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QUASINORMABLE SPACES AND THE PROBLEM OF TOPOLOGIES OF GROTHENDIECK

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Abstract. This article is dedicated to the study of quasinormable injective tensor products of locally convex spaces and quasinormable spaces of continuous linear operators. The stability of the quasinormability is obtained in the frame of the class of spaces which are quasinormable by operators; this class, introduced and studied here, contains many function spaces. The problems considered in the article are closely related to the problem of topologies of Grothendieck. A characterization of the quasinormable spaces which are (FBa)-spaces in the sense of Taskinen is obtained and new examples and counterexamples are given. In particular we show that the quasinormable space l_{p+} is a concrete example of a non-(FBa)-space.

Grothendieck (see [30, 31]) studied locally convex properties of function spaces, such as spaces of sequences, of differentiable functions, analytic functions, distributions, etc. There are a lot of examples of spaces of vector-valued functions which can be represented as tensor products or as spaces of continuous and linear mappings and it is convenient to know their topological structure.

The aim of this article is to study the stability of the property of being quasinormable under the formation of injective tensor products or of spaces of continuous and linear mappings. The class of quasinormable locally convex spaces was introduced and studied by Grothendieck as a class containing most of the usual function spaces. Banach spaces and Schwartz spaces are examples of quasinormable spaces. The typical examples of quasinormable spaces which are neither normable nor Montel are the following: the space $C(X)$ endowed with the compact open topology for every completely regular Hausdorff space X, the spaces $C^k(\Omega)$ ($k \in \mathbf{N}$, Ω an open subset of $\mathbf{R}^N$), every non-trivial quojection, every non-trivial (gDF)-space which is not Montel, the spaces $\mathscr{B}_0(\mathbf{R}^N)$ and $\mathscr{B}(\mathbf{R}^N)$ of Schwartz and the local spaces $\mathscr{B}^{\mathrm{loc}}_{p,k}(\Omega)$ of Hörmander.

More precisely we investigate the following questions:

(P1) If E and F are quasinormable l.c.s., when is $E \otimes_\varepsilon F$ also quasinormable?

(P2) If E'_b and F are quasinormable l.c.s., when is $L_b(E,F)$ quasinormable?

In [18] Bonet and the author proved that there are Banach spaces Z and quasinormable spaces E (Fréchet or (DF)) such that $Z \hat{\otimes}_\varepsilon E$ and $L_b(Z,E)$ are not quasinormable. Here we give conditions for the answer to (P1) to be positive

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if we fix the Banach space Z or the quasinormable space E. Our purpose is to introduce and investigate certain classes of l.c.s. which constitute a very general frame in which the problems (P1) and (P2) have a positive answer: the spaces which are quasinormable by operators, briefly (QNo), and the spaces that satisfy the strict Mackey condition by operators, briefly (QNo)'. These classes are related with the (FG) and (DFG)-spaces of Bonet, Díaz and Taskinen [14] which were introduced to give positive answers to the problem of topologies of Grothendieck and other related dual questions. In fact we show that the problem (P1) is somewhat equivalent to the problem of topologies of Grothendieck. We include some applications to infinite-dimensional holomorphy.

The article is divided in five sections. In Section 1 we introduce some notation and give preliminary results. Section 2 is devoted to the study of problems (P1) and (P2) when F is a fixed Banach space. In Section 3 we define and investigate the classes (QNo) and (QNo)', establishing some hereditary properties and showing that the answer to (P1) and (P2) is positive in this context. We also show that these classes are, in a certain sense, optimal. We provide examples of spaces in the classes (QNo) and (QNo)'. In particular we give examples of spaces of continuous functions, holomorphic functions and C^∞ functions. In Section 4 we study the problem of topologies of Grothendieck and some dual questions in the context of Fréchet spaces (QNo) and (DF)-spaces with condition (QNo)'. The quasinormable Fréchet spaces which are (FBa)-spaces in the sense of Taskinen [49] are characterized as the Fréchet spaces E such that E'_b satisfies (QNo)'. From this we conclude that the space l_{p+} is not an (FBa)-space. This answers a question of Taskinen and it is the first natural example of a Fréchet space which is not an (FBa)-space. In [45] the author showed that there are Fréchet Schwartz spaces which are not (QNo), thus not (FBa)-spaces. Finally, in Section 5 we apply the results of the previous sections to obtain new examples of spaces of holomorphic functions defined on Fréchet or (DF)-spaces which are quasinormable.

Our notation is standard. We refer the reader to [34, 38, 43] for locally convex spaces and to [24] for infinite holomorphy.

1. Definitions and preliminary results

If X is a Banach space, B_X denotes its closed unit ball. For a locally convex space (l.c.s.) E, $\mathrm{FIN}(E)$ denotes the set of all finite dimensional subspaces of E. If E is a l.c.s., $\mathscr{U}_0(E)$ and $\mathscr{B}(E)$ stand for the families of all absolutely convex 0-neighbourhoods and absolutely convex bounded sets in E respectively. The absolutely convex hull of a subset A of E is denoted by $\Gamma(A)$. If A is an absolutely convex (abx.) subset of E, we denote by p_A the Minkowski functional associated with A and $E_A := [A]/\ker p_A$ endowed with the norm induced by p_A. By E'_b, E'_{co} and E'_i we mean the strong dual of E, E' endowed with the topology of uniform convergence on abx. compact subsets of E, and the bornological space associated with E'_b, respectively. In what follows we will use the spaces C_p $(1 < p < \infty)$ of

Johnson as defined, e.g., in [34] except that we assume $C_p' = C_q$ $(1/p+1/q=1)$. This amounts to choosing a sequence $(F_k)_{k\in\mathbf{N}}$ of finite dimensional Banach spaces which is dense in the set of all finite dimensional Banach spaces endowed with the Banach Mazur distance and letting C_p be the l_p-direct sum of $\oplus_k F_k \times \oplus_k F_k'$. The space C_p $(1<p<\infty)$ is reflexive and has a Schauder basis (cf. [35]). We recall that a l.c. space E is called *quasinormable* if

$$\forall U \in \mathscr{U}_0(E) \qquad \exists V \in \mathscr{U}_0(E) \qquad \forall \varepsilon > 0 \qquad \exists B \in \mathscr{B}(E): \; V \subset \varepsilon U + B,$$

and E is said to satisfy the *strict Mackey condition* if

$$\forall B \in \mathscr{B}(E) \qquad \exists C \in \mathscr{B}(E) \qquad \forall \varepsilon > 0 \qquad \exists U \in \mathscr{U}_0(E): \; U \cap B \subset \varepsilon C.$$

If E is quasibarrelled then E is quasinormable if and only if E_b' satisfies the strict Mackey condition. The strong dual G_b' of a l.c.s. G with the strict Mackey condition is quasinormable, and the converse is true if G is quasibarrelled. Every quasinormable Fréchet space is distinguished, i.e. $E_b' = E_i'$. The class of quasinormable spaces has good stability properties, for instance quotients and countable inductive limits of quasinormable spaces are quasinormable, completions and products of quasinormable spaces are also quasinormable. Concerning tensor products, Grothendieck showed that the projective tensor product of two quasinormable spaces is also quasinormable. This is not true in general for the injective tensor product as was shown by Bonet and the author in [18]. One also has that the injective tensor product $G \otimes_\varepsilon H$ of two (DF)-spaces G and H with the strict Mackey condition always satisfies the strict Mackey condition, but there are Banach spaces X and strict (LB)-spaces G such that $G \otimes_\pi X$ does not satisfy the strict Mackey condition (see [18]).

A l.c.s. E is a *(gDF)-space* if it has a fundamental sequence $(B_n)_{n\in\mathbf{N}}$ of bounded subsets such that for every sequence $(U_n)_{n\in\mathbf{N}}$ of 0-neighbourhoods in E, there is $U \in \mathscr{U}_0(E)$ such that

$$U \subset \bigcap_{n\in\mathbf{N}} (U_n + B_n),$$

that is, the topology τ of E is the finest locally convex topology coinciding with τ on each B_n, $n \in \mathbf{N}$.

Every (gDF)-space is quasinormable and the (gDF)-spaces with the strict Mackey condition are bornological.

If E and F are l.c.s., by $L(E,F)$ we denote the space of all linear continuous mappings from E into F. If $A \subset E$, $B \subset F$ and M is a fixed linear subspace of $L(E,F)$ we write

$$W(A,B) := \{f \in M \;/\; f(A) \subset B\}.$$

The space $L(E,F)$ endowed with the topology of uniform convergence on the bounded subsets of E will be denoted by $L_b(E,F)$. There is a topological isomorphism between $L_b(E,F'_b)$ and the space $(E\otimes_\pi F)'$ endowed with the topology of uniform convergence on the elements of $\mathscr{A} := \big\{\overline{\Gamma(B\otimes C)}\ /\ B\in\mathscr{B}(E),\ C\in\mathscr{B}(F)\big\}$. In particular, if E and F are (DF)-spaces, then $L_b(E,F'_b)\cong(E\hat{\otimes}_\pi F)'_b$ (see [38, 41.4.(7)]), but this isomorphism is not true in general if E and F are Fréchet spaces as Taskinen showed in [48].

We recall the definitions of (FG)-spaces and (DFG)-spaces given by Bonet, Díaz and Taskinen in [14]:

Definition 1.1. *A Fréchet space E is said to be an (FG)-space if there is an increasing fundamental sequence of seminorms $(\|\cdot\|_k)_{k\in\mathbf{N}}$ such that for every sequence $(\alpha_k)_{k\in\mathbf{N}}$, $0<\alpha_k\leq 1$ $(k\in\mathbf{N})$ there is a sequence $(P_k)_{k\in\mathbf{N}}\subset L(E,E)$ satisfying*

(FG1) $x=\sum_{j\in\mathbf{N}}P_j(x)$, $\forall x\in E$,
(FG2) $\|P_k(x)\|_{k-1}\leq\alpha_k\|x\|_k$, $\forall x\in E$, $\forall k\geq 2$,
(FG3) $\forall j>k$, $\exists\lambda_{jk}\geq 1 : \|P_k(x)\|_j\leq\lambda_{jk}\|x\|_k$, $\forall x\in E$.

If we let $U_k := \{x\in E : \|x\|_k\leq 1\}$, condition (FG2) is equivalent to $P_k(U_k)\subset\alpha_k U_{k-1}$, $\forall k\geq 2$ and (FG3) is equivalent to $P_k(U_k)\in\mathscr{B}(E)$, $k\in\mathbf{N}$.

Definition 1.2. *A (DF)-space (G,t) is said to be a (DFG)-space if there is an increasing fundamental sequence $(B_k)_{k\in\mathbf{N}}$ of closed abx. bounded sets in G and there is a locally convex topology s in G weaker than t such that (G,t) has a basis of s-closed abx. 0-neighbourhoods and, for every sequence $(\alpha_k)_{k\in\mathbf{N}}$, $0<\alpha_k\leq 1$, there is a sequence of operators $(Q_k)_{k\in\mathbf{N}}$ in $L\big((G,t),(G,t)\big)$ such that*

(DFG1) $x=\sum_{j\in\mathbf{N}}Q_j(x)$, $\forall x\in G$, *where the series converges for the topology s,*
(DFG2) $Q_k(B_{k-1})\subset\alpha_k B_k$, $\forall k\geq 2$,
(DFG3) $Q_k^{-1}(B_k)$ *is a 0-neighbourhood in (G,t) for every $k\in\mathbf{N}$.*

If the topology s can be taken equal to t in the definition we will say that (G,t) is a strong (DFG)-space.

We need also some definitions and a technical lemma on tensor norms. We refer to [43] and [21] for the notations.

If E and F are l.c.s. and a is a tensor norm, the topology of $E\otimes_a F$ is given by the system of seminorms $(p_U\otimes_a p_V)(z) := a\big((\Phi_U\otimes\Phi_V)(z); E_U, F_V\big)$, $z\in E\otimes F$, $U\in\mathscr{U}_0(E)$, $V\in\mathscr{U}_0(F)$, where $\Phi_U\colon E\to E_U$, $\Phi_V\colon F\to F_V$ are the canonical maps.

If A and B are abx. subsets of E and F, respectively, we will denote by $a(A,B) := \{x\in E\otimes_a F\ /\ (p_A\otimes_a p_B)(x)\leq 1\} = \big\{x\in[A]\otimes[B]\ /\ a\big((\Phi_A\otimes\Phi_B)(x); E_A, E_B\big)\leq 1\big\}$.

Lemma 1.3. *If E_i and F_i are l.c.s., $T_i \in \mathscr{L}(E_i, F_i)$, $i = 1, 2$; then*

$$(T_1 \otimes T_2)\big(a(A,B)\big) \subset a\big(T_1(A), T_2(B)\big)$$

for every $A \subset E_1$, $B \subset E_2$ absolutely convex.

Proof. The following operators are canonically induced by T_i, $i = 1, 2$.

$$\tilde{T}_1\colon E_{1_A} \to F_{1_{T_1 A}} \qquad \tilde{T}_2\colon E_{2_B} \to F_{2_{T_2 B}}$$

and the following diagram is commutative

$$\begin{array}{ccc} [A] \otimes [B] & \xrightarrow{T_1 \otimes T_2} & [T_1 A] \otimes [T_2 B] \\ \Big\downarrow{\scriptstyle \Phi_A \otimes \Phi_B} & & \Big\downarrow{\scriptstyle \Phi_{T_1 A} \otimes \Phi_{T_2 B}} \\ E_{1_A} \otimes E_{2_B} & \xrightarrow{\tilde{T}_1 \otimes \tilde{T}_2} & F_{1_{T_1 A}} \otimes F_{2_{T_2 B}} \end{array}$$

Let $z \in a(A,B)$. By definition we have

$$a\big((\Phi_A \otimes \Phi_B)(z); E_{1_A}, E_{2_B}\big) \le 1.$$

This implies

$$a\big(\big[(\tilde{T}_1 \otimes \tilde{T}_2) \circ (\Phi_A \otimes \Phi_B)\big](z); F_{1_{T_1 A}} \otimes F_{2_{T_2 B}}\big) \le 1$$

and, by the commutativity of the diagram above,

$$(T_1 \otimes T_2)(z) \in a(T_1 A, T_2 B).$$

2. $\mathscr{L}_\infty$-spaces and quasinormability

In [18] Bonet and the author gave examples of quasinormable Fréchet or (DF)-spaces E and Banach spaces Z such that $E \hat{\otimes}_\varepsilon Z$ is not quasinormable. It is (now) possible to get a characterization, in terms of the Banach space Z, of the stability of the property of being quasinormable under the formation of injective tensor products. First we need a technical lemma.

For $\mathscr{L}_p$-spaces in the sense of Lindenstrauss and Pełczyński we refer to [39].

Lemma 2.1. *Let Z be a Banach space which is not an $\mathscr{L}_\infty$-space. Then there are reflexive Banach spaces X, Y such that Y is a topological subspace of X, X' and Y^o (polar in X') have the bounded approximation property and the canonical map*

$$j\colon Y^o \otimes_\pi Z' \to X' \otimes_\pi Z'$$

is not a monomorphism.

Proof. We will follow the idea of [22, Proposition 1.1]. If Z is not an $\mathscr{L}_\infty$-space then Z' is not an $\mathscr{L}_1$-space. This implies that there are Banach spaces G and M, with M a topological subspace of G, such that the canonical inclusion of $M \otimes_\pi Z'$ into $G \otimes_\pi Z'$ is not a monomorphism (see e.g. [28]). Since the projective tensor norm is determined by finite-dimensional spaces, there are finite-dimensional Banach spaces G_n with subspaces M_n and elements $z_n \in M_n \otimes Z'$, $n \in \mathbf{N}$, such that

$$(*) \qquad \|z_n\|_{G_n \otimes_\pi Z'} < 1 \qquad \text{but} \qquad \|z_n\|_{M_n \otimes_\pi Z'} > n.$$

Set $l_2\big((G_n)_{n\in\mathbf{N}}\big)$ and $l_2\big((M_n)_{n\in\mathbf{N}}\big)$ the l_2-sum of $(G_n)_{n\in\mathbf{N}}$ and $(M_n)_{n\in\mathbf{N}}$ respectively. We define

$$X := \big(l_2\big((G_n)_{n\in\mathbf{N}}\big)\big)'$$

and

$$Y := \big(l_2\big((M_n)_{n\in\mathbf{N}}\big)\big)^o = \big(l_2\big((G_n)_{n\in\mathbf{N}}\big)/l_2\big((M_n)_{n\in\mathbf{N}}\big)\big)'.$$

Let B and C be the unit balls of $l_2\big((M_n)_{n\in\mathbf{N}}\big)$ and $l_2\big((G_n)_{n\in\mathbf{N}}\big)$, respectively. If the canonical map $j\colon\ Y^o \otimes_\pi Z' \to X' \otimes_\pi Z'$ is a monomorphism, there is $\alpha > 0$ such that

$$(*)' \qquad \Gamma(C \otimes B_{Z'}) \cap (Y^o \otimes Z') \subset \alpha\Gamma(B \otimes B_{Z'})$$

Defining $x_n \in l_2\big((M_n)_{n\in\mathbf{N}}\big) \otimes Z'$ as z_n in the n^{th}-coordinate and zero in the other case, we have by $(*)$ that

$$x_n \in \Gamma(C \otimes B_{Z'}) \qquad \text{but} \qquad x_n \notin n\Gamma(B \otimes B_{Z'})$$

for every $n \in \mathbf{N}$. This contradicts $(*)'$. □

Definition 2.2. *Let λ be a normal Banach sequence space and let X, Y be Banach spaces such that Y is a topological subspace of X. The standard quojection of Moscatelli type associated with λ, X and Y is defined as:*

$$\lambda(X, X/Y) := \big\{\, (x_n)_{n\in\mathbf{N}} \in X^{\mathbf{N}} \,/\, (\|q(x_n)\|)_{n\in\mathbf{N}} \in \lambda \,\big\},$$

where $q\colon X \to X/Y$ is the quotient map. A basis of 0-neighbourhoods in E is given by $(k^{-1}W_k)_{k\in\mathbf{N}}$, where

$$W_k := \big\{\, (x_n)_{n\in\mathbf{N}} \in X^{\mathbf{N}} \,/\, \|\big((\|x_j\|)_{j<k}, (\|q(x_j)\|\big)_{j\geq k})\|_\lambda \leq 1 \,\big\}.$$

Denoting by F_n the n^{th}-coordinate subspace of E, we can identify F_n with X and

$$W_k \cap F_n = \begin{cases} B_X + Y & \text{if } k \leq n \\ B_X & \text{if } k > n. \end{cases}$$

For more details on Fréchet spaces of Moscatelli type we refer to [15].

Theorem 2.3. *Let Z be a Banach space. The following are equivalent*

(a) *Z is an $\mathscr{L}_\infty$-space,*
(b) *$Z\hat{\otimes}_\varepsilon E$ is quasinormable for every quasinormable space E,*
(c) *$Z\hat{\otimes}_\varepsilon E$ is quasinormable for every reflexive quojection E,*
(d) *$Z\hat{\otimes}_\varepsilon E$ is quasinormable for every reflexive strict LB-space E.*

Proof. (a) implies (b) follows from a result of Defant (see [19, 4.5.5]) while (b) implies (c) and (b) implies (d) are trivial.

(c) implies (a):

Take X and Y as in Lemma 2.1. Since X and X/Y have the approximation property and are reflexive, $E := l_2(X, X/Y)$ is a reduced projective limit of Banach spaces with the approximation property (i.e., E has the strict approximation property) and it is reflexive (see [15, 2.3, 2.6]). By [18, Lemma 2.6] we know that the canonical inclusion

$$\Phi\colon l_2(X, X/Y)'_b\hat{\otimes}_\pi Z' \to \big(l_2(X, X/Y)\hat{\otimes}_\varepsilon Z\big)'_b$$

is a continuous bijection such that, for every equicontinuous subset C of $\big(l_2(X, X/Y)\ \hat{\otimes}_\varepsilon Z\big)'_b$, there are $U \in \mathscr{U}_0\big(l_2(X, X/Y)\big)$ and $V \in \mathscr{U}_0(Z)$ satisfying

$$C \subset \overline{\Gamma(U^o \otimes V^o)}^{\,l_2(X,X/Y)'_b\hat{\otimes}_\pi Z'}.$$

We conclude as in [18, 2.5] that $l_2(X, X/Y)\hat{\otimes}_\varepsilon Z$ is not quasinormable.

(d) implies (a). Take also X and Y as in Lemma 2.1 and define $E := l_2(X, X/Y)'_b$. Now the conclusion follows from [22, Proposition 2.2] and the argument of [18, 3.4]. □

Remark. If Z is a Banach space which is not an $\mathscr{L}_\infty$-space, then there are quojections (respectively strict LB-spaces) E such that $L_b(Z', E)$ is not quasinormable (see [18, 2.2, 2.3]).

We can ask the following natural question, related with the Theorem 2.3: If Z is a Banach $\mathscr{L}_\infty$-space and E'_b is quasinormable, is $L_b(E, Z)$ necessarily quasinormable? This is not true in general and we will give a concrete example after some technical results.

Let X, Y, Z be Banach spaces such that Y is a topological subspace of X and consider the restriction map:

$$R\colon L_b(X, Z) \to L_b(Y, Z); \qquad f \mapsto f\,|_Y\,.$$

We define $Y^\perp := \{\, f \in L_b(X, Z)\ /\ f(Y) = 0\,\}$.

Lemma 2.4. *If there is $s > 0$ such that*

$$W(B_X, 2B_Z) \cap W(sB_Y, B_Z) \subset Y^\perp + W(B_X, B_Z),$$

then R is a homomorphism.

Proof. First, let us show that

$$(1)\qquad \big(Y^{\perp}+W(B_X,2B_Z)\big)\cap W(sB_Y,B_Z)\subset Y^{\perp}+W(B_X,B_Z).$$

If $f\in\big(Y^{\perp}+W(B_X,2B_Z)\big)\cap W(sB_Y,B_Z)$; $f=g+h$ with $g\in Y^{\perp}$ and $h\in W(B_X,2B_Z)$, then $h\in W(sB_Y,B_Z)+Y^{\perp}=W(sB_Y,B_Z)$. Therefore $h\in W(B_X,2B_Z)\cap W(sB_Y,B_Z)$ and, by hypothesis, $h\in Y^{\perp}+W(B_X,B_Z)$ as well as f.

Proceeding by induction the inclusion (1) implies

$$(2)\qquad \big(Y^{\perp}+W(B_X,2^nB_Z)\big)\cap W(sB_Y,B_Z)\subset Y^{\perp}+W(B_X,B_Z),\qquad \forall n\in\mathbf{N}.$$

Now (2) implies

$$W(sB_Y,B_Z)\subset Y^{\perp}+W(B_X,B_Z),$$

and we conclude $R\big(W(B_X,B_Z)\big)\supset R\big(W(sB_Y,B_Z)\big)\in\mathscr{U}_0(\operatorname{Im}R)$. □

By an injective Banach space we mean a Banach space which is complemented in every (Banach) space containing it. If $\lambda\geq 1$ we say that a Banach space X belongs to the class P_λ if, for every Banach space Y containing X, there is a projection from Y onto X with norm less than or equal to λ.

The following lemma was kindly provided by P. Domański.

Lemma 2.5. *There are Banach spaces X, Y, Z with Y a topological subspace of X such that Z is an $\mathscr{L}_\infty$-space but R is not a homomorphism.*

Proof. By [55, 5. Examples] we can find a sequence $(K_n)_{n\in\mathbf{N}}$ of compact spaces such that $C(K_n)$ is injective but $C(K_n)\notin P_n$. Then there is a sequence $(X_n)_{n\in\mathbf{N}}$ of Banach spaces such that $C(K_n)$ is a topological subspace of X_n and every projection from X_n onto $C(K_n)$ has norm greater than n, $n\in\mathbf{N}$. Define $X:=l_\infty\big((X_n)_{n\in\mathbf{N}}\big)$ and $Y=Z:=l_\infty\big((C(K_n))_{n\in\mathbf{N}}\big)$ (the l_∞-sum of the respective sequences of Banach spaces). Let us suppose that R is a homomorphism, then there is $\lambda>0$ such that

$$\lambda R\big(W(B_X,B_Z)\big)\supset \operatorname{Im}R\cap W(B_Y,B_Z).$$

Now take $f_n\colon Y\to Z$, $y\mapsto y\cdot e_n$ (e_n the n^{th}-coordinate vector of l_∞), $n\in\mathbf{N}$. It easily follows that $f_n\in\operatorname{Im}R\cap W(B_Y,B_Z)$, $n\in\mathbf{N}$, since $C(K_n)$ is injective. If there are elements $h_n\in W(B_X,B_Z)$ with $R(\lambda h_n)=f_n$, $n\in\mathbf{N}$, then choosing $n\in\mathbf{N}$ with $n>\lambda$ we have

$$\lambda h_n\,|_{X_n}\in L\big(X_n,C(K_n)\big),\qquad \lambda h_n\,|_{C(K_n)}=f_n\,|_{C(K_n)}=I_{C(K_n)},$$

that is, $g_n:=\lambda h_n\,|_{X_n}$ is a projection from X_n onto $C(K_n)$. But $\|g_n\|\leq\lambda<n$, which yields a contradiction. □

Proposition 2.6. *There are a Banach space X, a closed subspace Y, and a Banach $\mathscr{L}_\infty$-space Z such that $L_b(\lambda(X,X/Y),Z)$ is not quasinormable.*

Proof. Choose X, Y, Z as in Lemma 2.5. We can write $E := \lambda(X,X/Y)$ and consider the canonical 0-basis $(W_k)_{k\in\mathbf{N}}$ in E as in 2.2.

If $L_b(E,Z)$ is quasinormable then, given $B := \bigcap_{k\in\mathbf{N}} W_k \in \mathscr{B}(E)$, there is $C = \bigcap_{k\in\mathbf{N}} \lambda_k W_k \in \mathscr{B}(E)$ such that, for every $M > 0$, there exists $n \in \mathbf{N}$ satisfying

$$W(C, MB_Z) \subset nW(W_n, B_Z) + W(B, B_Z).$$

Take $k > n$ and intersect the above inclusion with $L(F_k, Z)$ to get

$$W(M,k) := W\big(\lambda_k B_X \cap (\lambda_1 B_X + Y), MB_Z\big) \subset nW(B_X + Y, B_Z) + W(B_X, B_Z).$$

We can now find a suitable (bigger) M such that there is $s > 0$ which satisfies

$$W(B_X, 2B_Z) \cap W(sB_Y, B_Z) \subset W(M,k) \subset Y^\perp + W(B_X, B_Z),$$

and we have a contradiction by Lemmas 2.4 and 2.5. □

Observation. Note that if Z is the dual of an $\mathscr{L}_1$-space and E is a Fréchet space, then it is known (cf. [22]) that $L_b(E,Z)$ is a (DF)-space (thus quasinormable). Proposition 2.6 shows that this assertion does not hold in general if Z is an $\mathscr{L}_\infty$-space which is not a dual space.

3. The classes (QNo) and (QNo)'

Definition 3.1. *A l.c. space E is said to be quasinormable by operators, briefly (QNo), if there is a basis $\mathscr{U}$ of absolutely convex closed 0-neighbourhoods in E such that:*

$$\forall U \in \mathscr{U}, \qquad \exists V \in \mathscr{U} \qquad \forall \varepsilon > 0 \qquad \exists P \in L(E,E):$$

(i) $P(V) \in \mathscr{B}(E)$,
(ii) $(I-P)(V) \subset \varepsilon U$.

Definition 3.2. *A l.c. space E satisfies the strict Mackey condition by operators, briefly (QNo)', if there is a fundamental system $\mathscr{B}$ of absolutely convex bounded subsets of E such that*

$$\forall B \in \mathscr{B}, \qquad \exists C \in \mathscr{B} \qquad \forall \varepsilon > 0 \qquad \exists P \in L(E,E):$$

(i) $P^{-1}(C) \in \mathscr{U}_0(E)$,
(ii) $(I-P)(B) \subset \varepsilon C$.

Remarks. (a) It follows easily from the definitions that the l.c.s. which are (QNo) (respectively satisfy (QNo)') are "a fortiori" quasinormable (respectively satisfy the strict Mackey condition).

(b) Definitions 3.1 and 3.2 do not depend on the respective fundamental systems $\mathscr{U}$ and $\mathscr{B}$ of 0-neighbourhoods and bounded subsets in E.

(c) A l.c.s. E is (QNo) if and only if, considering the corresponding projective spectrum $\{E_U, \Phi_{U,V}\}_{(\mathscr{U}_0(E),\subset)}$, the following property is satisfied: For every $U \in \mathscr{U}_0(E)$ there is $V \in \mathscr{U}_0(E)$, $(V \subset U)$, such that $\Phi_{U,V}$ can be approximated in the operator norm by linear maps of type $\{\Phi_U \circ P \,/\, P \in L(E_V, E)\}$. Analogously E satisfies (QNo)' if and only if, considering the corresponding inductive spectrum $\{E_B\,,\, \Phi_{C,B}\}_{(\mathscr{B}(E),\subset)}$, the following property is satisfied: For every $B \in \mathscr{B}(E)$, there is $C \in \mathscr{B}(E)$, $(B \subset C)$, such that $\Phi_{C,B}$ can be approximated in the operator norm by restrictions to E_B of continuous linear maps from E into E_C.

It is possible to establish the following easy hereditary properties of the classes (QNo) and (QNo)', which are inspired by the respective hereditary properties of quasinormability and the strict Mackey condition. We recall that a subspace F of a l.c.s. E is called *large* if, for every $B \in \mathscr{B}(E)$ there is $C \in \mathscr{B}(F)$ such that $B \subset \overline{C}$.

Proposition 3.3. (1) *The complemented subspaces of a l.c.s. E which is (QNo) (respectively satisfies (QNo)') are also (QNo) (respectively also satisfy (QNo)').*

(2) *Every normed space is (QNo) and satisfies (QNo)'.*

(3) *The product (respectively direct sum) of a family of spaces (QNo) (respectively satisfying (QNo)') is also (QNo) (respectively also satisfies (QNo)').*

(4) *The direct sum (respectively product) of a sequence of spaces (QNo) (respectively satisfying (QNo)') is also (QNo) (respectively also satisfies (QNo)').*

(5) *If E is locally complete and F is a dense (respectively large) subspace which is (QNo) (respectively satisfies (QNo)'), then E is also (QNo) (respectively also satisfies (QNo)').*

The main purpose of Definitions 3.1 and 3.2 is to obtain good stability properties of the quasinormability by taking tensor products and spaces of continuous and linear operators, as the following results show.

Proposition 3.4. *Let a be a tensor norm.*

(1) *If E and F are (QNo), then $E \otimes_a F$ is also (QNo).*

(2) *If G and H satisfy (QNo)' and the bounded subsets of $G \otimes_a H$ are "localizable", i.e., the family of bounded sets $\{\overline{a(A, B)}^{G\otimes_a H} \,/\, A \in \mathscr{B}(G),\ B \in \mathscr{B}(H)\}$ is fundamental in $G \otimes_a H$, then $E \otimes_a F$ also satisfies (QNo)'.*

Proof. We are going to prove (2). The proof of (1) follows the same pattern. Given $B \in \mathscr{B}(G \otimes_a H)$ there are $B_1 \in \mathscr{B}(G)$ and $B_2 \in \mathscr{B}(H)$ such that $B \subset$

$\overline{a(B_1,B_2)}^{G\otimes_a H}$. By hypothesis there are $C_1 \in \mathscr{B}(G)$, $C_2 \in \mathscr{B}(H)$ ($B_i \subset C_i$, $i = 1,2$) such that $\forall \lambda_1 > 0$, $\exists P_1 \in L(G,G)$ and $\forall \lambda_2 > 0$, $\exists P_2 \in L(H,H)$ satisfying:

(a) $U := P_1^{-1}(C_1) \in \mathscr{U}_0(G)$, $(I - P_1)(\lambda_1 B_1) \subset C_1$,

(b) $V := P_2^{-1}(C_2) \in \mathscr{U}_0(H)$, $(I - P_2)(\lambda_2 B_2) \subset C_2$.

Let $\lambda > 0$. We take $\lambda_1 := 2\lambda$ and $\lambda_2 := 2\lambda M$, where $M > 0$ satisfies $B_1 \subset MU$. Defining $P := P_1 \otimes P_2 \in L(G \otimes_a H, G \otimes_a H)$, $C := \overline{a(C_1,C_2)}^{G\otimes_a H}$ we conclude, on account of Lemma 1.3, that

(i) $P\big(a(U,V)\big) \subset a\big(P_1(U),P_2(V)\big) \subset C$,

(ii) $(I-P)\big(\lambda a(B_1,B_2)\big) = \big[(I-P_1)\otimes I + P_1 \otimes (I-P_2)\big]\big(\lambda a(B_1,B_2)\big)$
$\subset a\big((I-P_1)(\lambda B_1),B_2\big) + a\big(P_1(B_1),(I-P_2)(\lambda B_2)\big)$
$\subset \frac{1}{2}a(C_1,B_2) + \frac{1}{2}a(MC_1,M^{-1}C_2) \subset a(C_1,C_2)$.

Thus $(I-P)(\lambda B) \subset C$. □

Observation. The hypotheses of Proposition 3.4 (2) are satisfied if, for instance, $a = \varepsilon$ and G, H are both Fréchet or (DF)-spaces which satisfy (QNo)' (see [20]), or $a = \pi$ and G, H are both (DF)-spaces satisfying (QNo)' (cf. [38, 41.4.7]).

In the following proposition we will denote by $\mathscr{A}$ the ideal $\mathscr{F}$ of finite rank operators, $\mathscr{K}$ of compact operators, $\mathscr{M}$ of Montel operators (i.e. those operators that send bounded sets into relatively compact sets), LB of bounded operators or L of all operators and, if M and N are locally convex spaces, $\mathscr{A}_b(M,N)$ stands for the space of all operators $\Phi \in \mathscr{A}(M,N)$, endowed with the topology of uniform convergence on the bounded subsets of M.

Proposition 3.5. *If E satisfies (QNo)' and F is (QNo) then*

(a) *$\mathscr{A}_b(E,F)$ is (QNo),*

(b) *if the bounded subsets of $\mathscr{A}_b(F,E)$ are "localizable", i.e., the family of bounded sets $\{W(U,B)\,/\,U \in \mathscr{U}_0(F),\ B \in \mathscr{B}(E)\}$ is fundamental in $\mathscr{A}_b(F,E)$, then $\mathscr{A}_b(F,E)$ also satisfies (QNo)'.*

Proof. (a) Given $U \in \mathscr{U}_0\big(\mathscr{A}_b(E,F)\big)$, there are $U' \in \mathscr{U}_0(F)$ and $B \in \mathscr{B}(E)$ with $W(B,U') \subset U$. By hypothesis we can find $V' \in \mathscr{U}_0(F)$ and $C \in \mathscr{B}(E)$ ($V' \subset U'$, $B \subset C$) such that $\forall \varepsilon_1 > 0$, $\exists P_1 \in L(F,F)$, $\forall \varepsilon_2 > 0$, $\exists P_2 \in L(E,E)$ satisfying:

(1) $D := P_1(V') \in \mathscr{B}(F)$, $(I-P_1)(V') \subset \varepsilon_1 U'$,

(2) $\tilde{U} := P_2^{-1}(C) \in \mathscr{U}_0(E)$, $(I-P_2)(B) \subset \varepsilon_2 C$.

Given $\varepsilon > 0$, take $\varepsilon_1 := \frac{1}{2}\varepsilon$ and $\varepsilon_2 := \varepsilon/2M$, where $M > 0$ satisfies $D \subset MU'$, and define P: $\mathscr{A}_b(E,F) \to \mathscr{A}_b(E,F)$, $\Phi \mapsto P_1 \circ \Phi \circ P_2$. If $\Phi \in W(C,V')$, then

$$\text{(i)} \qquad \big[P(\Phi)\big](\tilde{U}) = (P_1 \circ \Phi \circ P_2)(\tilde{U}) \subset P_1(V') = D,$$

$$\begin{aligned}&\big[(I-P)(\Phi)\big](B)=(\Phi-P_1\circ\Phi\circ P_2)(B)\\ \text{(ii)}\quad&=\big[(I-P_1)\circ\Phi+P_1\circ\Phi\circ(I-P_2)\big](B)\subset(I-P_1)(V')+(P_1\circ\Phi)(\varepsilon_2C)\\&\subset\varepsilon_1U'+\frac{\varepsilon}{2M}P_1(V')\subset\frac{\varepsilon}{2}U'+\frac{\varepsilon}{2}\Big(\frac{1}{M}D\Big)\subset\varepsilon U'.\end{aligned}$$

Thus $P(\Phi)\in W(\tilde U,D)$ and $(I-P)(\Phi)\in\varepsilon U$. ▫

Corollary 3.6. (1) *If E satisfies (QNo)' then E'_b is (QNo).*

(2) *If E is (QNo) and quasibarrelled then E'_b satisfies (QNo)'.*

(3) *If E is a Fréchet space (respectively a quasibarrelled (DF)-space) (QNo) and F is (DF) (respectively Fréchet) with (QNo)', then $\mathscr{A}_b(E,F)$ satisfies (QNo)'.*

Proof. (1) and (2) are trivial. (3) is a consequence of 3.5 (b) and [20, Proposition 4]. ▫

As we have shown in 3.4 and 3.5, the classes (QNo) and (QNo)' constitute a good setting to get stability of the quasinormability and the strict Mackey condition by taking tensor products or spaces of continuous and linear mappings. But, how far are these classes from being optimal? Our next objective is to show that (QNo) and (QNo)' are, in certain sense, optimal. Our next lemma is an extension of [45, Proposition 1].

Lemma 3.7. *Let E be a quasinormable l.c.s. such that $E\otimes_\varepsilon C_2$ (respectively $L_b(C_2,E)$, $C_2\varepsilon E$) is quasinormable, then*

$\forall U\in\mathscr{U}_0(E)$ $\exists V\in\mathscr{U}_0(E)$ $\forall\varepsilon>0$ $\exists B\in\mathscr{B}(E)$ $\forall M\in FIN(E)$ $\exists P_M\in L(E,E)$:

(i) $P_M(M\cap V)\subset B$,

(ii) $(I-P_M)(M\cap V)\subset\varepsilon U$.

Proof. We are going to prove the result for the injective tensor product, the proof will be similar for $L_b(C_2,E)$ and $C_2\varepsilon E$. Setting $X:=C_2$, by hypothesis $E\otimes_\varepsilon X$ is quasinormable. Accordingly we have

$\forall U\in\mathscr{U}_0(E)$ $\exists V\in\mathscr{U}_0(E)$ $(V\subset U)$ $\forall\varepsilon>0$ $\exists B\in\mathscr{B}(E)$:

(1) $W(B_{X'},V)\subset W(B_{X'},B)+\varepsilon W(B_{X'},U)$.

Given $M\in FIN(E)$ we write $M=M'\oplus N$ with $N\subset\ker p_V$ and $M'\cap\ker p_V=\{0\}$. We select $k\in\mathbf{N}$ such that for the kth coordinate F_k of $X'=C_2$ we can find an isomorphism $T\colon F_k\to(M',p_V)$ satisfying $\|T\|\le1$ and $\|T^{-1}\|\le2$. We denote by $i_{M'}\colon(M',p_V)\to E$ the canonical inclusion (which is continuous since p_V is a norm on M') and we define $R\colon X'\to E$ by $R\big((x_n)_{n\in\mathbf{N}}\big):=i_{M'}(Tx_k)$. Clearly $R\in E\otimes X$ and $R\in W(B_{X'},V)$ since $\|T\|\le1$. By (1) we can find $S\colon X'\to E$ with finite rank such that $S\in W(B_{X'},B)$ and $R-S\in\varepsilon W(B_{X'},U)$.

Define $Q\colon M\to E$ by $Q(x+y):=S\big(j_k(T^{-1}(x))\big)$ for $x\in M'$, $y\in N$, where $j_k\colon F_k\to X'$ is the canonical inclusion. If $a\in M\cap V$ and $a=b+c$, $b\in M'$,

$c \in N \subset \ker p_V$, then $p_V(b) \leq 1$; hence $\|T^{-1}(b)\|_{F_k} \leq 2$ and $j_k\big(T^{-1}(b)\big) \in 2B_{X'}$. Accordingly $Q(M \cap V) \subset 2B$, since $S \in W(B_{X'}, B)$. On the other hand, if $x = x_1 + x_2 \in M \cap V$, $x_1 \in M'$, $x_2 \in N$, we get

$$x - Qx = x - S\big(j_k\big(T^{-1}(x_1)\big)\big) = (R - S)\big(j_k\big(T^{-1}(x_1)\big)\big) + x_2 \in 2\varepsilon U.$$

To conclude, if P_M is any continuous extension of Q, we have obtained (i) and (ii). □

Proposition 3.8. *Let E be a quasinormable l.c.s. such that $E \otimes_\varepsilon C_2$ (respectively $L_b(C_2, E)$, $C_2 \varepsilon E$) is quasinormable.*

(a) *If E is complemented in $(E'_b)'_e$ (e.g., if E is a dual space) then E is (QNo).*

(b) *If E is quasibarrelled then E'_b satisfies (QNo)'.*

Proof. By Lemma 3.7 we know that
$\forall U \in \mathscr{U}_0(E)\ \exists V \in \mathscr{U}_0(E)\ \forall \varepsilon > 0\ \exists B \in \mathscr{B}(E)\ \forall M \in FIN(E)\ \exists P_M \in L(E, E)$:

(i) $P_M(M \cap V) \subset B$,

(ii) $(I - P_M)(M \cap V) \subset \varepsilon U$.

Let $J := FIN(E)$ and consider $\mathscr{D}$ any ultrafilter on J containing the filter generated by the natural order of J. We define $P\colon E \to E''$ by setting $P(x) := \sigma(E'', E') - \lim_{\mathscr{D}} P_i(x)$, the limit taken for those $i \in J$ such that $x \in i$. Since B is $\sigma(E'', E')$-relatively compact in E'', $P(x)$ is a well defined element in E'', P is linear and (i) and (ii) above imply now

(1) $P(V) \subset B^{oo}$,

(2) $(i_E - P)(V) \subset \varepsilon U^{oo}$

(where $i_E\colon E \to E''$ is the canonical inclusion and the bipolars are taken in E'').

(a) If E is a complemented subspace of $(E'_b)'_e$, setting $\hat{P} := Q \circ P$ ($Q\colon (E'_b)'_e \to E$ is the projection), we obtain from (1) and (2) that

(I) $\hat{P}(V) \subset Q(B^{oo})$,

(II) $(I_E - \hat{P})(V) \subset \varepsilon Q(U^{oo})$,

which implies the quasinormability by operators in E.

(b) Let $P^t\colon E' \to E'$ be the transpose of P, then

(I)$'$ $P^t(B^o) \subset V^o$,

(II)$'$ $(I_{E'} - P^t)(U^o) \subset \varepsilon V^o$.

And, since E is quasibarrelled, E'_b satisfies (QNo)'. □

The classes (QNo) and (QNo)' are related to the (FG) and (DFG)-spaces of Bonet, Díaz and Taskinen [14] which were introduced to give positive answers to the problem of topologies of Grothendieck and other related dual questions. To establish the relation we first need the following lemmas. The first one is suggested by a characterization of the quasinormability due to J.C. Díaz (personal communication).

Lemma 3.9. (a) *A l.c.s. E is (QNo) if and only if there is a basis $\mathscr{U}$ of abx. 0-neighbourhoods in E such that*

$$\forall \lambda\colon \mathscr{U} \to \mathbf{R}_+ \setminus \{0\} \quad \forall U \in \mathscr{U} \quad \exists V \in \mathscr{U} \quad \exists P \in L(E,E):$$

(i)′ $P(V) \in \mathscr{B}(E)$,
(ii)′ $(I-P)\big(\lambda(V)V\big) \subset U$.

(b) *A l.c.s. G is (QNo)′ if and only if there is a fundamental system $\mathscr{B}$ of abx. bounded subsets in G such that*

$$\forall \alpha\colon \mathscr{B} \to \mathbf{R}_+ \setminus \{0\} \quad \forall B \in \mathscr{B} \quad \exists C \in \mathscr{B} \quad \exists P \in L(G,G):$$

(i)′ $P^{-1}(C) \in \mathscr{U}_0(G)$,
(ii)′ $(I-P)(B) \subset \alpha(C)C$.

Proof. We will only prove (a). If E is (QNo), then conditions (i)′ and (ii)′ in (a) are satisfied. For the converse, let us suppose that E is not (QNo). Since Definition 3.1 does not depend on the basis $\mathscr{U}$, we can obtain

$(*)$ $\exists U_0 \in \mathscr{U} \quad \forall V \in \mathscr{U} \quad \exists \lambda_V > 0 \quad \forall P \in L(E,E)$: (i) or (ii) of 3.1 are not satisfied.

We can define now $\lambda\colon \mathscr{U} \to \mathbf{R}_+ \setminus \{0\}$, $V \mapsto \lambda_V$. We find $V_0 \in \mathscr{U}$ and $P_0 \in L(E,E)$ such that (i)′ $P(V_0) \in \mathscr{B}(E)$, (ii)′ $(I-P)(\lambda_{V_0}V_0) \subset U_0$. This contradicts $(*)$. □

Lemma 3.10. (a) *If E is a Fréchet space then, for every mapping $\alpha\colon \mathscr{B}(E) \to \mathbf{R}^+ \setminus \{0\}$, there is a 0-neighbourhood U in E such that, for every $B \in \mathscr{B}(E)$, there is $C \in \mathscr{B}(E)$ satisfying $B \cap U \subset \alpha(C)C$.*

(b) *If G is a (gDF)-space then, for every mapping $\lambda\colon \mathscr{U}_0(G) \to \mathbf{R}^+ \setminus \{0\}$, there is a bounded subset B in G such that, for every $U \in \mathscr{U}_0(G)$, there is $V \in \mathscr{U}_0(G)$ satisfying $\lambda(V)V \subset B + U$.*

Proof. We are only going to prove (b). Let $(B_n)_{n\in\mathbf{N}}$ be a fundamental sequence of abx. bounded subsets in G. If (b) is not satisfied then there is $\lambda\colon \mathscr{U}_0(G) \to \mathbf{R}^+ \setminus \{0\}$ such that, for every $n \in \mathbf{N}$, there is $V_n \in \mathscr{U}_0(G)$ so that

$$(*) \qquad \lambda(U)U \not\subset B_n + V_n$$

for every $n \in \mathbf{N}$ and $U \in \mathscr{U}_0(G)$. Since G is (gDF), we can find $U \in \mathscr{U}_0(G)$ with $2U \subset \bigcap_{n\in\mathbf{N}}(B_n + V_n)$. Moreover G is quasinormable, then there is $V \in \mathscr{U}_0(G)$ and $m \in \mathbf{N}$ such that

$$\lambda(V)V \subset \tfrac{1}{2}B_m + U \subset B_m + V_m.$$

This contradicts $(*)$. □

Theorem 3.11. (a) *If E is a quasinormable (FG)-space (respectively a barrelled strong (DFG)-space which satisfies the strict Mackey condition) then E is (QNo) (respectively (QNo)').*

(b) *A Fréchet space (respectively (gDF)-space) E satisfies (QNo)' (respectively (QNo)) if and only if the space $LB(E,E)$ of bounded operators is dense in $L_b(E,E)$. In particular, every (FG)-space (respectively (DFG)-space) satisfies (QNo)' (respectively (QNo)).*

Proof. (a) If E is a quasinormable (FG)-space, let $(U_n)_{n\in\mathbf{N}}$ be a fundamental decreasing sequence of abx. and closed 0-neighbourhoods in E given by 1.1. Let $(\lambda_n)_{n\in\mathbf{N}}$ be any sequence of strictly positive scalars. Since E is quasinormable, we know that

$$\forall n\ \ \exists m(n)>n\ \ \exists B_n\in\mathscr{B}(E)\ \ (B_{n-1}\subset B_n):\quad \lambda_{m(n)}U_{m(n)}\subset B_n+U_n.$$

Let us consider a sequence $(M_n)_{n\in\mathbf{N}}$ of strictly positive scalars with $B_n\subset M_nU_n$, $\forall n\in\mathbf{N}$. Define $\alpha_k:=(2^kM_k)^{-1}$, $k\in\mathbf{N}$, and take $(P_k)_{k\in\mathbf{N}}\subset L(E,E)$ corresponding to $(\alpha_k)_{k\in\mathbf{N}}$ by 1.1. Since $\left\{\sum_s^\infty P_k\right\}_{s\in\mathbf{N}}$ is an equicontinuous subset of $L(E,E)$, then given $n\in\mathbf{N}$, there is $n'>n$ such that

$$\Big(\sum_s^\infty P_k\Big)(U_{n'})\subset \tfrac12 U_n\qquad \forall s\in\mathbf{N}.$$

Thus

$$\begin{aligned}\Big(\sum_{m(n')+1}^\infty P_k\Big)(\lambda_{m(n')}U_{m(n')})&\subset\Big(\sum_{m(n')+1}^\infty P_k\Big)(B_{n'}+U_{n'})\\ &\subset \tfrac12U_n+\Big(\sum_{m(n')+1}^\infty P_k\Big)(B_{m(n')})\subset \tfrac12U_n+\Big(\sum_{m(n')+1}^\infty P_k(M_kU_k)\Big)\\ &\subset \tfrac12U_n+\sum_{m(n')}^\infty \frac{1}{2^{k+1}}U_k\subset U_n.\end{aligned}$$

On the other hand

$$\begin{aligned}\Big(I-\Big(\sum_{m(n')+1}^\infty P_k\Big)\Big)(\lambda_{m(n')}U_{m(n')})&=\Big(\sum_1^{m(n')}P_k\Big)(\lambda_{m(n')}U_{m(n')})\\ &\subset\lambda_{m(n')}\Big(\sum_1^{m(n')}P_k(U_k)\Big)\in\mathscr{B}(E)\ \ (\text{since}\ \ P_k(U_k)\in\mathscr{B}(E)\ \ \forall k\in\mathbf{N}).\end{aligned}$$

Therefore, if we set $P := \sum_1^{m(n')} P_k$, we obtain conditions (i)$'$ and (ii)$'$ of Lemma 3.9 (a), concluding that E is (QNo).

If E is a barrelled strong (DFG)-space which satisfies the strict Mackey condition, let $(B_n)_{n\in\mathbf{N}}$ be a decreasing fundamental sequence of abx. closed bounded subsets of E given by 1.2. Let $(\lambda_n)_{n\in\mathbf{N}}$ be a sequence of strictly positive scalars. Since E'_b is quasinormable, then

$$\forall n \in \mathbf{N} \;\; \exists m(n) > n \;\; \exists U_n \in \mathscr{U}_0(E) \;\; (U_n \subset U_{n-1}): \quad \lambda_{m(n)} B^o_{m(n)} \subset U^o_n + B^o_n.$$

Let us consider a sequence $(M_n)_{n\in\mathbf{N}}$ of strictly positive scalars with $U^o_n \subset M_n B^o_n$, $n \in \mathbf{N}$. Define $\alpha_k := (2^k M_k)^{-1}$, $k \in \mathbf{N}$, and take $(Q_k)_{k\in\mathbf{N}} \subset L(E,E)$ corresponding to $(\alpha_k)_{k\in\mathbf{N}}$ by 1.2. Since E is barrelled $\left\{\sum_{j=n}^m Q_j;\ n,m \in \mathbf{N}\right\}$ is an equicontinuous subset of $L(E,E)$, hence $\left\{\sum_{j=n}^m Q'_j;\ n,m \in \mathbf{N}\right\}$ is an equicontinuous subset of $L(E'_b,E'_b)$. Moreover $\left\{\sum_{j=1}^m Q'_j\right\}_{m\in\mathbf{N}}$ converges pointwise, with respect to $\sigma(E',E)$, to the identity map. On the other hand, by polarity, $(Q'_k)_{k\in\mathbf{N}}$ satisfies (FG2) and (FG3) of 1.1. Now, following the argument above, we can see that

$\forall n \in \mathbf{N} \;\; \exists n' > n$ such that $P := \sum_{k=1}^{m(n')} Q'_k$ satisfies

(1) $P(B^o_{m(n')}) \in \mathscr{B}(E'_b)$,

(2) $(I-P)(\lambda_{m(n')} B^o_{m(n')}) \subset B^o_n$.

This implies that $Q := P'$ satisfies (i)$'$ and (ii)$'$ of 3.9 (b).

(b) Let E be a Fréchet space which satisfies (QNo)'. Let $B \in \mathscr{B}(E)$ and $U \in \mathscr{U}_0(E)$ be arbitrary. We can find $C \in \mathscr{B}(E)$ and a bounded operator P such that $(I-P)(B) \subset \varepsilon C \subset U$ for suitable $\varepsilon > 0$. Then $LB(E,E)$ is dense in $L_b(E,E)$. Conversely, if $LB(E,E)$ is dense in $L_b(E,E)$, given α: $\mathscr{B}(E) \to \mathbf{R}^+ \setminus \{0\}$ we find $U \in \mathscr{U}_0(E)$ as in Lemma 3.10 (a). Given $B \in \mathscr{B}(E)$, we take $P \in LB(E,E)$ and $B_1 \in \mathscr{B}(E)$ such that $(I-P)(B) \subset U$ and $(I-P)(B) \subset B_1$. By 3.10 (a), there is $B_2 \in \mathscr{B}(E)$ (with $P^{-1}(B_2) \in \mathscr{U}_0(E)$) such that

$$(I-P)(B) \subset U \cap B_1 \subset \alpha(B_2) B_2.$$

This condition, together with $P^{-1}(B_2) \in \mathscr{U}_0(E)$, implies (i)$'$ and (ii)$'$ of Lemma 3.9 (b), that is, E satisfies (QNo)'. Analogously, if E is a (gDF)-space, then $LB(E,E)$ is dense in $L_b(E,E)$ if and only if E is quasinormable by operators.

Finally, if E is an (FG)-space or a (DFG)-space, it is easy to see that $LB(E,E)$ is dense in $L_b(E,E)$.

Indeed, in the second case, given $B \in \mathscr{B}(E)$ and $U \in \mathscr{U}_0(E)$ which is s-closed (s the topology given in 1.2), we take $(\alpha_k)_{k\in\mathbf{N}}$ such that $\alpha_k B_k \subset 2^{-k} U$, $k \in \mathbf{N}$. With $(\alpha_k)_{k\in\mathbf{N}}$ we find $(Q_k)_{k\in\mathbf{N}}$ as in 1.2. By [14, Lemma 2.6], $V := \bigcap_{k\in\mathbf{N}} \alpha_k Q_k^{-1}(B_k) \in \mathscr{U}_0(E)$. Let $\varepsilon > 0$ with $\varepsilon B \subset V$ and $n \in \mathbf{N}$ satisfying

$2^{-n} < \varepsilon$. Define $P := \sum_{k=1}^{n} Q_k \in LB(E,E)$. If $x \in B$ then $(I-P)(x) = \sum_{k=n+1}^{\infty} Q_k(x)$, but

$$\sum_{k=n+1}^{l} Q_k(x) \in \varepsilon^{-1} \sum_{k=n+1}^{l} \alpha_k B_k \subset \varepsilon^{-1} \sum_{k=n+1}^{l} 2^{-k} U \subset U.$$

The series is s-convergent and U is s-closed, this yields $(I-P)(x) \in U$, which shows that $LB(E,E)$ is dense in $L_b(E,E)$. ◻

The last result provides a "good" collection of spaces (QNo) and (QNo)', it suffices to see the examples of Bonet, Díaz and Taskinen of (FG)-spaces and (DFG)-spaces [14]. For instance, the following spaces are (QNo) (respectively satisfy (QNo)'):

The quasinormable Fréchet spaces (respectively Fréchet spaces) which are:

- Banach valued Köthe echelon spaces of order p, $\lambda_p\big(A,(X_i)_{i\in\mathbf{N}}\big)$, $1 \le p < \infty$, $p = 0$,
- spaces of measurable functions introduced by Reiher $L_\varrho(A)$ (see [46]) with absolutely continuous ϱ (in particular the spaces $L_p((\mu_n)_{n\in\mathbf{N}})$ of Grothendieck, $1 \le p < \infty$, where μ_n are σ-finite measures),
- weighted spaces of continuous functions $CA_0(X)$,

and the (DF)-spaces (respectively (DF)-spaces with the strict Mackey condition) which are

- strong duals of FG-spaces,
- the weighted inductive limits of spaces of continuous functions $\operatorname{ind} C(v_n)_0(X)$, and the projective hulls $C\overline{V}(X)$, $C\overline{V}_0(X)$ of the weighted inductive limits of spaces of continuous functions.

We want to present more examples of spaces in the classes (QNo) and (QNo)', but mainly concentrating in the class (QNo).

If X is a completely regular topological space and E is a l.c.s., we will denote by $C_c(X,E)$ (respectively $C_{cm}(X,E)$) the space of all continuous functions from X into E endowed with the compact-open topology (respectively the topology of uniform convergence on metrizable compact subsets of X).

Proposition 3.12. (a) *If X is locally compact and E is quasinormable by operators then $C_c(X,E)$ is also (QNo).*

(b) *If E is (QNo) then $C_{cm}(X,E)$ is also (QNo).*

Proof. (a) A basis of abx. 0-neighbourhoods in $C_c(X,E)$ is given by

$$\{W(K,U) \;/\; K \text{ compact in } X \text{ and } U \in \mathscr{U}_0(E)\},$$

where $W(K,U) := \{f \in C(X,E) \,/\, f(K) \subset U\}$. Given $U \in \mathscr{U}_0(E)$ and K compact in X, we find $V \in \mathscr{U}_0(E)$ according to Definition 3.1 and K' compact in X with $K \subset \mathring{K'}$.

Given $\varepsilon > 0$ there is $P' \in L(E,E)$ which satisfies (i) and (ii) of 3.1. Take $\varphi \in C(X)$ such that the support of φ is contained in K', φ is equal to 1 in K and $0 \leq \varphi \leq 1$.

Setting $P \in \mathscr{L}\big(C_c(X,E), C_c(X,E)\big)$, $P(f) := P'(\varphi \cdot f)$ $\forall f \in C(X,E)$ we will prove

(i) $P\big(W(K',V)\big) \in \mathscr{B}\big(C_c(X,E)\big)$.

Given K'' compact in X and $U' \in \mathscr{U}_0(E)$, take $\lambda > 0$ such that $P'(V) \subset \lambda U'$. If $f \in W(K',V)$ then $\varphi \cdot f \in W(K'',V)$ by definition of φ; this implies

$$P(f) = P'(\varphi \cdot f) \in W\big(K'', P'(V)\big) \subset \lambda W(K'', U').$$

We also show

(ii) $(I-P)\big(W(K',V)\big) \subset 2\varepsilon W(K,U)$.

Let $f \in M(K',U)$, then $f - P(f) = (f - \varphi \cdot f) + \big(\varphi \cdot f - P'(\varphi \cdot f)\big)$ and $\varphi \cdot f - P'(\varphi \cdot f) \in \varepsilon W(K,U)$ (since $\varphi \cdot f \in W(K,V)$ and $(I-P')(V) \subset \varepsilon U$).

On the other hand $\varphi \equiv 1$ on K, then $f - \varphi \cdot f \in \varepsilon W(K,U)$ concluding the result.

(b) Let K' be a metrizable compact subset of X. By the Borsuk–Dugundji Theorem there exists a continuous extension operator $P' \in L\big(C(K',E), C(X,E)\big)$ such that, if $f \in C(K',E)$, then $P'(f)(X) \subset \Gamma\big(f(K')\big)$.

Let $U \in \mathscr{U}_0(E)$ and consider the basis of 0-neighbourhoods in $C_{cm}(X,E)$ given by $\{W(K,V)/K$ compact and metrizable in X, $V \in \mathscr{U}_0(E)\}$. Now define $P''(f) := P'(f\,|_{K'})$, $f \in C(X,E)$. P'' is linear and, since E is (QNo), there is $V \in \mathscr{U}_0(E)$ such that $\forall \varepsilon > 0$, $\exists P''' \in L(E,E)$ satisfying (i) and (ii) of Definition 3.1. Thus, if we set $P(f)(x) := P'''\big(P''(f)(x)\big)$, $x \in X$, $f \in C(X,E)$, it easily follows that P is linear and

(i) if $f \in W(K',V) \Rightarrow f(K') \subset V \Rightarrow P''(f)(X) \subset V$, then

$$P'''\big(P''(f)(X)\big) \subset P'''(V) \in \mathscr{B}(E),$$

consequently $P\big(W(K',V)\big) \in \mathscr{B}\big(C_{cm}(X,E)\big)$;

(ii) if $f \in W(K',V)$ and $x \in K'$ then

$$(I-P)(f)(x) = f(x) - P'''\big(P''(f)(x)\big) = f(x) - P'''\big(f(x)\big) \in \varepsilon U.$$

This implies $(I-P)\big(W(K',V)\big) \subset \varepsilon W(K',U)$. We obtain that $C_{cm}(X,E)$ is (QNo). □

Domański showed (see [26, Proposition 2.2] and [27, Corollary 4.2]) that for completely regular spaces X such that $C_c(X)$ is Fréchet and, moreover, X is locally compact or $C_c(X)$ is separable then $C_c(X)$ is isomorphic to a complemented subspace of a product of Banach spaces. In the second case, assuming a little bit more, Valdivia even obtained $C_c(X) \cong C_c([0,1])^{\mathbf{N}}$ [51, III.3.6]. We will later see an example of a Fréchet space $C(X)$ which is not (QNo).

For weighted spaces of continuous functions we refer to [8]. For Köthe sets we refer to [7].

Let X be a completely regular space and let V be a system of weights in X. F. Bastin and B. Ernst showed [2] that if $CV(X)$ is quasinormable then V satisfies condition (Q'''):

$\forall u \in V$, $\exists v \in V$ $(v \geq u)$ such that $\forall \varepsilon > 0$, $\exists \beta \in F(X, [0, +\infty[)$, β upper semicontinuous and bounded on V (i.e. $\sup_{x \in X} \beta(x)v(x) < \infty$, $\forall v \in V$) such that:

$$\frac{1}{v} \leq \beta + \frac{\varepsilon}{u}.$$

The relation between the quasinormability by operators and condition (Q''') is given in the following

Proposition 3.13. (a) *If the family of weights V has the continuous domination property (i.e. $\forall v \in V$, $\exists w \in V$ such that $w \geq v$ and w is continuous), satisfies (Q''') and $CV(X)$ has a fundamental family of bounded subsets generated by continuous functions, then $CV(X, E)$ is (QNo) for every l.c.s. E (QNo).*

(b) *If $\mathscr{P}$ is a Köthe set on I and $1 \leq p < \infty$, then $\lambda_p(I, \mathscr{P})$ is quasinormable if and only if it is quasinormable by operators.*

Proof. (a) By hypothesis $\forall u \in V \cap C(X)$, $\exists v \in V \cap C(X)$ $(v \geq u)$ such that $\forall \varepsilon > 0$, $\exists b \in C(X, [0, +\infty[)$ with b bounded on V satisfying

$$\frac{1}{v} \leq b + \frac{\varepsilon}{u}.$$

On the other hand, since E is (QNo), given $U \in \mathscr{U}_0(E)$, there are $V \in \mathscr{U}_0(E)$ and $P' \in L(E, E)$ satisfying (i) and (ii) of Definition 3.1 for some $\varepsilon' > 0$ with $\varepsilon' b(x) \leq \varepsilon / u(x)$ $\forall x \in X$.

Let $P(f)(x) := (bu)(x)/\big((bu)(x) + \varepsilon\big) \cdot P'\big(f(x)\big)$, $\forall f \in CV(X, E)$, $\forall x \in X$.

If $f \in \mathscr{U}_{(v,V)} := \big\{h \in CV(X, E) \,/\, h(x)v(x) \in V \ \forall x \in X\big\}$ then:

(i) Given $w \in V$ we get

$$\begin{aligned} w(x)P(f)(x) &= w(x)\frac{(bu)(x)}{(bu)(x) + \varepsilon}P'(f(x)) \in w(x)\frac{(bu)(x)}{(bu)(x) + \varepsilon}v(x)^{-1}P'(V) \\ &\subset w(x)b(x)P'(V) \subset \lambda P'(V) \in \mathscr{B}(E) \qquad \forall x \in X, \end{aligned}$$

where $\lambda > 0$ satisfies $w(x)b(x) \leq \lambda$ $\forall x \in X$. Therefore $P(\mathscr{U}_{(v,V)}) \in \mathscr{B}\big(CV(X, E)\big)$.

(ii) $(I - P)(\mathscr{U}_{(v,V)}) \subset 2\varepsilon \mathscr{U}_{(u,U)}$:

$$\begin{aligned} f(x) - P(f)(x) &= \left(f(x) - \frac{(bu)(x)}{(bu)(x) + \varepsilon}f(x)\right) + \left(\frac{(bu)(x)}{(bu)(x) + \varepsilon}f(x) - P(f)(x)\right) \\ &= \frac{\varepsilon}{(bu)(x) + \varepsilon}f(x) + \frac{(bu)(x)}{(bu)(x) + \varepsilon}(I - P')\big(f(x)\big) \\ &\in \varepsilon u^{-1}(x)V + b(x)\varepsilon' U \subset 2\varepsilon u^{-1}(x)U, \qquad \forall x \in X. \end{aligned}$$

(b) If $\lambda_p(I,\mathscr{P})$ is quasinormable then $\mathscr{P}$ satisfies condition (Q''') and, following the argument of (a) for $E = \mathbf{K}$, we conclude that $\lambda_p(I,\mathscr{P})$ is (QNo), since a basis of 0-neighbourhoods is given by:

$$U_\alpha := \left\{ x = (x_i)_{i\in I} \in \lambda_p(I,\mathscr{P}) \,/\, q_\alpha(x) := \left(\sum_{i\in I} |\alpha(i)x(i)|^p \right)^{1/p} \le 1 \right\}, \qquad \alpha \in \mathscr{P}$$

and a fundamental family of bounded subsets in $\lambda_p(I,\mathscr{P})$ is given by:

$$B_b := \left\{ x = (x_i)_{i\in I} \in \lambda_p(I,\mathscr{P}) \,/\, x(i) = b(i)z(i),\ i \in I : \sum_{i\in I} |z(i)|^p \le 1 \right\}, \quad b \in \mathscr{B}$$

where $\mathscr{B}$ is the family of all positive functions bounded on $\mathscr{P}$. □

In the above proposition, if $E = \mathbf{K}$ and L is a solid subspace of $CV(X)$ (respectively $\lambda_p(I,\mathscr{P})$), that is

$$f \in L, \qquad g : |g| \le |f| \implies g \in L,$$

then the spaces L and $CV(X)/L$ (respectively $\lambda_p(I,\mathscr{P})/L$) are also (QNo), since L is invariant by the operators defined. For instance this happens for the space $CV_0(X)$.

Part (b) of the following theorem was suggested by K.D. Bierstedt.

Theorem 3.14. *Let E be a Schwartz space which satisfies*
$(*)$ *$\forall U \in \mathscr{U}_0(E)$, $\exists V \in \mathscr{U}_0(E)$ such that there is a sequence of operators*

$$\{P_n\}_{n\in\mathbf{N}} \in \mathscr{F}(\widehat{E_V}, \widehat{E_U})$$

which converges, in the operator norm, to the canonical map $\Phi_{U,V}\colon \widehat{E_V} \to \widehat{E_U}$. Then E is quasinormable by operators.

In particular, $(*)$ is satisfied in the following cases:

(a) *E is a Schwartz space with the bounded approximation property.*
(b) *E is a Schwartz space such that $\forall U \in \mathscr{U}_0(E)$, $\exists V \in \mathscr{U}_0(E)$ such that $\Phi_{U,V}$ is compact and admits a (continuous) factorization through a l.c.s. with the approximation property.*

Proof. (Compare with [5, Theorem 1].) If E is a Schwartz space such that, for every $U \in \mathscr{U}_0(E)$, $\exists V \in \mathscr{U}_0(E)$ satisfying $(*)$, then, given $\varepsilon > 0$, $\exists n_0 \in \mathbf{N}$ with

$$(\Phi_{U,V} - P_{n_0})(\tilde{V}) \subset \frac{\varepsilon}{2}\tilde{U}$$

where $\tilde{V}$ and $\tilde{U}$ are the corresponding unit balls of $\widehat{E_V}$ and $\widehat{E_U}$.

We can write $P_{n_0} = \sum_1^m e_i \otimes f_i$ with $e_i \in \tilde{V}^o$ and $f_i \in \tilde{U}$, $i = 1 \cdots m$. Let $h_i \in E$ such that $\Phi_U(h_i) - f_i \in (\varepsilon/2m)\tilde{U}$, $i = 1 \cdots m$. Then $Q_{n_0} := \sum_1^m e_i \otimes h_i \in L(\widehat{E_V}, E)$. Let $P := Q_{n_0} \circ \Phi_V$, therefore:

(i) If $x \in V$

$$P(x) = Q_{n_0}(\Phi_V(x)) \in m\Gamma\Big(\bigcup_1^m \{h_i\}\Big) \in \mathscr{B}(E).$$

On the other hand:

(ii) If $x \in V$

$$\begin{aligned}\Phi_U \circ (I - P)(x) &= \Phi_U(x) - \sum_1^m e_i\big(\Phi_V(x)\big)\Phi_U(h_i)\\ &= (\Phi_{U,V} - P_{n_0}) \circ \Phi_V(x) + \sum_1^m e_i\big(\Phi_V(x)\big)\big(f_i - \Phi_U(h_i)\big)\\ &\in \frac{\varepsilon}{2}\tilde{U} + \frac{\varepsilon}{2}\tilde{U} = \varepsilon\tilde{U}.\end{aligned}$$

From (i) and (ii) we obtain that E is (QNo).

(a) If E is a Schwartz space with the bounded approximation property then there is an equicontinuous net $\{P_t\}_{t\in T}$ of finite rank operators in E which converges pointwise to the identity of E.

Let $U \in \mathscr{U}_0(E)$ and find $W \in \mathscr{U}_0(E)$ such that $(I - P_t)(W) \subset U$, $\forall t \in T$. Since E is Schwartz, there is $V \in \mathscr{U}_0(E)$ which satisfies

$$\forall \varepsilon > 0 \ \ \exists F_\varepsilon \subset E \ \text{ finite}: \qquad V \subset F_\varepsilon + \varepsilon W.$$

Now, given $n \in \mathbf{N}$, $\exists t_n \in T$ with

$$(I - P_{t_n})(F_{1/2n}) \subset \frac{1}{2n}U.$$

Thus

$$(I - P_{t_n})(V) \subset (I - P_{t_n})\Big(F_{1/2n} + \frac{1}{2n}W\Big) \subset \frac{1}{n}U,$$

and $(*)$ is verified.

(b) Let $U \in \mathscr{U}_0(E)$. By hypothesis we can find $W, V \in \mathscr{U}_0(E)$, $(V \subset W \subset U)$, such that $\Phi_{U,W}$ and $\Phi_{W,V}$ are compact and admit a factorization through a l.c.s. with the approximation property. Let F be a l.c.s. with the approximation property, $A \in L(\widehat{E_W}, F)$ and $B \in L(F, \widehat{E_U})$ such that $B \circ A = \Phi_{U,W}$. $K := (A \circ \Phi_{W,V})(\tilde{V})$ is a precompact subset of F, hence there are operators Q_n in F, $n \in \mathbf{N}$, of finite rank such that $Q_n(K) \subset (1/n)B^{-1}(\tilde{U})$.

Define $P_n := B \circ Q_n \circ A \circ \Phi_{W,V}$ to obtain

$$P_n(\tilde{V}) \subset \frac{1}{n}\tilde{U} \qquad \forall n \in \mathbf{N},$$

and conclude property $(*)$. ▫

Remarks. (1) Every nuclear l.c.s. is (QNo), since property (b) of Theorem 3.14 is satisfied.

(2) The above theorem extends a result of Nelimarkka [41, Theorem 15, Proposition 7], from Fréchet Schwartz spaces with the bounded approximation property to Schwartz spaces with the bounded approximation property.

We can characterize the property (QNo) in the context of the standard quojections of Moscatelli type. We first need the following

Lemma 3.15. *Let X and Y be Banach spaces with Y topological subspace of X and suppose that there is $P \in L(X,X)$ such that:*

(i) $Y \subset \ker P$,

(ii) $(I-P)(B_X) \subset \frac{1}{2}B_X + Y$.

Then Y is a complemented subspace of X.

Proof. Let $Q := I - P$, then $Q(y) = y \ \forall y \in Y$. We fix $x \in B_X$ and define $x_n := Q^n(x)$, $n \in \mathbf{N}$. The sequence $\{x_n\}_{n\in\mathbf{N}}$ converges:

$Q(x) = x_1 = \frac{1}{2}z_1 + y_1$, with $y_1 \in Y$, $z_1 \in B_X$; $Q(z_1) = \frac{1}{2}z_2 + y_2$, with $z_2 \in B_X$, $y_2 \in Y$. Proceeding by induction we find sequences $\{z_n\}_{n\in\mathbf{N}}$, $\{y_n\}_{n\in\mathbf{N}}$ with $z_n \in B_X$ and $y_n \in Y$, $n \in \mathbf{N}$, such that:

(a) $Q(z_n) = \frac{1}{2}z_{n+1} + y_{n+1}$,

(b) $x_n = z_n/2^n + \sum_{k=1}^n y_k/2^{k-1}$.

On the other hand, there is $\lambda > 0$ satisfying $\|y_n\| \le \lambda \ \forall n \in \mathbf{N}$, since

$$y_n = -\tfrac{1}{2}z_n + Q(z_{n-1}) \in (\tfrac{1}{2} + \|Q\|)B_X \qquad \forall n \in \mathbf{N}.$$

Fix $m \in \mathbf{N}$. If $p > m$ we get:

$$\|x_p - x_m\| = \left\| \frac{1}{2^p}z_p - \frac{1}{2^m}z_m + \sum_{k=m}^{p-1} \frac{1}{2^k}y_{k+1} \right\| \le \frac{1}{2^p} + \frac{1}{2^m} + \frac{\lambda}{2^{m-1}}.$$

This implies that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence.

Define $R(x) := \lim_n Q^n(x) \ \forall x \in X$, then $R \in L(X,X)$ and $R(X) \subset Y$, since $Q^n(x) \in (2^{-n}B_X) + Y$, $\forall n \in \mathbf{N}$, $\forall x \in B_X$.

Finally we only have to note that $y = Q^n(y)$, $\forall n \in \mathbf{N}$, $\forall y \in Y$. □

Proposition 3.16. *Let λ be a normal Banach sequence space, X Banach and Y a closed subspace of X. The standard quojection of Moscatelli type $E := \lambda(X, X/Y)$ is quasinormable by operators if and only if it is a complemented subspace of a countable product of Banach spaces.*

Proof. We can consider the canonical basis of 0-neighbourhoods in E $(W_k)_{k\in\mathbf{N}}$ defined in 2.2. If E is (QNo), then there are $k \in \mathbf{N}$ and $P \in L(E,E)$ such that

(i) $P(W_k) \in \mathscr{B}(E)$,

(ii) $(I-P)(W_k) \subset \frac{1}{2}W_1$.

We set $P_n := q_n \circ P \circ i_n$, where $i_n\colon F_n \to E$ and $q_n\colon E \to F_n$ are the respective inclusion and projection, for each $n \in \mathbf{N}$. Therefore, if $n > k$:

(a) $P_n(B_X + Y) \in \mathscr{B}(X)$,

(b) $(I_X - P_n)(B_X) = (q_n \circ i_n - P_n)(B_X) \subset q_n(\frac{1}{2}W_1) = \frac{1}{2}B_X + Y$.

From (a) we obtain that $Y \subset \ker P_n$ and, by Lemma 3.15, we conclude that Y is a complemented subspace of X. This yields the result. □

In contrast with 3.12, one can use an example due to Domański (see [26]) to obtain completely regular spaces X such that $C_c(X)$ is a standard quojection of Moscatelli type not isomorphic to a complemented subspace of a countable product of Banach spaces and, in consequence, it is not (QNo).

Finally we want also to give examples of "classical" spaces of vector valued differentiable and holomorphic mappings which are quasinormable by operators.

For the notation we refer to Schwartz [47].

Proposition 3.17. *Let E be a complete l.c.s. which is (QNo) and Ω an open subset of $\mathbf{R}^N$. The following spaces are quasinormable by operators as a consequence of the preceding results:*

(i) *$C^k(\Omega, E) \cong C^k([0,1]^N, E)^{\mathbf{N}}$ and $\mathscr{D}^k(\Omega, E) \cong C^k([0,1]^N, E)^{\mathbf{N}}$ for $k \in \mathbf{N}$. (See [10] for the representations and use that $C^k([0,1]^N, E) \cong C^k([0,1]^N)\hat{\otimes}_\varepsilon E$.)*

(ii) *$C^\infty(\Omega, E) \cong s(E)^{\mathbf{N}}$ and $\mathscr{D}(\Omega, E) \cong s(E)^{(\mathbf{N})}$. (We refer again to [10] for the representations.)*

(iii) *$\mathscr{B}_0(\Omega, E)$ and $\mathscr{B}_1(\Omega, E)$, which are isomorphic to $\lambda_0\big(s(E)\big)$ and $\lambda_\infty\big(s(E)\big)$, respectively (see [17]).*

(iv) *$\mathscr{D}_{L^p}(E) \cong l_p(s(E))$, $1 \le p < \infty$ (cf. [9]).*

(v) *If Ω is an open subset of $\mathbf{C}^N$, the space $\mathscr{H}(\Omega, E) = \mathscr{H}(\Omega)\hat{\otimes}_\varepsilon E$, endowed with the compact open topology is (QNo). (See [34] for the representation.)*

Proposition 3.18. *Let G be an open balanced subset of $\mathbf{C}^N$ and $v\colon G \to \mathbf{R}$ a radial weight in G (i.e. $v(\lambda z) = v(z)$ if $|\lambda| = 1$) such that $\mathscr{H}_{v_0}(G)$ contains the polynomials. Let E be a l.c.s. (QNo). The spaces $\mathscr{H}_{v_0}(G, E) \cong \mathscr{H}_{v_0}(G)\hat{\otimes}_\varepsilon E$ and $\mathscr{H}_v(G, E) \cong L_b\big(\mathscr{H}_{v_0}(G)'_b, E\big)$ are quasinormable by operators.*

For more information about the isomorphisms and notation above we refer to [4]. This result has some consequences on the quasinormability of the inductive limits $\mathscr{V}\mathscr{H}(G, E)$ and $\mathscr{V}_0\mathscr{H}(G, E)$ if E is a l.c.s. (QNo).

Note that if $N > 1$ and $k \in \mathbf{N}$, the space $C^k([0,1]^N)$ is not an $\mathscr{L}_\infty$-space as several authors showed (see [36]). There are also examples due to Kaballo of radial weights v on the disk $G = \mathbf{D}$ of $\mathbf{C}$ such that $\mathscr{H}_{v_0}(G)$ and $\mathscr{H}_v(G)$ are not $\mathscr{L}_\infty$-spaces (see [4]). Consequently, we can apply Theorem 2.3 and the remark after 2.3 to obtain examples of quasinormable Fréchet or (DF)-spaces E such that $C^k(\Omega, E)$, $k \in \mathbf{N}$, $\Omega \subset \mathbf{R}^N$ open, $N > 1$; $\mathscr{H}_{v_0}(G, E)$ and $\mathscr{H}_v(G, E)$ are not quasinormable. This should be compared with the examples above.

We conclude this section with other examples of spaces quasinormable by operators: the local spaces of Hörmander [33].

Let Ω be an open subset of $\mathbf{R}^N$. Let $\mathscr{F}$ be a semi-local linear subspace of $\mathscr{D}'(\Omega)$, i.e. $\varphi u \in \mathscr{F}$ for every $\varphi \in \mathscr{D}(\Omega)$, $u \in \mathscr{F}$. The local hull of $\mathscr{F}$ is defined as

$$\mathscr{F}^{\mathrm{loc}} := \big\{u \in \mathscr{D}'(\Omega) \,/\, \varphi u \in \mathscr{F} \ \ \forall \varphi \in \mathscr{D}(\Omega)\big\}.$$

Let us suppose that $\mathscr{F}$ is a normed space with a norm $\|\cdot\|$. We can consider the locally convex topology in $\mathscr{F}^{\mathrm{loc}}$ defined by the system of seminorms

$$\mathscr{P} := \big\{p_\varphi \,/\, p_\varphi(u) := \|\varphi u\| \ \ \varphi \in \mathscr{D}(\Omega), \ \ u \in \mathscr{F}^{\mathrm{loc}}\big\}.$$

We also assume that $\mathscr{F}^{\mathrm{loc}}$ is a Fréchet space.

Proposition 3.19. *$\mathscr{F}^{\mathrm{loc}}$ is isomorphic to a complemented subspace of a countable product of Banach spaces and, as a consequence, it is quasinormable by operators.*

Proof. We are done if we verify condition (c) of [27, Proposition 4.1]. To do this, we consider an increasing sequence of compact subsets $\{K_n\}_{n\in\mathbf{N}}$ of Ω such that $K_n \subset \mathring{K}_{n+1}$, $n \in \mathbf{N}$, and a sequence $\{\varphi_n\}_{n\in\mathbf{N}}$ in $\mathscr{D}(\Omega)$ such that $\varphi_{n+1} \equiv 1$ on K_n and $\operatorname{supp}\varphi_{n+1} \subset K_{n+1}$, $n \in \mathbf{N}$. The corresponding basis of 0-neighbourhoods in $E := \mathscr{F}^{\mathrm{loc}}$ given by

$$U_n := \big\{u \in E \,/\, p_n(u) := \|\varphi_n u\| \le 1/n\big\}, \qquad n \in \mathbf{N}.$$

We prove that there is an operator $r_n\colon E_{U_{n+1}} \to E$ such that $\Phi_n \circ r_n = \Phi_{n,n+1}$, where $\Phi_n\colon E \to E_{U_n}$ and $\Phi_{n,n+1}\colon E_{U_{n+1}} \to E_{U_n}$ are the canonical maps.

Let $n \in \mathbf{N}$ and define $P\colon E \to E$, $u \mapsto \varphi_{n+1} u$.

(i) $P(U_{n+1}) \in \mathscr{B}(E)$:

If $u \in U_{n+1}$ and $m > n+1$ then $p_m(P(u)) = \|\varphi_m \varphi_{n+1} u\| = \|\varphi_{n+1} u\| \le 1/n$.

(ii) $(I-P)(U_{n+1}) \subset \ker p_n$:

If $u \in U_{n+1}$, $p_n(u - P(u)) = \|\varphi_n(u - \varphi_{n+1}u)\| = \|\varphi_n u - \varphi_n u\| = 0$.

We conclude the result defining r_n as the canonical operator induced by P. ▫

The spaces $L^p_{\mathrm{loc}}(\Omega)$ and $B^{\mathrm{loc}}_{p,k}(\Omega)$ (see also [52]) are concrete examples of spaces which satisfy the hypotheses of Proposition 3.19. See the definition in [33].

More examples of spaces which are (QNo) (respectively (QNo)') or which are not will be provided later on.

4. The problem of topologies of Grothendieck

Grothendieck [30, 31] posed the following questions:

(a) Let E and F be Fréchet spaces. Is every bounded subset B of the projective tensor product $E\hat{\otimes}_\pi F$ *localizable*, i.e. are there bounded subsets C, D of E and F, respectively, with $B \subset \overline{\Gamma(C \otimes D)}$? This is the so-called problem of topologies of Grothendieck (see [31, question non resolue no. 2]).

(b) Let G and H be (DF)-spaces. Is $G \otimes_\varepsilon H$ a (DF)-space? (See [31, question non resolue no. 10]).

(c) Let E be a Fréchet space and let G be a (DF)-space. Is $L_b(E, G)$ a (DF)-space? (See [30, question non resolue no. 7]).

Taskinen showed (see [48] and [50]) that the answer to these problems is negative. Partial positive answers were given in [13], [14], [48] and [49]. The purpose of this section is to show that the answer to the problems of Grothendieck is also positive in the context of the classes (QNo) and (QNo)'. Moreover it is possible to establish an equivalence between the problem of the stability of the quasinormability by the injective tensor product in the context of Fréchet spaces and the problem of topologies of Grothendieck for quasinormable Fréchet spaces. The results of this section are also relevant for the conmutativity of inductive limits and the injective tensor product.

Following Taskinen [48] we say that a pair (E, F) of Fréchet spaces satisfies property (BB) if the problem of topologies of Grothendieck has a positive answer for $E\hat{\otimes}_\pi F$. If (E, F) has property (BB) then $L_b(E, F'_b) = (E\hat{\otimes}_\pi F)'_b$ holds topologically. A Fréchet space E is an (FBa)-space (see [49]) if (E, X) satisfies property (BB) for every Banach space X.

We want to give positive answers to the problems of Grothendieck in the context of the classes (QNo) and (QNo)' even when we deal with spaces of compact operators.

Proposition 4.1. (1) *If E is a metrizable l.c.s. (QNo), then there is a fundamental decreasing sequence $(U_k)_{k\in\mathbf{N}}$ of abx. 0-neighbourhoods in E such that, for every increasing sequence $(\lambda_k)_{k\in\mathbf{N}}$ of scalars, $\lambda_k > 1$, $k \in \mathbf{N}$, there exists $B \in \mathscr{B}(E)$ so that, for every $n \in \mathbf{N}$, there is a finite collection $(P_k)_{k=1}^{n+1}$ of operators in E satisfying*

(Fo1) $x = \sum_{k=1}^{n+1} P_k(x), \quad \forall x \in E$,

(Fo2) $P_k(\lambda_k U_k) \subset B + U_n, \quad k = 1 \dots n+1$.

(2) *If G is a (DF)-space which satisfies (QNo)', then there is an increasing fundamental sequence $(B_k)_{k\in\mathbf{N}}$ of abx. bounded subsets in E such that, for every decreasing sequence $(\alpha_k)_{k\in\mathbf{N}}$ of strictly positive scalars, $\alpha_k < 1$, $k \in \mathbf{N}$, there exists a 0-neighbourhood U in G so that, for every $n \in \mathbf{N}$, there is a finite collection $(Q_k)_{k=1}^{n+1}$ of operators in G satisfying*

(DFo1) $x = \sum_{k=1}^{n+1} Q_k(x), \quad \forall x \in G$, (DFo2) $Q_k(U \cap B_n) \subset \alpha_k B_k$, $k = 1 \ldots n+1$.

Proof. We will show (2); (1) is analogous. We can suppose, without loss of generality, that there is an increasing fundamental sequence $(B_k)_{k\in\mathbf{N}}$ of bounded subsets in G such that, for every decreasing sequence $(\alpha_k)_{k\in\mathbf{N}}$ of positive scalars, $\alpha_k < 1$, $k \in \mathbf{N}$, there are sequences $(P_k)_{k\in\mathbf{N}} \subset L(G,G)$ and $(U_k)_{k\in\mathbf{N}} \subset \mathscr{U}_0(G)$ satisfying

(a) $P_k^{-1}(\alpha_{k+1}B_{k+1}) \supset 2U_k \quad (U_{k+1} \subset U_k)$,

(b) $(I - P_k)(2B_k) \subset \alpha_{k+1}B_{k+1}$, $k \in \mathbf{N}$.

We set $U := \left[\frac{1}{4}\bigcap_{k\geq 2}(B_{k-1} + U_k)\right] \cap U_1 \in \mathscr{U}_0(G)$ and $C_k := B_{k+1}$, $k \in \mathbf{N}$. Given $n \in \mathbf{N}$, we define $Q_1 := P_1$, $Q_k := P_k - P_{k-1}$, $k = 2, \ldots, n$, $Q_{n+1} := I - P_n$ to conclude:

(DFo1) $\sum_{k=1}^{n+1} Q_k(x) = x$, $\forall x \in G$,

(DFo2) $Q_1(U) \subset P_1(U_1) \subset \alpha_2 B_2 \subset \alpha_1 C_1$.

For $2 \leq k \leq n$

$$\begin{aligned} Q_k(U) &= (P_k - P_{k-1})(U) \subset (P_k - P_{k-1})\big(\tfrac{1}{4}(B_{k-1} + U_k)\big) \\ &\subset \tfrac{1}{4}(P_k - P_{k-1})(B_{k-1}) + \tfrac{1}{4}P_k(U_k) + \tfrac{1}{4}P_{k-1}(U_{k-1}) \\ &\subset \tfrac{1}{4}(P_k - I)(B_k) + \tfrac{1}{4}(I - P_{k-1})(B_{k-1}) + \tfrac{1}{4}\alpha_k B_{k+1} \subset \tfrac{1}{2}\alpha_k C_k. \end{aligned}$$

Finally

$$\begin{aligned} Q_{n+1}(U \cap C_n) &\subset (I - P_{n+1})(B_{n+1}) + (P_{n+1} - P_n)(U) \\ &\subset \tfrac{1}{2}\alpha_{n+2}B_{n+2} + \tfrac{1}{2}\alpha_{n+1}C_{n+1} \subset \alpha_{n+1}C_{n+1}. \end{aligned}$$

(For the estimate $(P_{n+1} - P_n)(U) \subset \frac{1}{2}\alpha_{n+1}C_{n+1}$ one proceeds as above.) And the proof is complete. □

In the following theorem we will denote by $\mathscr{A}$ the ideal $\mathscr{F}$ of finite rank operators, $\mathscr{K}$ of compact operators or LB of bounded operators.

Moreover, when we write $E = \mathrm{proj}_k E_k$ (respectively $G = \mathrm{ind}_k G_k$) for a metrizable l.c.s. (QNo) (respectively for a (DF)-space which satisfies (QNo)') we suppose $E_k = E_{U_k}$ (respectively $G_k = G_{B_k}$), $k \in \mathbf{N}$, where $(U_k)_{k\in\mathbf{N}}$ (respectively $(B_k)_{k\in\mathbf{N}}$) is the basis of 0-neighbourhoods in E given by Proposition 4.1.(1) (respectively is the fundamental sequence of bounded subsets in G given by Proposition 4.1.(2)).

Theorem 4.2. *Let $E = \mathrm{proj}_k E_k$ be a metrizable l.c.s. (QNo) and let $G = \mathrm{ind}_k G_k$ be a (DF)-space which satisfies (QNo)' then, if F is any l.c.s., we have:*

(1) The canonical mapping R: $\mathscr{A}_b(F, \bigoplus_k G_k) \to \mathscr{A}_b(F, G)$; $\Phi \mapsto q \circ \Phi$ is surjective and open.

(2) The canonical mapping S: $\mathscr{A}_b(\prod_k E_k, F) \to \mathscr{A}_b(E, F)$; $\Phi \mapsto \Phi \circ i$ is surjective and open.

Proof. (1) Let $B \in \mathscr{B}(F)$ and let $(\alpha_k)_{k\in\mathbf{N}}$ be a decreasing sequence of strictly positive scalars, $\alpha_k < 1$, $k \in \mathbf{N}$. What we want to know is just if $R\big(W\big(B, \bigcup_m \big(\bigoplus_{k=1}^m \alpha_k B_k\big)\big)\big)$ is a neighbourhood of zero in $\mathscr{A}_b(F,G)$.

By 4.1 there exists a 0-neighbourhood U in G such that, for every $n \in \mathbf{N}$, there is a finite collection $(Q_k)_{k=1}^{n+1}$ of operators in G satisfying (DFo1) and (DFo2). Let $T \in W(B,U)$, by hypothesis T is bounded and we can find $V \in \mathscr{U}_0(F)$ ($B \subset V$) and $n \in \mathbf{N}$ such that $T \in W(V, B_n)$. Moreover, if $\mathscr{A} = \mathscr{K}$, we can suppose that $T(V)$ is relatively compact in G_n, since G satisfies the strict Mackey condition. Let $(Q_k)_{k=1}^{n+1} \subset L(G,G)$ satisfy (DFo1) and (DFo2). By (DFo2), $T_k := Q_k \circ T \in \mathscr{A}(F, G_k)$, $k = 1, \dots, n+1$ (if $\mathscr{A} = \mathscr{K}$, $T(V)$ is relatively compact in G_n, hence $Q_k\big(T(V)\big)$ is relatively compact in G_k) and we can define the operator $\tilde{T} := \bigoplus_{k=1}^{n+1} T_k \in \mathscr{A}(F, \bigoplus_k G_k)$ which satisfies

(a) $R(\tilde{T}) = q \circ \tilde{T} = \big(\sum_{k=1}^{n+1} Q_k\big) \circ T = T$ by (DFo1),

(b) $\tilde{T}(B) = \bigoplus_{k=1}^{n+1} Q_k(T(B)) \subset \bigoplus_{k=1}^{n+1} Q_k(U \cap B_n) \subset \bigoplus_{k=1}^{n+1} \alpha_k B_k$ by (DFo2), that is, $\tilde{T} \in W\big(B, \bigcup_m \big(\bigoplus_{k=1}^m \alpha_k B_k\big)\big)$. This implies that R is open.

(2) Let $U \in \mathscr{U}_0(F)$, $\tilde{U}_k$ the unit ball of E_k and $(\lambda_k)_{k\in\mathbf{N}}$ an increasing sequence of positive scalars. We set $\mathscr{U} := W(\prod_k \lambda_k \tilde{U}_k, U)$ and claim that $S(\mathscr{U})$ is open in $\mathscr{A}_b(E,F)$.

By 4.1 there exists an abx. bounded subset B in E associated with $\{2^{k+1}\lambda_k\}_k$ such that, for every $n \in \mathbf{N}$, there is a finite collection $(P_k)_{k=1}^{n+1}$ of operators in E satisfying (Fo1) and (Fo2). Let $T \in W(B,U)$. By hypothesis there is $n \in \mathbf{N}$ with $T(U_n)$ bounded in F $\big(T(U_n) \subset U\big)$, hence by (Fo2) $T \circ P_k$ induces an operator $T_k \in \mathscr{A}(E_k, F)$ with $T_k \circ i = T \circ P_k$, $k = 1 \cdots n+1$, and we can define the operator $\tilde{T} := \sum_{k=1}^{n+1} T_k \in \mathscr{A}\big(\prod_k E_k, F\big)$ which satisfies

(a) $S(\tilde{T}) = \tilde{T} \circ i = T \circ \big(\sum_{k=1}^{n+1} P_k\big) = T$ by (Fo1),

(b) $\tilde{T}\big(\prod_k \lambda_k \tilde{U}_k\big) = \sum_{k=1}^{n+1} \lambda_k T_k(\tilde{U}_k) = \sum_{k=1}^{n+1} \lambda_k T\big(P_k(U_k)\big)$
$\subset \sum_{k=1}^{n+1} T\big(2^{-k-1}(B + U_n)\big) \subset \sum_{k=1}^{n+1} 2^{-k} U \subset U$ by (Fo2).
And we conclude that S is open. □

Corollary 4.3. (1) *If $E = \mathrm{proj}_k E_k$ is a metrizable l.c.s. (QNo), $G = \mathrm{ind}_k G_k$ satisfies (QNo)' and if $\mathscr{A} = \mathscr{F}$, $\mathscr{K}$ or L, then $\mathscr{A}_b(E,G)$ is a bornological (DF)-space.*

(2) *If G and H are (DF)-spaces which satisfy (QNo)' then $G \hat{\otimes}_\varepsilon H$ is a bornological (DF)-space.*

(3) *If E and F are Fréchet spaces (QNo) then the pair (E,F) satisfies the property (BB).*

Proof. (1) By Theorem 4.2 $\mathscr{A}_b(E,G)$ is a quotient of $\mathscr{A}_b\big(\prod_k E_k, \bigoplus_k G_k\big)$ ($L(E,G) = LB(E,G)$ by [20, Proposition 4]) which is isomorphic to $\bigoplus_k \mathscr{A}_b(E_k, G_k)$, hence $\mathscr{A}_b(E,G)$ is a (DF)-space and, moreover, it satisfies the strict Mackey condition by Proposition 3.5 (b). This implies that $\mathscr{A}_b(E,G)$ is bornological.

(2) The spaces $\mathscr{K}_b(C_2,G) = C_2\hat{\otimes}_\varepsilon G$ and $\mathscr{K}_b(C_2,H) = C_2\hat{\otimes}_\varepsilon H$ are bornological (DF)-spaces by (1). We conclude (2) by [12, 1.6.(1)].

(3) We only have to note that (E,F) satisfies the property (BB) if $L_b(E,F_b')$ is a (DF)-space (see [14, Proposition 2]) and the result follows from (1). □

It is possible to characterize when we have positive answers for the problems of Grothendieck in the context of quasinormable Fréchet spaces and (DF)-spaces with the strict Mackey condition. First we need the following result due to Kaballo and Vogt [37].

Theorem 4.4 ([37, 1.1]). *Let* $0 \to H \xrightarrow{J} G \xrightarrow{K} Q \to 0$ *be a topological exact sequence of locally convex spaces. The following are equivalent:*

(1) $I_X\otimes_\varepsilon K\colon X\otimes_\varepsilon G \to X\otimes_\varepsilon Q$ *is a homomorphism for every Banach space* X.

(2) $I_X\otimes_\pi J\colon X\otimes_\pi H \to X\otimes_\pi G$ *is a monomorphism for every Banach space* X.

Following Kaballo, Vogt [37] a topologically exact sequence $0 \to H \xrightarrow{J} G \xrightarrow{K} Q \to 0$ of locally convex spaces is called a *tensor sequence* if one of the equivalent conditions (1)–(2) of the theorem above are satisfied. In this case the sequences

$$0 \to E\otimes_\varepsilon H \to E\otimes_\varepsilon G \to E\otimes_\varepsilon Q \to 0,$$

$$0 \to E\otimes_\pi H \to E\otimes_\pi G \to E\otimes_\pi Q \to 0,$$

are topologically exact for every locally convex space E (see [37, 1.1]).

We mainly concentrate in the following algebraically exact sequence

$$(*) \qquad 0 \to \bigoplus_{n\in\mathbf{N}} G_n \xrightarrow{\sigma} \bigoplus_{n\in\mathbf{N}} G_n \xrightarrow{q} G \to 0,$$

where $(G_n)_{n\in\mathbf{N}}$ is an increasing sequence of normed spaces, $G := \mathrm{ind}_n G_n$, $q(x) := \sum_n x_n$, $\sigma(x) := (x_n - x_{n-1})_{n\in\mathbf{N}}$, $x_0 := 0$, for all $x \in \bigoplus_n G_n$.

The map q is a homomorphism, but σ need not to be a monomorphism. Following Palamodov [42], the spectrum $G_1 \hookrightarrow G_2 \hookrightarrow \cdots$ is called *acyclic* if σ is a monomorphism. By a version of Palamodov of a result of Retakh (see [42, Theorem 6.2]), if σ is a monomorphism then G satisfies the strict Mackey condition and, if $\mathrm{ind}_n G_n$ is regular, the converse is also true (cf. [42, Theorem 6.1]). Obviously $G_1 \hookrightarrow G_2 \hookrightarrow \cdots$ is acyclic if and only if $\hat{G}_1 \hookrightarrow \hat{G}_2 \hookrightarrow \cdots$ is acyclic, and this implies that $\hat{G} = \mathrm{ind}_n \hat{G}_n$, since G is a topological subspace of $\mathrm{ind}_n \hat{G}_n$, which is complete (see [42, Corollary 7.1]). For a detailed study of acyclic (LF)-spaces we refer to [53].

In the following theorem $(G_n)_{n\in\mathbf{N}}$ is an increasing sequence of normed spaces with continuous embeddings such that $G := \mathrm{ind}_n G_n$ is a regular inductive limit. This generalizes a proposition in [4] (where the inductive limit was supposed to be a (DFS)-space, i.e., the strong dual of a Fréchet–Schwartz space).

Theorem 4.5. *Let G satisfy the strict Mackey condition.*
(a) *Then the following are equivalent:*
(1) *$G\hat{\otimes}_\varepsilon X$ is a bornological (DF)-space for every Banach space X,*
(2) *$G\otimes_\varepsilon X = \mathrm{ind}_n(G_n\otimes_\varepsilon X)$ holds topologically for every Banach space X; that is, the space $G = \mathrm{ind}_n G_n$ is an inductive limit with local partition of unity in the sense of Hollstein* [32, 2.2, 3.2],
(3) *$G\otimes_\varepsilon E = \mathrm{ind}_n(G_n\otimes_\varepsilon E)$ holds topologically for every locally convex space E having the countable neighbourhood property,*
(4) *$G\hat{\otimes}_\varepsilon X = \mathrm{ind}_n(G_n\hat{\otimes}_\varepsilon X)$ holds algebraically and topologically for every Banach space X.*
(b) *Properties* (1)–(4) *imply the following equivalent conditions:*
(5) *$G\hat{\otimes}_\pi X$ satisfies the strict Mackey condition for every X Banach,*
(6) *G'_b is quasinormable by operators.*
(c) *If G is a complemented subspace of its bidual we have that* (1)–(6) *are equivalent.*

Proof. (a) (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). We note that $\hat{G}\varepsilon C_2 = G\hat{\otimes}_\varepsilon C_2$ since C_2 has the approximation property. Moreover, as $\mathrm{ind}_n\hat{G}_n$ is compactly regular, it follows from [6, 3.13] that the spaces $\hat{G}\varepsilon C_2$ and $\mathrm{ind}_n(\hat{G}_n\varepsilon C_2) = \mathrm{ind}_n(G_n\hat{\otimes}_\varepsilon C_2)$ are equal algebraically. We then have the topological identity $G\hat{\otimes}_\varepsilon C_2 = \mathrm{ind}_n(G_n\hat{\otimes}_\varepsilon C_2)$ by *Grothendieck's factorization theorem.* Now the density of $G_n\otimes_\varepsilon C_2$ in $G_n\hat{\otimes}_\varepsilon C_2$ for each $n\in\mathbf{N}$ implies the topological identity $G\otimes_\varepsilon C_2 = \mathrm{ind}_n(G_n\otimes_\varepsilon C_2)$ (use [43, 6.3.1]). Hollstein showed (see [32, 3.2]) that this is equivalent to (2) and (3).

(2) implies (4). $G_1\otimes_\varepsilon X \hookrightarrow G_2\otimes_\varepsilon X \hookrightarrow \cdots$ is an acyclic spectrum, since $\mathrm{ind}_n G_n\otimes_\varepsilon X$ is regular and $G\otimes_\varepsilon X$ satisfies the strict Mackey condition (see [42, Theorem 6.1]). Then $G_1\hat{\otimes}_\varepsilon X \hookrightarrow G_2\hat{\otimes}_\varepsilon X \hookrightarrow \cdots$ is also an acyclic spectrum and, by [42, Corollary 7.1], $\mathrm{ind}_n G_n\hat{\otimes}_\varepsilon X$ is complete. But, as $G\otimes_\varepsilon X = \mathrm{ind}_n(G_n\otimes_\varepsilon X)$ holds topologically, $\mathrm{ind}_n(G_n\hat{\otimes}_\varepsilon X)$ induces in $G\otimes X$ the ε-topology and $G\otimes X$ is a dense subspace of $\mathrm{ind}_n(G_n\hat{\otimes}_\varepsilon X)$; this implies the topological identity $G\hat{\otimes}_\varepsilon X = \mathrm{ind}_n(G_n\hat{\otimes}_\varepsilon X)$ (see [8, 1.2]).

(4) implies (1) is trivial.

(b) (5) $\Longleftrightarrow$ (6). Taking duals, $G\otimes_\pi X$ satisfies the strict Mackey condition for every X Banach if and only if $L_b(X, G'_b)$ is quasinormable for every X Banach, which is equivalent to (6) by Propositions 3.5 and 3.8.

(2) $\Longrightarrow$ (6). Let us consider the canonical topologically exact sequence:

$$0 \to \bigoplus_{n\in\mathbf{N}} G_n \xrightarrow{\sigma} \bigoplus_{n\in\mathbf{N}} G_n \xrightarrow{q} G \to 0.$$

Hollstein showed (see [32, 1.2, 2.4, 3.2]) that (2) holds if and only if

$$(*) \qquad 0 \to \bigoplus_{n\in\mathbf{N}}(G_n \otimes_\pi C_2) \overset{\sigma\otimes I_{C_2}}{\longrightarrow} \bigoplus_{n\in\mathbf{N}}(G_n \otimes_\pi C_2) \overset{q\otimes I_{C_2}}{\longrightarrow} \mathrm{ind}_n(G_n \otimes_\pi C_2) \to 0$$

is a topologically exact sequence, which is equivalent to the acyclicity of the following spectrum: $G_1 \otimes_\pi C_2 \hookrightarrow G_2 \otimes_\pi C_2 \hookrightarrow \cdots$. This implies that

$$G \otimes_\pi C_2 = \mathrm{ind}_n(G_n \otimes_\pi C_2)$$

satisfies the strict Mackey condition and, in consequence, $L_b(C_2, G'_b)$ is quasinormable. By Proposition 3.8 (a) G'_b is quasinormable by operators.

(c) If G is a complemented subspace of its bidual and G'_b is (QNo) then G satisfies (QNo)' by Corollary 3.6 (2) and Proposition 3.3 (1). We conclude (1) by Corollary 4.3 (2). ▫

Lemma 4.6. *Let E and F be Fréchet spaces such that F is quasinormable and reflexive, E'_b or F'_b has the strict approximation property (cf. [38]) and $E'_b\hat{\otimes}_\pi F'_b$ is bornological. Then $(E\hat{\otimes}_\varepsilon F)'_i = E'_b\hat{\otimes}_\pi F'_b$.*

Proof. We know that $\{\overline{\Gamma(U^o \otimes V^o)}^{E'_b\hat{\otimes}_\pi F'_b} / U \in \mathscr{U}_0(E),\ V \in \mathscr{U}_0(F)\}$ is a fundamental family of bounded subsets of $E'_b\hat{\otimes}_\pi F'_b$ (see [38, 41.4.(7)]). Now we conclude the topological identity $(E\hat{\otimes}_\varepsilon F)'_i = E'_b\hat{\otimes}_\pi F'_b$ by [18, Lemma 2.6]. ▫

Theorem 4.7. *If E is a quasinormable Fréchet space then the following assertions are equivalent:*

(a) *E is an (FBa)-space,*
(b) *$E\hat{\otimes}_\varepsilon X$ is quasinormable for every X Banach,*
(c) *E'_b satisfies (QNo)'.*

Proof. (a) implies (b). If E is an (FBa)-space, taking duals we have that $L_b(X, E'_b)$ is (DF) (then (gDF)) for every Banach space X. If $\{B_n\}_{n\in\mathbf{N}}$ is a fundamental sequence of abx. bounded subsets in E'_b and $\{U_n\}_{n\in\mathbf{N}}$ is a sequence of abx. 0-neighbourhoods in E'_b, following the argument of 3.7 and 3.8, it is possible to find a sequence $\{P_n\}_{n\in\mathbf{N}}$ of continuous operators from E'_b into E'_b and an abx. 0-neighbourhood U in E'_b such that

(i) $P_n(U) \subset B_n$,
(ii) $(I - P_n)(U) \subset U_n$.

Thus, as in the proof of 3.4, we have that $X\hat{\otimes}_\varepsilon E'_b$ is a (gDF)-space for every X Banach. Moreover $X\hat{\otimes}_\varepsilon E'_b$ satisfies the strict Mackey condition and this implies that $X\hat{\otimes}_\varepsilon E'_b$ is bornological for every X Banach. By Theorem 4.5 $E'_b\hat{\otimes}_\pi X$ satisfies the strict Mackey condition for every X Banach, in particular for $X = C_2$. C_2 is reflexive and it has the approximation property, hence, by Lemma 4.6, we have that $(E\hat{\otimes}_\varepsilon C_2)'_i = E'_b\hat{\otimes}_\pi C_2$ and, in consequence, $E\hat{\otimes}_\varepsilon C_2$ is quasinormable (see [11]), therefore $E\hat{\otimes}_\varepsilon X$ is quasinormable for every X Banach by [12, 1.6.(1)].

(b) implies (c) follows from Proposition 3.7.

(c) implies (a). If E'_b satisfies (QNo)' then Corollary 4.3 implies that $L_b(X, E'_b)$ is a bornological (DF)-space for every X Banach and, by the quasinormability of E, we obtain that E is an (FBa)-space (see [14, 1.2.(i)]). □

Now, with the characterizations and precedent results, we can get new examples which answer positively or negatively the problem of topologies of Grothendieck and the related questions.

Positive examples. (1) If G is a Silva space (i.e. the strong dual of a Fréchet Schwartz space) which is an inductive limit of Banach spaces with injective linking maps such that the linking maps are approximable, in the operator norm, by finite rank operators, then G satisfies (QNo)' by Theorem 3.14 and $G \otimes_\varepsilon X$ is bornological for every X normed. This result was proved directly in [5] and was successfully applied to the study of vector valued germs of holomorphic mappings on Fréchet Schwartz spaces by Bierstedt, Bonet and the author in [5].

(2) The known examples of Fréchet Schwartz spaces which are (FBa)-spaces always satisfy the approximation property, and it is a natural question if every (FBa)-space which is Fréchet Schwartz necessarily satisfies the approximation property. This is not true because we can find examples of Fréchet Schwartz spaces F which are (QNo) but without the approximation property (see [45]), using examples, due to Willis [54], of Banach spaces with the compact approximation property but without the approximation property (compare with counterexample (3) below).

(3) If E (respectively G) is a quasinormable hilbertizable Fréchet space (respectively a countable inductive limit of Hilbert spaces which satisfies the strict Mackey condition) then it is reflexive and, by [12, 1.7], it is an (FBa)-space (respectively $G \hat{\otimes}_\varepsilon X$ is a bornological (DF)-space for every Banach space X). Theorem 4.7 (respectively Theorem 4.5) implies that E is (QNo) (respectively G satisfies (QNo)'). We refer to [44] for a direct proof.

Counterexamples. (1) Let λ be a normal Banach sequence space such that φ is dense in λ and let Y be a Banach space such that Y' has the approximation property. By Theorem 4.7 the space $\lambda(X, X/Y) \hat{\otimes}_\varepsilon Z$ is quasinormable for every Banach space Z if and only if $\lambda(X, X/Y)$ is an (FBa)-space, and this is equivalent to the property that $\lambda(X, X/Y)'_b$ is a complemented subspace of a countable direct sum of Banach spaces (see [23]). Compare with 3.16. We refer to [44] for a direct proof.

(2) The space $l_{p+} = \mathrm{proj}_n l_{p+1/n}$, $1 \leq p < \infty$ is a quasinormable reflexive Fréchet space (see [40]) which is not (QNo); thus, by 4.7, it is not an (FBa)-space. To see that l_{p+} does not satisfy (QNo) we use the facts that every bounded operator in l_{p+} is compact (cf. [16, Example 6]) and that, if there is a compact

operator P in l_{p+} such that $(I-P)(e_i) \in \frac{1}{2}B_n$ (B_n the unit ball of $l_{p+1/n}$ and $\{e_i\}_{i\in\mathbf{N}}$ the canonical basis of l_p) then $\{P(e_i)\}_{i\in\mathbf{N}}$ has a cluster point which, necessarily, must be zero, since $\{e_i\}_{i\in\mathbf{N}}$ is weakly convergent to zero and P is compact. But $\|P(e_i)\|_n \geq \frac{1}{2}$, $i \in \mathbf{N}$, (where $\|\cdot\|_n$ is the norm of $l_{p+1/n}$) which is a contradiction. This is the first concrete (and natural) example (as far as the author knows) of a Fréchet space which is not an (FBa)-space.

(3) It was an open problem of Bierstedt, Meise [6] and Hollstein [32] (respectively Taskinen [49]) if, for every Silva space G and for every Banach space X (respectively every Fréchet Schwartz E) the space $G\hat{\otimes}_\varepsilon X$ is bornological (respectively E is an (FBa)-space). In [45] we gave an example of a Fréchet Schwartz space F without the compact approximation property (hence not satisfying (QNo)), obtaining as a consequence that F is not an (FBa)-space (respectively $F'_b\hat{\otimes}_\varepsilon C_2$ is not a (DF)-space).

5. Infinite holomorphy

In this section we suppose that E is a complex locally convex space. $\mathscr{H}(E)$ stands for the space of (all) holomorphic functions on E. The compact-open topology in $\mathscr{H}(U)$ is denoted by τ_0. We write $\mathscr{H}_b(E)$ for the space of (all) holomorphic functions bounded on the bounded subsets of E, endowed with the topology τ_b of uniform convergence on the bounded subsets of E.

$\mathscr{P}(^nE)$ stands for the space of the $\mathbf{C}$-valued continuous polynomials on E, $n \in \mathbf{N}$. If $f \in \mathscr{H}(E)$ we denote by $\sum_1^\infty \hat{d}^n f(0)/n!$ the Taylor series of f in 0. We have $\hat{d}^n f(0)/n! \in \mathscr{P}(^nE)$, $n \in \mathbf{N}$. Define

$$S := \{\, (\alpha_n)_{n\in\mathbf{N}} \,/\, \alpha_n \in \mathbf{C} \quad \text{and} \quad \limsup_{n\to\infty} |\alpha_n|^{1/n} \leq 1 \,\}.$$

For every element f in $\mathscr{H}(E)$ (respectively $\mathscr{H}_b(E)$) and every $(\alpha_n)_{n\in\mathbf{N}} \in S$, the series

$$\sum_1^\infty \alpha_n \frac{\hat{d}^n f(0)}{n!}$$

belongs to $\mathscr{H}(E)$ (respectively $\mathscr{H}_b(E)$) and, if p is a continuous seminorm in $(\mathscr{H}(E), \tau_0)$ (respectively $(\mathscr{H}_b(E), \tau_b)$), then the seminorm

$$\tilde{p}(f) := \sum_1^\infty \alpha_n p\Big(\frac{\hat{d}^n f(0)}{n!}\Big)$$

is also continuous in $(\mathscr{H}(E), \tau_0)$ (respectively in $(\mathscr{H}_b(E), \tau_b)$) (see [24, Chapter 3] and [25, 1.3]). That is, $\{(\mathscr{P}(^nE), \tau_0)\}_{n\in\mathbf{N}}$ (respectively $\{(\mathscr{P}(^nE), \tau_b)\}_{n\in\mathbf{N}}$) is an S-absolute decomposition for $(\mathscr{H}(E), \tau_0)$ (respectively $(\mathscr{H}_b(E), \tau_b)$) (cf. [24, p. 114]).

If $f \in \mathscr{H}(E)$ and $A \subset E$, we write $\|f\|_A := \sup\{\, |f(x)| \,/\, x \in A \,\}$.

There are recent studies about spaces of holomorphic functions which are quasinormable (see e.g. [25]). We want to present a result which, in view of the examples obtained in Section 3, extends some results of Dineen in [25]. We will make use of the following lemma, which is a particular case of [29, Lemma 1.3].

Lemma 5.1. *Let E be a l.c.s. and $n \in \mathbf{N}$. There exists a linear extension operator*

$$R_n\colon \mathscr{P}(^nE) \to \mathscr{P}(^nE'')$$

such that $\|P\|_A = \|R_n(P)\|_{A^{oo}}$, $\forall P \in \mathscr{P}(^nE)$, for every abx. subset A of E.

Theorem 5.2. *If E is a l.c.s., then*

(a) *E'_b is (QNo) if and only if $\big(\mathscr{H}_b(E), \tau_b\big)$ is (QNo).*

(b) *E'_{co} is (QNo) if and only if $\big(\mathscr{H}(E), \tau_0\big)$ is (QNo).*

Proof. We will show (a). ((b) is analogous replacing bounded sets by compact subsets of E.)

If $\big(\mathscr{H}_b(E), \tau_b\big)$ is (QNo) then E'_b is also (QNo) because it is a complemented subspace of $\big(\mathscr{H}_b(E), \tau_b\big)$.

Conversely, let us suppose that E'_b is (QNo). Then, for every $B \in \mathscr{B}(E)$, there is $B' \in \mathscr{B}(E)$ $(B \subset B')$ such that $\forall \varepsilon > 0$, $\exists P \in L(E'_b, E'_b)$ satisfying

(i) $P(B'^o) =: C \in \mathscr{B}(E'_b)$,

(ii) $(I - P)(B'^o) \subset \varepsilon B^o$.

Let us consider the τ_b-norm

$$p(f) := |f(0)| + \sum_1^\infty \frac{1}{n!}\|\hat{d}^n f(0)\|_{2B'}; \qquad f \in \mathscr{H}_b(E).$$

Given $\varepsilon > 0$ $(\varepsilon < 1)$, take $n_o \in \mathbf{N}$ with $2^{-n_o} < \varepsilon$ and $P \in L(E'_b, E'_b)$ satisfying (i) and (ii). Now define

$$\begin{aligned} Q\colon \mathscr{H}_b(E) &\to \mathscr{H}_b(E) \\ f &\mapsto Q(f) := f(0) + \sum_1^{n_o} T_n(f) \circ P^t\,|_E \end{aligned}$$

where $T_n(f) := R_n\big(\hat{d}^n f(0)/n!\big)$, $f \in \mathscr{H}_b(E)$, and $P^t\,|_E$ is the transpose of P restricted to E.

Let $A \in \mathscr{B}(E)$ and $\lambda > 1$ with $A \subset \lambda C^o$, then, given $f \in \mathscr{H}_b(E)$:

$$\begin{aligned} \|Q(f)\|_A &\le |f(0)| + \sum_1^{n_o} \big\|T_n(f) \circ P^t\,|_E\big\|_A \le |f(0)| + \sum_1^{n_o} \lambda^n \big\|T_n(f) \circ P^t\,|_E\big\|_{C^o} \\ &\le |f(0)| + \sum_1^{n_o} \lambda^n \|T_n(f)\|_{B'^{oo}} = |f(0)| + \sum_1^{n_o} \frac{\lambda^n}{n!}\|\hat{d}^n f(0)\|_{B'} \le \lambda^{n_o} p(f), \end{aligned}$$

and we obtain that $Q\big(\{f \in \mathscr{H}_b(E)/p(f) \le 1\}\big) \in \mathscr{B}\big((\mathscr{H}_b(E), \tau_b)\big)$. On the other hand,

$$\begin{aligned}\|(I-Q)(f)\|_B &\le \sum_1^{n_o} \big\|T_n(f)\circ(I-P)^t\,|_E\big\|_B + \sum_{n_o+1}^{\infty} \frac{1}{n!}\|\hat{d}^n f(0)\|_B \\ &\le \sum_1^{n_o} \varepsilon^n \|T_n(f)\|_{B'^{oo}} + \sum_{n_o+1}^{\infty} \frac{1}{2^n n!}\|\hat{d}^n f(0)\|_{2B'} \\ &\le \varepsilon \sum_1^{n_o} \frac{1}{n!}\|\hat{d}^n f(0)\|_{B'} + \frac{1}{2^{n_o}} \sum_{n_o+1}^{\infty} \frac{1}{n!}\|\hat{d}^n f(0)\|_{2B'} \le \varepsilon p(f).\end{aligned}$$

This yields that $\big(\mathscr{H}_b(E), \tau_b\big)$ is (QNo). □

Ansemil and Ponte showed [1] that, for E normed, the space $\big(\mathscr{H}_b(E), \tau_b\big)$ is quasinormable. In recent work of Dineen [25] it is shown that, for instance, the space $\big(\mathscr{H}(E), \tau_0\big)$ is Schwartz if E is a compact or strict inductive limit of Fréchet spaces. Dineen also characterized (under some mild restrictions) the standard strict LB-spaces of Moscatelli type E such that $\big(\mathscr{H}_b(E), \tau_b\big)$ is quasinormable.

Example. If E is a Fréchet space which has the approximation property then E'_{co} is a Schwartz (gDF)-space with the approximation property and, by Theorem 3.11 (b), it is (QNo). This implies that $\big(\mathscr{H}(E), \tau_0\big)$ is also (QNo).

Corollary 5.3. *If E is a l.c.s. such that $\big(\mathscr{H}_b(E), \tau_b\big)$ is not (QNo), then there is a Banach space X such that $\big(\mathscr{H}_b(E \times X), \tau_b\big)$ is not quasinormable.*

Proof. If E is a l.c.s. such that $\big(\mathscr{H}_b(E), \tau_b\big)$ is not quasinormable by operators then, by the theorem above, E'_b is not (QNo) and, by Proposition 3.8, there is a Banach space X such that $L_b(X, E'_b)$ is not quasinormable. Define $F := E \times X$. By [18, Lemma 8] we obtain (taking duals) that $L_b(X, E'_b)$ is a complemented subspace of $\big(\mathscr{P}(^2F), \tau_b\big)$, hence $\big(\mathscr{P}(^2F), \tau_b\big)$ is not quasinormable and, consequently, $\big(\mathscr{H}_b(F), \tau_b\big)$ is also not quasinormable. □

It is possible to formulate more applications to infinite-dimensional holomorphy. We will come back to this topic somewhere else.

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