(\mathbf{T})$. By the well-known Calderon result ([14], [23]) this means that $\log f'$ belongs to the Hardy space $H^1(\mathbf{D})$. Thus, f' is an outer function ([14], [16]). □

3.3. Rectifiability of the boundary. Under which condition on μ does the image $G=f(\mathbf{D})$ of f have a rectifiable boundary $\Gamma=\partial f(\mathbf{D})=f(\mathbf{T})$? This property is equivalent to the inclusion $f'\in H^1(\mathbf{D})$.

It is known ([7]) that $\partial f(\mathbf{D})$ is rectifiable if

$$\int_0^1\frac{K(t)^2}{t}\,dt<+\infty, \tag{3.1}$$

where K is defined by (1.6).

A more general rectifiability condition follows from Astala–Zinsmeister's result [5]. It requires that the measure

$$\frac{|\mu(z)|^2}{|z|-1}\,dx\,dy \tag{3.2}$$

satisfies the Carleson condition in $\mathbf{C}\setminus\mathbf{D}$ with a small constant.

Evidently, (3.1) implies the Carleson condition on the measure (3.2).

Here, we prove a stronger result, providing a new condition of the rectifiability of Γ.

Theorem 7. *There exists $a>0$ such that if $\exp aQ(\mu)^2\in L^1(\mathbf{T})$, then $f'\in H^1(\mathbf{D})$ and the boundary of $G=f(\mathbf{D})$ is rectifiable.*

Proof. According to Theorem 6, $\exp bS^2 \in L^1(\mathbf{T})$ with $b = a/C_2^2$.

Bañuelos and Moore ([6]) proved the following "good λ-inequality" between $S = S_\alpha$, $\alpha > 1$, and the non-tangential maximal function

$$N(z) = \sup\{|\log f'(\zeta)| : \zeta \in \Gamma_1(z)\}.$$

For any $\varepsilon > 0$ and $\lambda > 0$

$$\operatorname{mes}\{z \in \mathbf{T} : N(z) > 2\lambda,\ S(z) < \varepsilon\lambda\} \le c_1 e^{-c_2/\varepsilon^2} \operatorname{mes}\{z \in \mathbf{T} : N(z) > \lambda\}.$$

Set here

$$\varepsilon = \varepsilon(\lambda) = \Big(\frac{c_2}{\lambda + c_3}\Big)^{1/2},$$

where $c_3 > 1$ is large enough. Then

$$\begin{aligned}\int_{\mathbf{T}} (e^N - 1) &= \int_0^\infty e^\lambda \operatorname{mes}\{N > \lambda\}\, d\lambda = 2\int_0^\infty e^{2\lambda} \operatorname{mes}\{N > 2\lambda\}\, d\lambda \\ &\le 2\int_0^\infty e^{2\lambda} \operatorname{mes}\{S > \varepsilon\lambda\}\, d\lambda + 2c_1 \int_0^\infty e^{-c_2/\varepsilon^2} e^{2\lambda} \operatorname{mes}\{N > \lambda\}\, d\lambda.\end{aligned}$$

The last integral does not exceed

$$2c_1 e^{-c_3} \int_0^\infty e^\lambda \operatorname{mes}\{N > \lambda\}\, d\lambda = 2c_1 e^{-c_3} \int_{\mathbf{T}} (e^N - 1)$$

due to the definition of ε. Choosing c_3 so large that $4c_1 e^{-c_3} < 1$ one obtains

$$\int_{\mathbf{T}} (e^N - 1) \le 4\int_0^\infty e^{2\lambda} \operatorname{mes}\{S > \varepsilon\lambda\}\, d\lambda$$

if the left-hand side is finite. The latter is true for the truncated function $f(rz)$, $r < 1$. But both sides of the estimate are monotonic in r in this case and so the estimate holds for the initial function f as well. However, a straightforward calculation shows that the last integral does not exceed $C \int_{\mathbf{T}} \exp bS^2$, where $b = 2c_3/c_2$. Therefore if $\exp aQ(\mu)^2 \in L^1$, $a = C_2^2 b$, then $\int_{\mathbf{T}} \exp N < +\infty$, and so $f' \in H^1(\mathbf{D})$. □

The case where (3.1) holds corresponds to *bounded* $Q(\mu)$.

In the next section, we shall prove that under the Carleson condition with a small constant, on the measure (3.2), the quadratic function $Q(\mu)^2$ belongs to $\mathrm{BMO}(\mathbf{T})$ with small BMO norm. Therefore, $\exp aQ(\mu)^2$ is integrable in this case by the John–Nirenberg theorem. So, Theorem 7 covers both known cases of the rectifiability.

Furthermore, we present, at the end of the following section, an example of a function μ, which satisfies the assumptions of Theorem 7 and does not satisfy the Carleson condition.

3.4. Absolute continuity in the Carleson problem. Theorem 7 has a corollary in the Carleson problem setting. Again, let Φ be a quasiconformal self-mapping of $\mathbf{C}\setminus\mathbf{D}$, $\Phi(\infty)=\infty$, with the complex dilatation μ. According to Section 0.6 we may assume that μ vanishes outside the annulus $\{z : 1<|z|<2\}$.

Corollary 3. *There exists $a>0$ such that if $\exp aQ(\mu)^2\in L^1(\mathbf{T})$ then Φ is absolutely continuous on $\mathbf{T}$.*

Proof. In view of Theorem 7, $\Gamma=f(\mathbf{T})$ is rectifiable in this case. According to the well-known properties of conformal mappings ([16], [21]) both f and g are absolutely continuous at the boundary. Therefore, $\Phi=g\circ f$ is also absolutely continuous on $\mathbf{T}$. □

4. The inclusion $\log f'\in\mathrm{BMOA}$

4.1. Astala–Zinsmeister's theorem. Theorem 1 leads to a new proof of Astala–Zinsmeister's result [5], proved originally by a certain estimate for the Schwarzian derivative of f.

As before, let f be a univalent function in $\mathbf{D}$ admitting a quasiconformal extension with complex dilatation μ, vanishing outside the annulus $\{z : 1<|z|<2\}$.

Theorem 8 [5]. *If μ satisfies in $\mathbf{C}\setminus\mathbf{D}$ the Carleson condition*

$$\frac{1}{|I|}\int_{\square_e(I)}\frac{|\mu(z)|^2}{|z|-1}\,dx\,dy\le A, \tag{4.1}$$

for any arc $I\subset\mathbf{T}$, then

$$\log f'\in\mathrm{BMOA}(\mathbf{D}).$$

Remark. It follows from the proof that if the left hand side of (4.1) tends to 0 as $|I|\to 0$ then $\log f'\in\mathrm{VMOA}(\mathbf{D})$.

Proof. According to Section 0.9, we have to prove that the measure

$$(1-|z|)\left|\frac{f''(z)}{f'(z)}\right|^2dx\,dy$$

in the unit disc is a Carleson measure.

Let $I\subset\mathbf{T}$ be an arc. In view of (1.1)

(4.2)

$$\begin{aligned}\int_{\square_i(I)}(1-|z|)\left|\frac{f''}{f'}\right|^2dx\,dy &\le C\int_{\square_i(I)}(1-|z|)\left\{\frac{1}{(1-|z|)^{\varkappa}}\left[1+\int_{1-|z|}^{1}\frac{\omega(z,t)}{t^{2-\varkappa}}\,dt\right]\right\}^2dx\,dy\\ &\le C|I|^{3-2\varkappa}+C\int_I d\theta\int_0^{|I|}r^{-\varkappa}\,dr\int_r^{\infty}\frac{\omega((1-r)e^{i\theta},t)^2}{t^{2-\varkappa}}\,dt\\ &\le C|I|^{3-2\varkappa}+C\int_{1<|\zeta|<2}|\mu(\zeta)|^2\,d\xi\,d\eta\int_{\Omega(\zeta)}\frac{dx\,dy\,dt}{(1-|z|)^{\varkappa}t^{4-\varkappa}},\end{aligned}$$

where the domain of integration is

$$\Omega(\zeta) = \{(z,t) : z \in \square_i(I),\ t > |\zeta - z|\}.$$

If $\zeta \notin \square_e(4I)$ then a straightforward calculation (similar to those from Sections 2.2 and 3.1) shows that the inner integral does not exceed $C \cdot (|I|^{2-\varkappa})/(|\zeta - z_I|^{3-\varkappa})$, where z_I is the central point of I. Therefore, the corresponding contribution to (4.2) does not exceed

$$C\,|I|^{2-\varkappa} \int_{\{1<|\zeta|<2\}\setminus\square_e(4I)} \frac{|\mu(\zeta)|^2}{|\zeta - z_I|^{3-\varkappa}}\, d\xi\, d\eta$$
$$\leq C\,|I|^{2-\varkappa} \sum_{k=2}^{\infty} \frac{1}{(2^k|I|)^{3-\varkappa}} \cdot 2^k|I| \int_{\square_e(2^kI)} \frac{|\mu(\zeta)|^2}{|\zeta|-1}\, d\xi\, d\eta.$$

In view of (4.1) this is less than

$$C \cdot A \sum_{2}^{\infty} \frac{1}{(2^k)^{2-\varkappa}}\, 2^k|I| \leq CA|I|.$$

If now $\zeta \in \square_e(4I)$ then, in the same way, the inner integral in (4.2) does not exceed $C/(|\zeta| - 1)$, and the corresponding contribution to (4.2) is less than

$$C \int_{\square_e(4I)} \frac{|\mu(\zeta)|^2}{|\zeta|-1}\, d\xi\, d\eta \leq CA|I|. \ \square$$

4.2. Rectifiability. Suppose that A in (4.1) is small enough. Then, due to Section 0.9 and to the John–Nirenberg theorem, $f' \in H^1(\mathbf{D})$ and the boundary of $f(\mathbf{D})$ is rectifiable.

This rectifiability result is weaker, however, than Theorem 7 of the previous section, as the following lemma shows.

Lemma. *If* (4.1) *holds then the quadratic function* $Q(\mu)^2$ *belongs to* $\mathrm{BMO}(\mathbf{T})$ *and its* BMO*-norm does not exceed* CA.

Now, if A in (4.1) is small enough, then by the John–Nirenberg theorem $\exp aQ(\mu)^2$ belongs to $L^1(\mathbf{T})$ and so (4.1) implies the assumption of Theorem 7.

Proof of the lemma. Let $I \in \mathbf{T}$ be an arc centered at a point z_I. Set

$$c_I = \int_{\{1<|\zeta|<2\}\setminus\square_e(2I)} \frac{|\mu(\zeta)|^2}{|\zeta - z_I|^2}\, d\xi\, d\eta.$$

Now for any $z \in I$

$$|Q(\mu)(z)^2 - c_I| \leq \int_{\square_e(2I)} \frac{|\mu(\zeta)|^2}{|\zeta - z|^2}\, d\xi\, d\eta + \int_{\{1<|\zeta|<2\}\setminus\square_e(2I)} \left| \frac{|\mu(\zeta)|^2}{|\zeta - z|^2} - \frac{|\mu(\zeta)|^2}{|\zeta - z_I|^2} \right| d\xi\, d\eta = P(z) + R(z).$$

Furthermore, in view of (4.1)

$$\frac{1}{|I|}\int_I P(z)\,|dz| \leq \frac{C}{|I|}\int_{\square_e(2I)} \frac{|\mu(\zeta)|^2}{|\zeta| - 1}\, d\xi\, d\eta \leq CA.$$

As to R, we have

$$R(z) \leq C \int_{\{1<|\zeta|<2\}\setminus\square_e(2I)} |\mu(\zeta)|^2 \frac{|I|}{|\zeta - z_I|^3}\, d\xi\, d\eta \leq C \sum_1^\infty \int_{\square_e(2^{k+1}I)\setminus\square_e(2^k I)}$$
$$\leq C \sum_1^\infty \frac{|I|}{(2^k|I|)^2} \int_{\square_e(2^{k+1}I)} \frac{|\mu(\zeta)|^2}{|\zeta| - 1}\, d\xi\ d\eta \leq CA \sum_1^\infty \frac{1}{2^{2k}|I|} \cdot 2^k|I| \leq CA. \ \square$$

4.3. An example. In this section, an example of a function μ, satisfying the assumptions of Theorem 7 but not the Carleson condition (4.1), is given.

Let E be a closed subset of $\mathbf{T}$ of zero Lebesgue measure. Consider the following function μ:

$$\mu(z) = b\Big(\frac{|z| - 1}{\rho(z, E)}\Big)^{1/2}, \quad 1 < |z| < 2, \qquad \mu(z) = 0 \quad \text{otherwise.} \tag{4.3}$$

Here the small constant b is to be chosen later.

For this μ

$$\frac{|\mu(z)|^2}{|z| - 1}\, dx\, dy = b^2\, \frac{dx\, dy}{\rho(z, E)}.$$

It is known (see, for example, [9]) that such a measure satisfies the Carleson condition if and only if the following *porosity condition* on E holds: there exists a constant $q > 0$ such that for any arc $I \subset \mathbf{T}$, $\sup_{z\in I} \rho(z, E) \geq q|I|$. On the other hand, if $z \in \mathbf{T}$ then the contribution of the disc $\{\zeta : |\zeta - z| < \frac{1}{2}\rho(z, E)\}$ to $Q(\mu)(z)^2$ is less than

$$\frac{4b^2}{\rho(z, E)} \int_{|\zeta - z|<\rho(z,E)/2} \frac{|\zeta| - 1}{|\zeta - z|^2}\, d\xi\, d\eta \leq 4\pi b^2.$$

The contribution of the rest of the annulus $\{1 < |\zeta| < 2\}$ does not exceed

$$b^2 \int_{\rho(z,E)/2<|\zeta-z|<3} \frac{d\xi\, d\eta}{|\zeta - z|^2} \le 2\pi b^2 \log \frac{6}{\rho(z,E)}.$$

Therefore, for our μ

$$Q(\mu)^2 \le 2\pi b^2 \cdot (2 + \log 6 + \log \frac{1}{\rho(z,E)}),$$

and $\exp aQ(\mu)^2 \in L^1(\mathbf{T})$ provided $(1/\rho(z,E)^{2\pi a b^2}) \in L^1(\mathbf{T})$.

Suppose now that the set E does not satisfy the porosity condition above but, nevertheless, there exists $\varepsilon > 0$ such that $(1/\rho(z,E)^{\varepsilon}) \in L^1$. A possible example of such a set is

$$E = \{e^{i/n} : n = 1, 2, \ldots\} \cup \{1\},$$

where any $\varepsilon < \frac{1}{2}$ fits.

Then the function μ (4.3) with $b < \left(\varepsilon/2\pi a\right)^{1/2}$ meets the rectifiability condition of Theorem 7 and does not meet the Carleson condition (4.1).

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