Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 22, 1997, 349–364

ANALYTICITY OF HAUSDORFF DIMENSION OF LIMIT SETS OF KLEINIAN GROUPS

James W. Anderson and André C. Rocha

University of Southampton, Faculty of Mathematical Studies
Highfield, Southampton SO17 1BJ, United Kingdom; jwa@maths.soton.ac.uk
Universidade Federal de Pernambuco, Departamento de Matemática
BR-50740-540 Recife-PE, Brazil; acr@dmat.ufpe.br

Abstract. We show that the Hausdorff dimension of the limit set is a real analytic function on the deformation space of a class of convex co-compact Kleinian groups which includes all convex co-compact function groups. This extends a result of Ruelle [20] for quasifuchsian groups.

1. Introduction

Ruelle [20], making use of Bowen's characterization of the Hausdorff dimension of the limit set of quasifuchsian and Schottky groups [7], showed that the Hausdorff dimension of the limit set is a real analytic function on the deformation space of a quasifuchsian or Schottky group. Sullivan [22] asked whether this holds true for any expanding, that is convex co-compact, Kleinian group Γ. In this note, we extend the class of groups for which the answer to Sullivan's question is affirmative to a class which includes all convex co-compact function groups as a proper subset.

Our work builds on Ruelle's approach. Roughly, given an expanding Markov map for Γ, Ruelle shows that the Hausdorff dimension of the limit set is a real analytic function on the deformation space of Γ. In Sections 2 and 3, we summarize his approach and give a brief survey of basic facts about deformation spaces of Kleinian groups.

It remains only to construct expanding Markov maps. Following Bowen and Series [8], Rocha [19] gives a condition on the fundamental polyhedron P of Γ in $\mathbf{H}^3$ which yields an expanding Markov map. This condition is that, in the tiling of $\mathbf{H}^3$ by translates of P, the hyperplane supporting a side of P lies in $\Gamma(\partial P)$; that is, P is an even-cornered fundamental polyhedron for Γ. We summarize Rocha's work in Section 4.

We then go on to show, in Section 5, that up to suitable modification by quasiconformal deformation, the Klein combination of two groups with even-cornered

---

1991 Mathematics Subject Classification: Primary 30F40, 58F20.

The first author was supported in part by an NSF-NATO Postdoctoral Fellowship and the second author by the Brazilian Research Council, CNPq.

fundamental polyhedra itself has an even-cornered fundamental polyhedron. Combining Bowen's observation that, up to quasiconformal deformation, quasifuchsian groups always possess such fundamental polyhedra with the observation that loxodromic cyclic groups always possess such polyhedra gives our result.

The natural setting for this question is holomorphically varying families of geometrically finite Kleinian groups. For Kleinian groups which are finitely generated but not geometrically finite, Bishop and Jones [6] have shown that the Hausdorff dimension of the limit set is always equal to 2.

Astala and Zinsmeister [3], [4] have investigated this question for infinitely generated quasifuchsian groups, and in particular have constructed a holomorphically varying family of infinitely generated quasifuchsian groups for which the Hausdorff dimension of the limit set is not real analytic.

Our methods do not apply to the case of geometrically finite groups with parabolics, though it is known that the Hausdorff dimension of the limit set is a continuous function on the deformation space of any geometrically finite Kleinian group; this follows from estimates of Gehring and Väisälä [14] from the theory of quasiconformal mappings. We would like to thank Edward Taylor for bringing this to our attention.

There is a large literature relating the Hausdorff dimension of the limit set of a Kleinian group to properties of the corresponding hyperbolic manifold. In particular, there is Sullivan's exploration of the very deep relationship between the Hausdorff dimension of the limit set of a convex co-compact Kleinian group and the critical exponent of the Poincaré series of the group [23], [24]. Also of interest is recent work of Canary, Minsky, and Taylor [9], which relates the Hausdorff dimension of limit sets to the spectral theory and topology of the quotient 3-manifolds.

We would also like to mention Furusawa [13] and Canary and Taylor [10], who consider the relationship between Hausdorff dimension of limit sets, critical exponents of Poincaré series, and Klein combination.

One can see the problem from a dynamical systems point of view if one uses the work of Sullivan [24]. He shows that, for convex co-compact Kleinian groups, the Hausdorff dimension of the limit set is equal to the topological entropy of the geodesic flow, restricted to those geodesics whose end points are in the limit set. Katok, Knieper, Pollicott and Weiss studied real analytic pertubations of Anosov and geodesic flows on closed Riemannian manifolds. They showed that the topological entropy varies as smoothly as the pertubations.

Acknowledgements. Both authors would like to thank the Mathematics Institute at the University of Warwick, where many of the early discussions leading to this work were held. We would also like to thank Martin Dunwoody for pointing out the proof of Lemma 5.5, and the referee for his/her useful comments.

2. Markov partitions and deformation spaces

In this section, we describe the notion of an expanding Markov map for a convex co-compact Kleinian group, and we show how such a map propogates through the deformation space of the group.

We begin by defining our terms. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, which acts both on hyperbolic 3-space $\mathbf{H}^3$ via isometries and on the Riemann sphere $\overline{\mathbf{C}}$ via conformal homeomorphisms. We assume throughout that a Kleinian group is *torsion-free*, that is, it contains no non-trivial elements of finite order. The action of Γ partitions $\overline{\mathbf{C}}$ into two sets. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open subset of $\overline{\mathbf{C}}$ on which Γ acts properly discontinuously. Throughout this work, we always assume that $\Omega(\Gamma)$ is non-empty and that $\infty \in \Omega(\Gamma)$.

The *limit set* $\Lambda(\Gamma)$ is the smallest non-empty closed subset of $\overline{\mathbf{C}}$ which is invariant under the action of Γ, and is the home of much of the interesting dynamical behavior of Γ. If $\Lambda(\Gamma)$ is finite, it contains at most two points, and we say Γ is *elementary.* If $\Lambda(\Gamma)$ is infinite, it is a closed, perfect, nowhere dense subset of $\overline{\mathbf{C}}$, and we say that Γ is *non-elementary.*

The *convex hull* of Γ is the smallest non-empty convex subset of $\mathbf{H}^3$ which is invariant under Γ, and contains all lines in $\mathbf{H}^3$ both of whose endpoints lie in $\Lambda(\Gamma)$. The *convex core* of $\mathbf{H}^3/\Gamma$ is the quotient of the convex hull by Γ. Say that Γ is *convex co-compact* if its convex core is compact.

In the language of Sullivan, convex co-compact Kleinian groups are *expanding*, in the sense that, for every $x \in \Lambda(\Gamma)$, there is some $\gamma \in \Gamma$ so that $|\gamma'(x)| > 1$; actually, there are infinitely many such elements γ for each point of $\Lambda(\Gamma)$.

We wish to encode the dynamics of the action of a convex co-compact Kleinian group Γ on its limit set $\Lambda(\Gamma)$ by a single map, and then study this map. Motivated by the constructions given for quasifuchsian and Schottky groups in [7], we wish to break $\overline{\mathbf{C}}$ into a finite collection of closed sets which are well-behaved with respect to the action of Γ. We then define a map $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ using Γ, in such a way as to record the dynamics of the action of Γ.

To that end, let $\{c_1, \dots, c_s\}$ be a finite collection of (not necessarily disjoint) Jordan curves in $\overline{\mathbf{C}}$, such that $\overline{\mathbf{C}} - (c_1 \cup \cdots \cup c_s)$ has finitely many components. Let $P_1, \dots, P_p$ be the closures of the components which contain points of $\Lambda(\Gamma)$ in their interiors, and let P_0 be the union of the closures of the remaining components. Let $\mathscr{P} = \{P_0, \dots, P_p\}$. Note that P_j is connected for $1 \le j \le p$, that the interior $\mathrm{int}(P_i)$ of P_i is disjoint from the interior $\mathrm{int}(P_j)$ of P_j for $i \ne j$, and that $P_0 \cup \cdots \cup P_p = \overline{\mathbf{C}}$.

In order that $\mathscr{P}$ should reflect the action of Γ, we impose some conditions.

(MP0): P_0 contains the closure of a fundamental domain for the action of Γ on $\Omega(\Gamma)$.

(MP1): It is necessary to impose some condition on the size of the intersection

$\partial P_j \cap \Lambda(\Gamma)$ for all $0 \leq j \leq p$, as these intersections cause ambiguity in the coding of the action of Γ on $\Lambda(\Gamma)$. For the purposes of this note, we assume that $\partial P_j \cap \Lambda(\Gamma)$ is finite for all $0 \leq j \leq p$.

We also suppose it is possible to define a map $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ which respects $\mathscr{P}$ and which records the dynamics of Γ, and so we require that

(MP2): for $1 \leq j \leq p$, there exists $\gamma_j \in \Gamma$ so that $f|_{P_j} = \gamma_j|_{P_j}$; we set $f|_{P_0}$ to be the identity.

(MP3): the image under f of any element in the partition is a union of elements of the partition; that is, given $0 \leq i \leq p$, there exist $j_1, \ldots, j_n$ so that $f(P_i) = P_{j_1} \cup \cdots \cup P_{j_n}$.

We say that such an f has the *Markov property* with respect to the partition $\mathscr{P}$.

We use the Markov map f to get a coding for the points in the limit set $\Lambda(\Gamma)$ of Γ. Specifically, given $x \in \overline{\mathbf{C}}$, consider a sequence $(p_0, p_1, \ldots)$ given by $x \in P_{p_0}$, $f(x) \in P_{p_1}$, $f^2(x) \in P_{p_2}$, etc. Notice that, if $x \in \Omega(\Gamma)$, then $f^n(x) \in P_0$ for some $n \geq 0$, and then $p_k = p_n = 0$ for all $k \geq n$. Hence, we ignore those points which end up in P_0.

Set

$$\Sigma_f = \left\{ \underline{p} = (p_k) \in \prod_{k=0}^{\infty} \{1, \ldots, n\} : f(P_{p_k}) \supseteq P_{p_{k+1}} \right\},$$

and consider the (one-sided) *subshift of finite type* Σ_f having as alphabet the elements in $\mathscr{P}$.

Define $\sigma_f\colon \Sigma_f \to \Sigma_f$ by $(\sigma_f \underline{p})_k = p_{k+1}$, and equip $\{1, \ldots, n\}$ with the discrete topology, so that Σ_f with the product topology is compact.

A finite sequence $(p_0 \cdots p_m)$ is *admissible* if $f(P_{p_k}) \supseteq P_{p_{k+1}}$ for $0 \leq k \leq m-1$, so that

$$f^m(P_{p_0}) \supseteq f^{m-1}(P_{p_1}) \supseteq \cdots \supseteq P_{p_m}.$$

If $(p_0 \cdots p_m)$ is admissible, define

$$P(p_0 \cdots p_m) = \bigcap_{i=0}^{m} f^{-i}(P_{p_i}).$$

Note that, if $(p_0, \ldots, p_m)$ is admissible, then the restriction of f^{m+1} to the interior of $P(p_0, \ldots, p_m)$ is equal to the restriction of $\gamma = \gamma_{p_m} \cdots \gamma_{p_0}$.

There are two additional conditions that we require a Markov partition satisfy.

(MP4): For each $\underline{p} \in \Sigma_f$, we have that $\operatorname{diam}\bigl(P(p_0 \ldots p_n)\bigr) \longrightarrow 0$ as $n \to \infty$, where $\operatorname{diam}(X)$ is the Euclidean diameter of X.

(MP5): f is *expanding*; that is, there exists an integer $N > 0$ and a constant $\beta > 1$ such that $|(f^N)'(x)| \geq \beta$ for all $x \in P(p_0, \ldots, p_N)$ for any admissible sequence $(p_0, \ldots, p_N)$.

These two conditions imply that the map $\pi_f\colon \Sigma_f \to \Lambda(\Gamma)$ defined by

$$\pi_f(\underline{p}) = \bigcap_{m=0}^{\infty} P(p_0 \cdots p_m)$$

is continuous and surjective. The relation between the shift map σ_f, the Markov map f, and π_f is that $f \circ \pi_f = \pi_f \circ \sigma_f$.

We say that a convex co-compact Kleinian group Γ *supports an expanding Markov map* if there exists a partition $\mathscr{P}$ and a map $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ which satisfy conditions (MP0)–(MP5).

We now give a brief description of the deformation space $\mathscr{T}(\Gamma)$ of Γ, and review some basic facts which we will make use of later. Much of the background material relating to this can be found in Bers [5] and Maskit [16].

Let $\mathscr{M}$ be the open unit ball in the complex Banach space of all complex valued measurable functions on $\overline{\mathbf{C}}$, with the norm $\|\mu\| = \operatorname{ess\,sup}(\mu)$. Given a Kleinian group Γ, set

$$\mathscr{M}(\Gamma) = \bigl\{\mu \in \mathscr{M} \mid \mu(\gamma(z)) \circ \overline{\gamma'(z)} = \mu(z) \circ \gamma'(z) \text{ for all } \gamma \in \Gamma,\ \text{a.e. } z \in \Omega(\Gamma)\bigr\},$$

and note that $\mathscr{M}(\Gamma)$ is again a complex Banach space, as it is a closed subspace of $\mathscr{M}$.

For each $\mu \in \mathscr{M}(\Gamma)$, let $\omega = \omega^\mu$ be the unique solution of the Beltrami equation $\omega_{\overline{z}} = \mu\omega_z$ fixing 0, 1, and ∞. Note that the condition on the elements of $\mathscr{M}(\Gamma)$ is exactly the condition needed to assert that $\omega \circ \gamma \circ \omega^{-1}$ is again an element of $\mathrm{PSL}_2(\mathbf{C})$. Hence, the group $\Gamma^\mu = \omega^\mu \circ \Gamma \circ (\omega^\mu)^{-1}$ is again a subgroup of $\mathrm{PSL}_2(\mathbf{C})$. The discreteness of Γ^μ follows from the fact that there does not exist a sequence in Γ converging to the identity, and so no such sequence can exist in Γ^μ. We call Γ^μ a *quasiconformal deformation* of Γ. As we have assumed that $\infty \in \Omega(\Gamma)$, we have that $\infty \in \Omega(\Gamma^\mu)$ for all $\mu \in \mathscr{M}(\Gamma)$.

Say that an element $\sigma \in \mathscr{M}(\Gamma)$ is *trivial* if $\omega^\sigma \circ \gamma \circ (\omega^\sigma)^{-1} = \gamma$ for all $\gamma \in \Gamma$. Note that, if σ is trivial, then ω^σ is the identity when restricted to $\Lambda(\Gamma)$, as ω^σ fixes the fixed points of all non-trivial elements of Γ and these fixed points are dense in $\Lambda(\Gamma)$.

Say that two elements μ and τ of $\mathscr{M}(\Gamma)$ are *equivalent*, denoted $\mu \sim \tau$, if there is a trivial $\sigma \in \mathscr{M}(\Gamma)$ so that

$$\omega^\mu = \omega^\tau \circ \omega^\sigma.$$

In particular, if $\mu \sim \tau$, then $\omega^\mu|_{\Lambda(\Gamma)} = \omega^\tau|_{\Lambda(\Gamma)}$.

Denote the equivalence class of $\mu \in \mathscr{M}(\Gamma)$ by $[\mu]$. The quotient of $\mathscr{M}(\Gamma)$ by this equivalence relation is the *deformation space* $\mathscr{T}(\Gamma)$ of Γ. The complex structure on $\mathscr{T}(\Gamma)$ can be seen by embedding $\mathscr{T}(\Gamma)$ as an open set in a complex manifold. Given $\mu \in \mathscr{M}(\Gamma)$, consider the discrete, faithful representation $\rho^\mu\colon \Gamma \to$

$\mathrm{PSL}_2(\mathbf{C})$ given by $\rho^\mu(\gamma) = \omega^\mu \circ \gamma \circ (\omega^\mu)^{-1}$. Note that the representation depends only on the equivalence class of μ, as trivial elements of $\mathscr{M}(\Gamma)$ induce the identity representation. With the choice of a generating set $\gamma_1, \dots, \gamma_k$ for Γ, this gives a holomorphic embedding of $\mathscr{T}(\Gamma)$ as a complex submanifold of $X = (\mathrm{PSL}_2(\mathbf{C}))^k$, namely

$$[\mu] \mapsto \big(\rho^\mu(\gamma_1), \dots, \rho^\mu(\gamma_k)\big).$$

In particular, we may view the entries of the $\rho^\mu(\gamma_j)$ as (complex) coordinates on $\mathscr{T}(\Gamma)$. The holomorphicity of this embedding follows from the Ahlfors–Bers measurable Riemann mapping theorem [2].

We now have the necessary material to construct Markov maps associated to groups in the deformation space $\mathscr{T}(\Gamma)$ of Γ. We use ω^μ to transport the partition $\mathscr{P}$ to a partition $\mathscr{P}^\mu$ associated to $\Gamma^\mu = \rho^\mu(\Gamma)$.

Given $\mu \in \mathscr{M}(\Gamma)$, define $f^\mu = \omega^\mu \circ f \circ (\omega^\mu)^{-1}$ and $\mathscr{P}^\mu = \{P_i^\mu = \omega^\mu(P_i) : P_i \in \mathscr{P}\}$; note that the elements of $\mathscr{P}^\mu$ are the complementary components of a finite collection of Jordan curves, and that the satisfaction of conditions (MP0) through (MP4) is unchanged by conjugation by a homeomorphism of $\overline{\mathbf{C}}$. The proof that f^μ is expanding mirrors the proof of Lemma 3 in Bowen [7]. Hence, f^μ is an expanding Markov map for Γ^μ with respect to the partition $\mathscr{P}^\mu$.

If we choose $\tau \sim \mu$, then ω^τ and ω^μ are equal when restricted to $\Lambda(\Gamma)$; in particular, $f^\mu = f^\tau$ as maps of $\overline{\mathbf{C}}$ and $\rho^\mu(\gamma) = \rho^\tau(\gamma)$ for all $\gamma \in \Gamma$, so that Γ^μ and Γ^τ are equal, and f^μ and f^τ are the same Markov map when restricted to $\Lambda(\Gamma^\mu)$. Hence, f^μ depends only on the equivalence class $[\mu] \in \mathscr{M}(\Gamma)$.

3. The Hausdorff dimension of the limit set

Denote the Hausdorff dimension of a subset X in $\mathbf{C}$ by $\dim_{\mathrm{H}}(X)$; for a definition and basic properties of Hausdorff dimension, see Falconer [12]. Given a convex co-compact Kleinian group Γ, consider the function $\delta_{\mathrm{H}}\colon \mathscr{T}(\Gamma) \to \mathbf{R}$, given by $\delta_{\mathrm{H}}([\mu]) = \dim_{\mathrm{H}}\big(\Lambda(\Gamma^\mu)\big)$. The purpose of this section is to demonstrate, given an expanding Markov map f for Γ with respect to the partition $\mathscr{P}$, that δ_{H} is a real analytic function on $\mathscr{T}(\Gamma)$. The argument follows closely the argument given by Ruelle [20] for Julia sets of hyperbolic rational maps and for quasifuchsian groups, and is given for the sake of completeness.

Let $\mathscr{C}(\Sigma_f)$ be the space of complex valued continuous functions on Σ_f, and let $\mathscr{C}(\Sigma_f, \mathbf{R}) \subset \mathscr{C}(\Sigma_f)$ denote the subspace of all real valued functions. Define the *pressure function* $P\colon \mathscr{C}(\Sigma_f, \mathbf{R}) \to \mathbf{R}$ by

$$(1) \qquad P(u) = \sup\left\{ h(\eta, \sigma) + \int u \, d\eta : \eta \text{ is a } \sigma\text{-invariant probability measure} \right\},$$

where $h(\eta, \sigma)$ is the measure theoretic entropy of σ with respect to η.

Define $\phi_\mu : \Sigma_{f^\mu} \to \mathbf{R}$ by $\phi_\mu(\underline{p}) = -\log\big|(f^\mu)'\big(\pi(\underline{p})\big)\big|$. We make use of the fact that ϕ_μ is Hölder continuous on Σ_{f^μ}; that is, there are constants $c > 0$ and

$\alpha < 1$ such that $|\phi_\mu(\underline{p}) - \phi_\mu(\underline{q})| \le c\alpha^n$ for $\underline{p}$ and $\underline{q}$ in Σ_{f^μ} with $p_j = q_j$ for $0 \le j \le n$. The proof of this fact mirrors the proof of Lemma 4 of Bowen [7].

We now show that δ_H is a real analytic function on $\mathscr{T}(\Gamma)$ when Γ supports an expanding Markov map.

Theorem 3.1. *Let Γ be a convex co-compact Kleinian group which supports an expanding Markov map. Then, the Hausdorff dimension of the limit set is a real analytic function on the deformation space $\mathscr{T}(\Gamma)$ of Γ.*

Proof. Let f be an expanding Markov map for Γ with respect to the partition $\mathscr{P}$. Given $[\mu] \in \mathscr{T}(\Gamma)$, let f^μ be the corresponding expanding Markov map for Γ^μ with respect to the partition $\mathscr{P}^\mu$.

We follow the argument in [20]. Consider the Ruelle zeta function

$$\zeta_\mu(u) = \exp \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{x \in \mathrm{Fix}(f^\mu)^n} \exp\biggl(\sum_{k=0}^{n-1} \phi_\mu\bigl((f^\mu)^k(x)\bigr)\biggr). \tag{2}$$

One then has that $\zeta_\mu(u)$ converges for $|u|$ sufficiently small, and extends to a meromorphic function of u in the entire complex plane $\mathbf{C}$ with a unique simple pole at $\exp\bigl(P(\phi_\mu)\bigr) > 0$.

Recall, for $[\mu] \in \mathscr{T}(\Gamma)$, the isomorphism $\rho^\mu\colon \Gamma \to \Gamma^\mu$ given by $\rho^\mu(\gamma) = \omega^\mu \circ \gamma \circ (\omega^\mu)^{-1}$, and let $\gamma^\mu = \rho^\mu(\gamma)$. Given a fixed point x of f^n, it is easy to see that there is some $\gamma \in \Gamma$ so that $x = f^n(x) = \gamma(x)$; in particular, x is also a fixed point of γ. The same holds true for the fixed points of $(f^\mu)^n$.

Given a loxodromic $\gamma \in \mathrm{PSL}_2(\mathbf{C})$, let $\mathscr{L}(\gamma)$ denote the real translation length of γ along its axis in $\mathbf{H}^3$. It is an easy calculation that

$$\mathscr{L}(\gamma) = \log|\gamma'(x)|,$$

where x is the repelling fixed point of γ.

We use these facts to see that (2) can be rewritten as

$$\zeta_\mu(u) = \exp \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{\gamma^\mu} e^{-\mathscr{L}(\gamma^\mu)n}, \tag{3}$$

where the second sum is taken over over all non-conjugate primitive loxodromic elements γ^μ of Γ^μ; recall that an element is *primitive* if it is not a proper power of any element in the group.

Fixing $\gamma \in \Gamma$, consider the function $[\mu] \mapsto \mathscr{L}(\gamma^\mu)$. It follows from the measurable Riemann mapping theorem [2] that $w^\mu(z)$ varies as nicely as μ varies for a fixed $z \in \mathbf{C}$, and so the entries and multiplier of γ^μ vary as nicely as $[\mu]$. In particular, $\mathscr{L}(\gamma^\mu)$ varies real analytically with $[\mu]$, as $\mathscr{L}(\gamma^\mu)$ is the real part of an analytic function of the entries of γ^μ. Hence, the coefficients of $\zeta_\mu(u)$ depend analytically on $[\mu]$, and it follows that so does $\zeta_\mu(u)$.

Applying the implicit function theorem to the function $([\mu], u) \mapsto 1/\zeta_\mu(u)$, we see that the function $[\mu] \mapsto P(\phi_\mu)$ is real analytic. Bowen [7] showed that there is a unique real number t_μ such that $P(t_\mu \phi_\mu) = 0$, and Ruelle [20] showed that the Hausdorff dimension of $\Lambda(\Gamma^\mu)$ is equal to t_μ.

Finally, consider the mapping from $\mathbf{R} \times \mathscr{T}(\Gamma)$ given by $(a, \mu) \mapsto P(a\phi_\mu)$. Since the pressure function P has a unique zero at $a = t_\mu$, the implicit function theorem yields that this zero varies real analytically with $[\mu]$, and we are done. □

We close this section with the following remark.

Remark 3.2. Let Γ be a convex co-compact Kleinian group, and let Γ^o be a finite index subgroup of Γ. If Γ^o supports an expanding Markov map f with respect to a partition $\mathscr{P}$, then Γ also supports the same expanding Markov map f with respect to the same partition $\mathscr{P}$. This mirrors the fact that the limit set of a finite index subgroup of Γ is equal to the limit set of Γ.

4. Even corners and expanding Markov maps

In this section, we describe the condition of having an even-cornered fundamental polyhedron, and show that the existence of such a fundamental polyhedron yields the existence of an expanding Markov map. This section is an exposition of some of the results appearing in Rocha [19].

This construction generalizes a construction originally given by R. Bowen and C. Series [8] in their work on Markov partitions for Fuchsian groups. They show, among other results, the existence of an expanding Markov map associated to certain finitely generated Fuchsian groups Φ acting on the hyperbolic plane $\mathbf{H}^2$, namely those having a fundamental polygon D in $\mathbf{H}^2$ with the *even corners property*, that is, $\Phi(\partial D)$ is a union of lines. Note that the fundamental group of any compact surface of genus $g \geq 2$ has a Fuchsian realization having a fundamental polygon with this property [8].

A *polyhedron* in $\mathbf{H}^3$ is the intersection of a locally finite collection of closed half spaces; in particular, polyhedra are convex. A *fundamental polyhedron* P for a Kleinian group Γ is a polyhedron in $\mathbf{H}^3$ so that every point of $\mathbf{H}^3$ is a translate of a point of P, the interior $\mathrm{int}(P)$ of P is disjoint from all its translates, and the sides of P are paired by elements of Γ. By this last condition, we mean that, for every side s of P, there is a side s' of P and an element $\gamma_s \in \Gamma$ with $\gamma_s(s) = s'$; we also require that $\gamma_{s'} = \gamma_s^{-1}$. We call γ_s a *side pairing* transformation.

A Kleinian group is *geometrically finite* if there exists a finite sided fundamental polyhedron for its action on $\mathbf{H}^3$. It is a consequence of the Poincaré polyhedron theorem (see [13]) that a geometrically finite Kleinian group is generated by its side pairing transformations; in particular, geometrically finite Kleinian groups are finitely generated. Convex co-compact Kleinian groups are geometrically finite.

A Kleinian group Γ acting on $\mathbf{H}^3$ has the *even corners property* if there is a fundamental polyhedron P for Γ so that

$$\Gamma(\partial P) = \bigcup_{\gamma \in \Gamma} \gamma(\partial P)$$

is a union of geodesic planes. We sometimes refer to a fundamental polyhedron P satisfying this condition as an *even cornered* fundamental polyhedron for Γ.

Given an even cornered fundamental polyhedron $\mathscr{R}$ for a convex co-compact Kleinian group Γ, we use the planes in $\Gamma(\partial\mathscr{R})$ to construct a partition of $\overline{\mathbf{C}}$, and use the generating set for Γ which comes from $\mathscr{R}$ to construct the map f.

Let $\mathscr{R}$ be a fundamental polyhedron for the convex co-compact Kleinian group Γ, and set

$$\Gamma_{\mathscr{R}} = \{\gamma \in \Gamma : \gamma(\mathscr{R}) \cap \mathscr{R} \text{ is a side of } \mathscr{R}\}.$$

This is a finite subset of Γ and is symmetric, in the sense that $\Gamma_{\mathscr{R}}$ is closed under inverses. Since $\Gamma_{\mathscr{R}}$ consists of the side pairing transformations for Γ, it is a generating set for Γ (though perhaps not minimal).

Let s be a side of $\mathscr{R}$, let P_s be the geodesic plane containing s, and let γ_s be the side pairing transformation for s. Label the side s of $\mathscr{R}$ by writing γ_s on the inside of $\mathscr{R}$ and $\overline{\gamma_s}$ on the outside, where $\overline{\gamma_s}$ denotes γ_s^{-1}. Each plane $P(\gamma_s)$ is the boundary of a half space $H(\gamma_s)$ which does not contain $\mathscr{R}$. Observe that $\mathbf{H}^3 = \mathscr{R} \bigcup \cup_{\gamma\in\Gamma_{\mathscr{R}}} H(\gamma)$.

Let $\Gamma_{\mathscr{R}} = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Define a map $f\colon \mathbf{H}^3 \to \mathbf{H}^3$ as follows. Since each $P(\gamma_s)$ contains a side s of $\mathscr{R}$, we use the corresponding side identification γ_s to send a point $x \in H(\gamma_s)$ to $\gamma_s^{-1}x$, if the external label of s is γ_s. Clearly this transformation is far from being well defined due to the fact that the point x may be in more than one of the half-spaces $H(\gamma_s)$. In order to get a well defined map, we choose an order on the generating set $\Gamma_{\mathscr{R}}$, say $\gamma_1 \prec \gamma_2 \cdots \prec \gamma_n$, then, if $x \in H(\gamma_{s_1}) \cap \cdots \cap H(\gamma_{s_l})$, define $f(x) = \gamma_{s_k}^{-1}x$ where γ_{s_k} is the least among $\gamma_{s_1}, \dots, \gamma_{s_l}$. If $x \in \mathscr{R}$, define $f(x) = x$.

Consider now the set of all planes in $\Gamma(\partial\mathscr{R})$ that intersect $\mathscr{R}$ in either a side, an edge, or a vertex. The intersection of each plane in this collection with $\overline{\mathbf{C}}$ is a circle, and the closures of the components of the complement of the union of these circles covers $\overline{\mathbf{C}}$, giving us a finite partition $\mathscr{P} = \{P_i\}$ of $\overline{\mathbf{C}}$. Observe that the intersection of $\mathscr{R}$ with $\overline{\mathbf{C}}$ is a union of elements of $\mathscr{P}$.

It is immediate to verify that the map f defined in the last paragraph can be extended to $\overline{\mathbf{C}}$ in the same way it was defined. One is also able to show that this partition satisfies conditions (MP4) and (MP5), and so is an expanding Markov map.

Summarizing the analysis carried out in the last few paragraphs, we have the following.

Theorem 4.1 (see Rocha [19]). *Let Γ be a convex co-compact Kleinian group which has an even-cornered fundamental polyhedron. Then, Γ supports an expanding Markov map.*

5. Even corners, quasiconformal deformations, and Klein combination

The purpose of this section is two-fold. We begin by exploring some connections between Klein combination, described below, and quasiconformal deformations. Specifically, we show that, if Γ is formed from Γ_1 and Γ_2 by Klein combination, and if Γ_j' is a quasiconformal deformation of Γ_j, then under some mild conditions, Γ and the group Γ' formed by Klein combination of Γ_1' and Γ_2' are quasiconformally conjugate. We go on to apply this to show that a large class of convex co-compact Kleinian groups, including all such function groups, are quasiconformally conjugate to groups with even-cornered fundamental polyhedra. This is sufficient to guarantee that the Hausdorff dimension of the limit set is a real analytic function on the deformation space of such a Kleinian group.

As has already been stated, we work only with torsion-free convex co-compact Kleinian groups. In the constructions we consider in this section, these restrictions are not essential, but are made primarily for ease of exposition.

Let Γ_1 and Γ_2 be convex co-compact Kleinian groups. We wish to impose topological conditions which imply that $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is again a Kleinian group, and which yield information about the structure of Γ. One of the first results of this type is the Klein combination theorem. The version we give below is tailored to the applications of this paper. Recall that a set $X \subset \overline{\mathbf{C}}$ is *precisely invariant under the identity* in a Kleinian group Γ if $X \cap \gamma(X) = \emptyset$ for all $\gamma \in \Gamma - \{1\}$.

Theorem 5.1 (Klein combination [18]). *Let Γ_1 and Γ_2 be convex co-compact Kleinian groups.*

In the case that Γ_1 and Γ_2 are both non-elementary, suppose there exists a Jordan curve $c \subset \Omega(\Gamma_1) \cap \Omega(\Gamma_2)$ which bounds closed discs E_1 and E_2 in $\overline{\mathbf{C}}$, where E_j is precisely invariant under the identity in Γ_j. Then, $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to $\Gamma_1 * \Gamma_2$; moveover, c is contained in $\Omega(\Gamma)$, and is precisely invariant under the identity in Γ.

In the case that $\Gamma_2 = \langle \gamma_2 \rangle$ is elementary, suppose there exists a pair of disjoint Jordan curves c_1 and c_2 in $\Omega(\Gamma_1)$ which bound disjoint closed discs E_1 and E_2 in $\Omega(\Gamma_1)$, so that E_1 and E_2 are both precisely invariant under the identity in Γ_1, so that no translate of E_1 by an element of Γ_1 intersects E_2, and so that $\gamma_2(c_1) = c_2$ and $\gamma_2(\mathrm{int}(E_1)) \cap \mathrm{int}(E_2) = \emptyset$. Then, $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to $\Gamma_1 * \Gamma_2$; moreover, c_1 and c_2 are contained in $\Omega(\Gamma)$ and each is precisely invariant under the identity in Γ.

As an illustrative example of the use of the Klein combination theorem, consider a Schottky group of genus two. That is, let c_1, c_1', c_2, and c_2' be disjoint

Jordan curves which bound a common region D, and let γ_j be a Möbius transformation which maps c_j to c'_j so that $\gamma_j(D) \cap D = \emptyset$. Let $\Gamma_j = \langle \gamma_j \rangle$ and $\Gamma = \langle \gamma_1, \gamma_2 \rangle$.

To build Γ using the first part of the Klein combination theorem, let c be a Jordan curve in D which separates $c_1 \cup c'_1$ from $c_2 \cup c'_2$. It is easy to see that c lies in $\Omega(\Gamma_1) \cap \Omega(\Gamma_2)$ and bounds closed discs E_1 and E_2 in $\overline{\mathbf{C}}$, where E_j is precisely invariant under the identity in Γ_j.

To build Γ using the second part of the Klein combination theorem, note that c_2 and c'_2 satisfy the hypotheses of the combination theorem. That is, c_2 and c'_2 bound closed discs E_2 and E'_2, respectively, in $\Omega(\Gamma_1)$, namely the closed discs determined by c_2 and c'_2 which are disjoint from D, both E_2 and E'_2 are precisely invariant under the identity in Γ_1, no translate of E_2 by an element of Γ_1 intersects E'_2, and $\gamma_2(c_2) = c'_2$ and $\gamma_2\bigl(\operatorname{int}(E_2)\bigr) \cap \operatorname{int}(E'_2)$ is empty.

Recall that a *function group* is a finitely generated Kleinian group Φ whose domain of discontinuity contains a component Δ which is invariant under the action of Φ. The stabilizer of a component of the domain of discontinuity of a finitely generated Kleinian group is the archetypal example of a function group.

For the duration of this section, we use the notation established in the statement of Theorem 5.1.

Theorem 5.2. *Let Γ be a Kleinian group formed from convex co-compact Kleinian groups Γ_1 and Γ_2 using Klein combination. Let Γ'_j be a quasiconformal deformation of Γ_j.*

In the case that Γ_1 and Γ_2 are non-elementary, we are in the first case of the Klein combination theorem. Let Δ_j be the component of $\Omega(\Gamma_j)$ containing the Jordan curve c, and let Δ'_j be the corresponding component of $\Omega(\Gamma'_j)$. If Γ' is formed from Γ'_1 and Γ'_2 by Klein combination along a Jordan curve c' in $\Delta'_1 \cap \Delta'_2$, then Γ' is a quasiconformal deformation of Γ.

In the case that $\Gamma_2 = \langle \gamma_2 \rangle$ is elementary, we are in the second case of the Klein combination theorem. Let Δ_1 and Δ_2 be the components of $\Omega(\Gamma_1)$ containing the Jordan curves c_1 and c_2, respectively, and let Δ'_1 and Δ'_2 be the corresponding components of $\Omega(\Gamma'_1)$. If Γ' is formed from Γ'_1 and Γ'_2 by Klein combination along Jordan curves c'_1 and c'_2 in Δ'_1 and Δ'_2, respectively, then Γ' is a quasiconformal deformation of Γ.

Proof. Let $w_j\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be the quasiconformal mapping which conjugates Γ_j to Γ'_j and let $\rho_j\colon \Gamma_j \to \Gamma'_j$ be the induced isomorphism.

Consider the isomorphism

$$\rho\colon \Gamma = \Gamma_1 * \Gamma_2 \to \Gamma' = \Gamma'_1 * \Gamma'_2$$

defined by setting $\rho(\gamma_j) = \rho_j(\gamma_j)$ for $\gamma_j \in \Gamma_j$ and by setting $\rho(\gamma_n \cdots \gamma_1) = \rho_2(\gamma_n) \cdots \rho_1(\gamma_1)$, where $\gamma_{\mathrm{even}} \in \Gamma_2$ and $\gamma_{\mathrm{odd}} \in \Gamma_1$. We show that there exists a quasiconformal homeomorphism w of $\overline{\mathbf{C}}$ such that $\rho(\gamma) = w \circ \gamma \circ w^{-1}$.

Marden's isomorphism theorem [15] yields that an isomorphism between convex co-compact Kleinian groups is induced by a quasiconformal homeomorphism of $\overline{\mathbf{C}}$ if there exists a quasiconformal homeomorphism between the domains of discontinuity which induces the isomorphism. A theorem of Maskit [16] on isomorphisms between function groups states that an isomorphism between convex co-compact function groups is induced by a homeomorphism between their invariant components. Hence, it suffices to show, for any component Δ of $\Omega(\Gamma)$, that $\rho(\mathrm{st}_\Gamma(\Delta))$ is the stabilizer of a component of Γ'.

In the case that Γ_1 and Γ_2 are non-elementary, any component Δ of $\Omega(\Gamma_j)$ which is not equivalent to Δ_j is a component of $\Omega(\Gamma)$. Let Δ' be the component of Γ_j' corresponding to Δ. Then, the isomorphism ρ takes the stabilizer of Δ to the stabilizer of Δ'.

The stabilizer of the component Δ of $\Omega(\Gamma)$ containing c is the Klein combination of $\mathrm{st}_{\Gamma_1}(\Delta_1)$ and $\mathrm{st}_{\Gamma_2}(\Delta_2)$ along c, while the stabilizer of the component of Γ' containing c' is the Klein combination of $\mathrm{st}_{\Gamma_1'}(\Delta_1')$ and $\mathrm{st}_{\Gamma_2'}(\Delta_2')$ along c'. Since $\rho(\mathrm{st}_{\Gamma_j}(\Delta_j)) = \mathrm{st}_{\Gamma_j'}(\Delta_j')$ for both j, ρ takes the stabilizer of Δ in Γ to the stabilizer of Δ' in Γ'.

Hence, in the case that Γ_1 and Γ_2 are non-elementary, ρ is geometric.

In the case that $\Gamma_2 = \langle\gamma_2\rangle$ is elementary, any component of $\Omega(\Gamma_1)$ which is not equivalent to either Δ_1 or Δ_2 is a component of $\Omega(\Gamma)$, and ρ takes the stabilizer of such a component of Γ to the stabilizer of such a component of Γ'.

If $\Delta_1 = \Delta_2$, then c_1 and c_2 lie in the same component Δ of $\Omega(\Gamma)$, and the stabilizer of this component is formed from $\mathrm{st}_{\Gamma_1}(\Delta_1)$ and Γ_2 using the second part of the Klein combination theorem. Since $\Delta_1' = \Delta_2'$, c_1' and c_2' lie in the component Δ' of $\Omega(\Gamma')$, and the stabilizer of this component is the Klein combination of $\mathrm{st}_{\Gamma_1'}(\Delta_1')$ and Γ_2' using the second part of the Klein combination theorem, and so ρ takes the stabilizer of Δ in Γ to the stabilizer of Δ' in Γ'. If Δ_1 and Δ_2 are equivalent under Γ_1, we can make a change of generators, modifying Γ_2, so that $\Delta_1 = \Delta_2$, and use the conjugating quasiconformal map to carry this to Γ_1'.

If $\Delta_1 \neq \Delta_2$, then γ_2 does not stabilize any component of $\Omega(\Gamma)$. In this case, the stabilizer of the component Δ of $\Omega(\Gamma)$ containing c_1 is the Klein combination of $\mathrm{st}_{\Gamma_1}(\Delta_1)$ and $\gamma_2^{-1}\,\mathrm{st}_{\Gamma_1}(\Delta_2)\gamma_2$ using the first part of the Klein combination theorem. As before, the stabilizer of the corresponding component Δ' of $\Omega(\Gamma')$ is similarly constructed, and ρ takes the stabilizer of Δ in Γ to the stabilizer of Δ' in Γ'.

Hence, ρ is induced by a quasiconformal homeomorphism of $\overline{\mathbf{C}}$. □

As an application of Theorem 5.2, we establish a link between Klein combination and quasiconformal deformations, and the existence of even-cornered fundamental polyhedra.

Proposition 5.3. *Let Γ be a Kleinian group which is formed from convex co-compact Kleinian groups Γ_1 and Γ_2 by Klein combination. Suppose, for both*

j, that Γ_j is quasiconformally conjugate to a Kleinian group Γ'_j which has an even-cornered fundamental polyhedron. Then, Γ is quasiconformally conjugate to a convex co-compact Kleinian group Γ' with an even-cornered fundamental polyhedron.

Proof. Let $P'_j \subset \mathbf{H}^3$ be an even-cornered fundamental polyhedron for Γ'_j. Let D'_j be the interior of the boundary at infinity of P'_j, and note that D'_j is a fundamental domain for the action of Γ'_j on $\Omega(\Gamma'_j)$.

Suppose Γ_1 and Γ_2 are non-elementary, and let Δ_j be the component of $\Omega(\Gamma_j)$ containing the Jordan curve c. Taking a translate of P'_j, if necessary, suppose that Δ'_j contains a component of D'_j. Conjugating Γ'_2 by a Möbius transformation, if necessary, we may assume that the circle $c = \{|z| = 1\}$ induces the Klein combination of Γ'_1 and Γ'_2, and that the closed discs bounded by c lie in D'_1 and D'_2. Theorem 5.2 gives that Γ' is a quasiconformal deformation of Γ. Moreover, the intersection of the fundamental polyhedra of Γ'_1 and Γ'_2 is a fundamental polyhedron for Γ', and it is easy to see that this polyhedron is even-cornered.

There is a similar argument in the case that Γ_2 is elementary. □

It is known that the Klein combination theorem gives a decomposition of a convex co-compact Kleinian group into subgroups which either have connected limit set or are elementary, that is loxodromic cyclic. We begin the discussion of this decomposition with a description of those convex co-compact Kleinian groups whose limit sets are connected.

A *quasifuchsian* group is a convex co-compact Kleinian group Γ whose limit set $\Lambda(\Gamma)$ is a Jordan curve and which contains no element interchanging the components of $\overline{\mathbf{C}} - \Lambda(\Gamma)$. An *extended quasifuchsian* group is a convex co-compact Kleinian group Γ whose limit set $\Lambda(\Gamma)$ is a Jordan curve and which contains some element interchanging the components of $\overline{\mathbf{C}} - \Lambda(\Gamma)$. Every extended quasifuchsian group Γ contains a canonical quasifuchsian subgroup of index 2, consisting of those elements which do not interchange the components of $\overline{\mathbf{C}} - \Lambda(\Gamma)$.

A *web* group is a finitely generated Kleinian group Γ whose domain of discontinuity $\Omega(\Gamma)$ contains infinitely many components so that the stabilizer $\mathrm{st}_\Gamma(\Delta)$ of any component Δ is quasifuchsian.

It follows from work of Abikoff and Maskit [1] that, given a convex co-compact Kleinian group Γ, there exists a finite collection $\Phi_1, \dots, \Phi_s$ of convex co-compact subgroups, where each Φ_j is either loxodromic cyclic, quasifuchsian, extended quasifuchsian, or web, so that Γ is formed from $\Phi_1, \dots, \Phi_s$ by $s-1$ applications of the Klein combination theorem. Moreover, this decomposition corresponds to a maximal free product decomposition of Γ. Combining this with Proposition 5.3, we have the following.

Proposition 5.4. *Let Γ be a convex co-compact Kleinian group with a maximal free product decomposition $\Gamma = \Phi_1 * \cdots * \Phi_p$. Suppose that each Φ_j is* *quasiconformally conjugate to a group Φ_j' which has an even-cornered fundamental polyhedron. Then, Γ is quasiconformally conjugate to a group Γ' which has an even-cornered fundamental polyhedron.*

Each loxodromic cyclic Kleinian group $\langle\gamma\rangle$ has the even corners property; let H be any hyperplane in $\mathbf{H}^3$ which is orthogonal to the axis of γ and consider the polyhedron whose sides are H and $\gamma(H)$. As noted by Bowen [7], every quasifuchsian group is quasiconformally conjugate to a Fuchsian group which has the even corners property.

It is not known whether an extended quasifuchsian or web group has a quasiconformal deformation which has an even-cornered fundamental polyhedron, though by Remark 3.2 an extended quasifuchsian group always possesses an expanding Markov map. We are unable to directly construct expanding Markov maps for Klein combinations of groups from expanding Markov maps of the factors. To handle groups with extended quasifuchsian factors, we use the following lemma.

Lemma 5.5. *Let $\Gamma = \Gamma_1 * \cdots * \Gamma_p$ be a non-trivial free product, and let Γ_j^0 be a finite index normal subgroup of Γ_j. Then, there exists a finite index normal subgroup Γ^0 of Γ which is a free product $\Gamma^0 = \Phi_1 * \cdots * \Phi_q * \Theta$, where each Φ_k is conjugate to some Γ_j^0, and Θ is a free group.*

Proof. Consider the homomorphism from Γ to the direct product $(\Gamma_1/\Gamma_1^0) \times \cdots \times (\Gamma_p/\Gamma_p^0)$. This direct product is finite, as each Γ_j^0 has finite index in Γ_j. By the Kurosh subgroup theorem, the kernel has the desired form. □

We are now ready to prove the main result of this section.

Theorem 5.6. *Let Γ be a convex co-compact Kleinian group which is formed from loxodromic cyclic, quasifuchsian, and extended quasifuchsian groups by Klein combination. Then, the Hausdorff dimension of the limit set is a real analytic function on $\mathscr{T}(\Gamma)$.*

Proof. The hypothesis gives that Γ has a maximal free product decomposition $\Gamma = \Phi_1 * \cdots * \Phi_p$, where each Φ_j is either loxodromic cyclic, quasifuchsian, or extended quasifuchsian. By Lemma 5.5, Γ has a finite index subgroup Γ^o which is the free product of loxodromic cyclic and quasifuchsian groups. By Proposition 5.4, Γ^o is quasiconformally conjugate to a group Γ' which has an even-cornered fundamental polyhedron, and hence supports an expanding Markov map. Combining Theorem 4.1 and Theorem 3.1, we see that δ_{H} is real analytic on $\mathscr{T}(\Gamma)$. □

As convex co-compact function groups can always be constructed from loxodromic cyclic and quasifuchsian groups using Klein combination, we have the following corollary.

Corollary 5.7. *Let Γ be a convex co-compact function group. Then, Γ is quasiconformally conjugate to a Kleinian group with an even cornered fundamental polyhedron. In particular, the Hausdorff dimension of the limit set is a real analytic function on $\mathscr{T}(\Gamma)$.*

References

[1] Abikoff, W., and B. Maskit: Geometric decomposition of Kleinian groups. - Amer. J. Math. 99, 1977, 687–697.

[2] Ahlfors, L., and L. Bers: Riemann's mapping theorem for variable metrics. - Ann. Math. 72, 1960, 385–404.

[3] Astala, K., and M. Zinsmeister: Holomorphic families of quasi-Fuchsian groups. - Ergodic Theory Dynamical Systems 14, 1994, 207–212.

[4] Astala, K., and M. Zinsmeister: Abelian coverings, Poincaré exponent of convergence, and holomorphic deformations. - Ann. Acad. Sci. Fenn. Ser. A I Math. 20, 1995, 81–86.

[5] Bers, L.: Spaces of Kleinian groups. - In: Several Complex Variables I, edited by J. Horváth, Lecture Notes in Math. 155, Springer-Verlag, Berlin, 1970, 9–34.

[6] Bishop, C.J., and P.W. Jones: Hausdorff dimension and Kleinian groups. - Preprint.

[7] Bowen, R.: Hausdorff dimension of quasi-circles. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 1–25.

[8] Bowen, R., and C. Series: Markov maps associated to Fuchsian groups. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 153–170.

[9] Canary, R.D., Y.N. Minsky, and E.C. Taylor: Spectral theory, Hausdorff dimension, and the topology of hyperbolic 3-manifolds. - J. Geom. Anal. (to appear).

[10] Canary, R.D., and E.C. Taylor: Kleinian groups with small limit sets. - Duke Math. J. 73, 1994, 371–381.

[11] Epstein, D.B.A., and C. Petronio: An exposition of Poincaré's polyhedron theorem. - Enseign. Math. 40, 1994, 113–170.

[12] Falconer, K.J.: The Geometry of Fractal Sets. - Cambridge University Press, Cambridge, 1985.

[13] Furusawa, H.: The exponent of convergence of Poincaré series of combination groups. - Tôhoku Math. J. 43, 1991, 1–7.

[14] Gehring, F.W., and J. Väisälä: Hausdorff dimension and quasiconformal mappings. - J. London. Math. Soc. 6, 1973, 504–512.

[15] Marden, A.:: The geometry of finitely generated Kleinian groups. - Ann. Math. 99, 1974, 383–462.

[16] Maskit, B.: Self-maps of Kleinian groups. - Amer. J. Math. 93, 1971, 840–856.

[17] Maskit, B.: Isomorphisms of function groups. - J. Analyse Math. 32, 1977, 63–82.

[18] Maskit, B.: Kleinian Groups. - Springer-Verlag, Berlin, 1989.

[19] Rocha, A.C.: Meromorphic extension of the Selberg zeta function via thermodynamic formalism. - Math. Proc. Cambridge Philos. Soc. 119, 1996, 179–190.

[20] Ruelle, D.: Repellers for real analytic maps. - Ergodic Theory Dynamical Systems 2, 1982, 99–107.

[21] Series, C.: Geometrical Methods of Symbolic Dynamics. - In: Ergodic Theory, Symbolic Dynamics and Hyperbolic Geometry, edited by T. Bedford, M. Keane and C. Series, Oxford University Press, Oxford, 1991, 125-151.

[22] Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 171–209.

[23] Sullivan, D.: Conformal dynamical systems. - In: Geometric Dynamics, edited by J. Palis, Jr., Lecture Notes in Math. 1007, Springer-Verlag, Berlin, 1983, 725–752.

[24] Sullivan, D.: Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. - Acta Math. 153, 1984, 259–277.

Received 20 December 1995