Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 19, 1994, 83–93

QUASICONFORMAL EQUIVALENCE OF SPHERICAL CR MANIFOLDS

Robert R. Miner

University of Oklahoma, Department of Mathematics
Norman, OK 73019, U.S.A.; rminer@nsfuvax.math.uoknor.edu

Abstract. In this paper we consider some basic existence questions about quasiconformal mappings between spherical CR manifolds. A spherical CR manifold is locally modeled on the Heisenberg group. The manifolds we consider are actually quotients of the 3-dimensional Heisenberg group by cyclic groups of automorphisms. When the group consists of loxodromic elements, the quotient manifold is compact, and if two such manifolds are homeomorphic, there is in fact a quasiconformal homeomorphism. By contrast, when the group consists of parabolics, the manifolds are noncompact, and there are exactly two quasiconformal equivalence classes.

1. Introduction

The theory of quasiconformal mapping on the Heisenberg group has a unique appeal both for the amount of similarity with the classical theory, and for the interesting ways in which the similarity breaks down. For example, a quasiconformal mapping on the Heisenberg group satisfies a system of equations formally very similar to the classical Beltrami equation. But unlike the classical Beltrami equation, in general these equations have no solution. Thus it is possible that given two compact, smooth, homeomorphic manifolds M and N modeled on the Heisenberg group, no quasiconformal homeomorphism exists between them.

The intuition behind this curious phenomenon runs as follows. The Heisenberg group has a natural contact structure, and any quasiconformal mapping on the Heisenberg group which is sufficiently smooth (C^2 will do) must preserve this structure. Consequently, if M and N are manifolds modeled on the Heisenberg group, and the induced contact structures on M and N are not homotopic, it follows that at least there are can be no C^2 quasiconformal maps between them. If we restrict our attention to compact manifolds, the converse is also true: given a contactomorphism, it is automatically quasiconformal. Thus, if we are content with reasonably smooth mappings and compact spaces, the question of quasiconformal equivalence is a question about homotopy classes of contact structures, or equivalently, about constructing contact mappings.

1991 Mathematics Subject Classification: Primary 30C65, 32C16; Secondary 53C15.

The author would like to acknowledge the hospitality of the University of Berne, and partial support from NSF Grant No. 9202437.

For noncompact manifolds, however, the situation is more involved. Even if a contact mapping exists, it need not be quasiconformal because of the noncompactness. In any event, because of the regularity issue, contact mappings are useful only for demonstrating existence of q.c. mappings, not their non-existence. However, as this article demonstrates, it is still possible to produce examples of noncompact manifolds modeled on the Heisenberg group which are homeomorphic but not quasiconformal.

To illustrate these ideas, we need only consider the simplest non-simply connected quotients of the Heisenberg group available. Namely, we will investigate quotients by cyclic groups of automorphisms in dimension 3. In analogy with the theory of conformal structures on the boundary of real hyperbolic space, the one point compactification of the Heisenberg group can be regarded as the boundary of complex hyperbolic space. Similarly, automorphisms of complex hyperbolic space can be classified as elliptic, parabolic or loxodromic according to their fixed points properties. We will consider cyclic subgroups generated by loxodromic and parabolic elements, which act properly and discontinously on the complement of their limit sets. The resulting quotient spaces are compact for loxodromic subgroups and noncompact for parabolic ones. Using techniques due to Korányi and Reimann [1], we will show that all loxodromic quotients are quasiconformally equivalent by exhibiting contactomorphisms. By contrast, we will show that there are exactly two quasiconformal equivalence classes of purely parabolic quotients, even though there is only one homeomorphism class.

The organization of the paper runs as follows. In Section 2, we present some basic facts about complex hyperbolic space and the Heisenberg group viewed as its boundary. In Section 3, we recall the fundamentals of quasiconformal mapping on the Heisenberg group. In Section 4, we analyze loxodromic quotients, and in Section 5, we treat the parabolic case.

The author would like to thank H.M. Reimann for his expert guidance and support on this project.

2. Complex hyperbolic space and its boundary

A useful way of introducing complex hyperbolic spaces is by way of the projective model. Consider $\mathbf{C}^{n+1}$ with the indefinite Hermitian inner product:

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1}.$$

Let V_- denote the set of negative vectors with respect to this metric, and define $H^n_{\mathbf{C}}$ to be $P(V_-)$, that is the projectivization of the negative vectors. The $(n,1)$-Hermitian form induces a Riemannian metric of negative curvature on $H^n_{\mathbf{C}}$ via a procedure analogous to the construction of the Fubini–Study metric on $\mathbf{C}P^n$. The holomorphic automorphism group of $H^n_{\mathbf{C}}$ with this metric is the Lie group $PU(n,1)$.

It is easy to see $H^n_{\mathbf{C}}$ is homeomorphic to $B^n \subset \mathbf{C}^n$, so its boundary is S^{2n-1}. The automorphisms of $H^n_{\mathbf{C}}$ extend to the boundary by continuity. The stabilizer of a boundary point is a group of automorphisms analogous to the group of Euclidean similarity transformations. Abstractly, this group is isomorphic to $\mathscr{H}^{2n-1} \rtimes (\mathbf{R}^+ \times U(n))$, where $\mathscr{H}^{2n-1}$ is the $2n-1$- dimensional Heisenberg group. We shall denote this group by $\mathrm{Sim}(\mathscr{H}^{2n-1})$. The Heisenberg group is the maximal parabolic subgroup of the stabilizer, and it acts simply transitively on the complement of the fixed point on the boundary.

For the remainder of the paper, we shall focus on complex dimension 2, so the boundary has real dimension 3. For notational simplicity, we will write $\mathscr{H}$ for $\mathscr{H}^3$. In this case, the Heisenberg group can be described as the set $\mathbf{C} \times \mathbf{R}$ with the group law:

$$(\zeta, v) * (\zeta', v') = \big(\zeta + \zeta', v + v' + 2\,\mathrm{Im}(\zeta\bar{\zeta}')\big).$$

Since the Heisenberg group acts simply transitively on the complement of a boundary point, we may identify the two by choosing a point. In the resulting coordinates, the action of the whole stabilizer, $\mathrm{Sim}(\mathscr{H})$, is affine. The action of the Heisenberg group on itself by left translation is given by the preceding formula. The action of $\mathbf{R}^+$ is given by:

$$r \cdot (\zeta, v) = (r\zeta, r^2 v).$$

The action of $U(1)$ is given by

$$\lambda(\zeta, v) = (\lambda\zeta, v)$$

where λ is a unit complex number.

Elements of $PU(2,1)$ are elliptic, parabolic or loxodromic according to their fixed point behavior. Elements with a fixed point in the interior of $H^2_{\mathbf{C}}$ are elliptic. In the stabilizer of a boundary point, the only elliptics are conjugate to elements in the $U(1)$ factor. Elements with exactly one fixed point on the boundary are called parabolic. We will refer to unipotent parabolics as purely parabolic automorphisms. (The other parabolics are products of purely parabolic and elliptic elements.) Elements of $\mathscr{H}$ acting on itself are all purely parabolic. Again thinking of $\mathscr{H}$ as $\mathbf{C} \times R$, it is useful to distinguish the elements in the two factors. Elements of the form $(0, v)$ are central in $\mathscr{H}$. Finally, the loxodromic elements have exactly two fixed points on the boundary. In $\mathrm{Sim}(\mathscr{H})$, loxodromics are conjugate to a product of an element in the $\mathbf{R}^+$ factor and an element in the $U(1)$ factor. Elements without a rotational part are called purely hyperbolic.

Since the dimension of the boundary is odd, it has no complex structure. However, in each tangent space, there is a maximal subspace stabilized by the complex structure of the ambient space $\mathbf{C}^2$. Specifically, define a hyperplane bundle $E \subset TS^3$ by setting $E_p = T_pS^3 \cap JT_pS^3$. This hyperplane field is totally non-integrable and defines a contact structure on $\partial H^n_{\mathbf{C}}$. Since it admits an almost complex structure, it in fact defines a CR structure as well. Using the affine coordinates described above, the plane field is concisely described as the kernel of $\omega\colon T\mathscr{H} \longrightarrow \mathbf{R}$ where:

$$\omega = dv + i\zeta d\bar{\zeta} - i\bar{\zeta} d\zeta = dv + 2x\,dy - 2y\,dx.$$

In addition to the CR structure, there is a natural metric space structure on the boundary. While the Riemannian metric on $H^n_{\mathbf{C}}$ is degenerate on the boundary, it induces an infinitesimal metric on the plane field E [2]. Using this tensor, one can define the length of a curve tangent to E, and hence a path metric, often called the Carnot metric, on the Heisenberg group. This metric is equivariant with respect to the action of $\mathrm{Sim}(\mathscr{H})$. Alternatively, one may define an equivariant metric space structure by using a *gauge* as follows (see [2] for details). Define:

$$|\zeta, v| = (|\zeta|^4 + v^2)^{1/4}.$$

Then define a metric d by the formula:

$$d\big((\zeta, v), (\zeta', v')\big) = |(\zeta, v) * (\zeta', v')^{-1}|.$$

It is not hard to show that d defines a metric on $\mathscr{H}$ which is equivariant with respect to $\mathrm{Sim}(\mathscr{H})$. Moreover, it is possible to show that d is equivalent to the Carnot metric. Thus we will usually use the d metric for simplicity, rather than the better known, but more unwieldly, Carnot metric.

3. Quasiconformal mappings on the Heisenberg group

Suppose that $f\colon \mathscr{H} \longrightarrow \mathscr{H}$ is a homeomorphism. Define

$$\begin{aligned} a(x,r) &= \inf_{\{y: d(x,y)=r\}} d\big(f(x), f(y)\big) \\ b(x,r) &= \sup_{\{y: d(x,y)=r\}} d\big(f(x), f(y)\big). \end{aligned}$$

Set $H(x,r) = b(x,r)/a(x,r)$ and define $Hf(x) = \limsup_{r\to 0} H(x,r)$. The quantity $Hf(x)$ is called the dilatation of f at x. The mapping f is called k-quasiconformal on a set $U \subset \mathscr{H}$ provided Hf is uniformly bounded on U by k.

We shall require only a few basic results from the theory of quasiconformal mappings on the Heisenberg group. For a systematic treatment, see [3].

The first result we need describes the relationship between contactomorphisms and quasiconformal maps. A contactomorphism is a diffeomorphism f which preserves the contact bundle E. That is, $f_*(E_p) = E_{f(p)}$. Equivalently, $f^*\omega = \mu\omega$ for some nonvanishing, real valued function μ. The following statement is proved in [1].

Proposition 3.1. *If $f\colon U \subset \mathscr{H} \longrightarrow \mathscr{H}$ is a C^2 quasiconformal mapping, then f is a contactomorphism. Conversely, if f is a contactomorphism, and U is compact, then f is quasiconformal.*

The collection of contactomorphisms is quite large. In particular, it is easy to produce flows of contactomorphisms. Consider the vector fields

$$\begin{aligned} Z &= \frac{\partial}{\partial z} + i\bar{\zeta}\frac{\partial}{\partial v} \\ \bar{Z} &= \frac{\partial}{\partial \bar{z}} - i\zeta\frac{\partial}{\partial v} \\ V &= \frac{\partial}{\partial v} \end{aligned}$$

on $\mathscr{H}$. These vector fields form a left invariant frame for $T\mathscr{H}$. The following theorem is also proved in [1].

Proposition 3.2. *Let $f\colon \mathscr{H} \longrightarrow \mathbf{R}$ be any differentiable function. Define:*

$$\xi_f = \frac{i}{2}\big((\bar{Z}f)Z - (Zf)\bar{Z}\big) + fV.$$

Then the flow Ψ_s of ξ_f is contact for all times s.

The function f is sometimes called a *contact potential* function.

In addition to satisfying an infinitesimal distortion bound, quasiconformal mappings also satisfy a global distortion estimate. For proof see [3].

Proposition 3.3. *There exists a constant C such that for any k-quasiconformal mapping $f\colon \mathscr{H} \longrightarrow \mathscr{H}$,*

$$H(x,r) \leq \exp(kC).$$

Finally, we shall make use of the fact that the extremal length, or modulus, of a curve family is a quasi-invariant. Let Γ denote a family of piecewise C^1 curves, tangent to the plane field E. Denote by Σ_Γ the collection of Borel measurable functions $\sigma\colon \mathscr{H} \longrightarrow \mathbf{R}$ such that

$$\int_\gamma \sigma \geq 1$$

for all $\gamma \in \Gamma$. Then define the modulus of Γ by:

$$M(\Gamma) = \inf_{\sigma \in \Sigma_\Gamma} \int_{\mathscr{H}} \sigma^4 \,\mathrm{dvol}.$$

For the following theorem see [3] and [4].

Proposition 3.4. *If $f\colon \mathscr{H} \longrightarrow \mathscr{H}$ is k-quasiconformal, then:*

$$\frac{1}{k^2}M(f\Gamma) \leq M(\Gamma) \leq k^2 M(f\Gamma).$$

4. Loxodromic quotients

Let D_{α_1} and D_{α_2} be two Heisenberg dilations, i.e. $D_{\alpha_i}(\zeta, v) = (\alpha_i\zeta, \alpha_i^2 v)$. To produce a quasiconformal mapping from $\mathscr{H} \setminus \{0\}/\langle D_{\alpha_1}\rangle$ to $\mathscr{H} \setminus \{0\}/\langle D_{\alpha_2}\rangle$, we must produce a quasiconformal mapping $\phi\colon \mathscr{H} \longrightarrow \mathscr{H}$ which commutes with the group action. Since these manifolds are compact, it suffices to produce a contactomorphism ϕ such that:

$$D_{\alpha_1} \circ \phi = \phi \circ D_{\alpha_2}.$$

To accomplish this, we will produce a contact vector field whose flow is essentially given by:

$$\phi_s(\zeta, v) = |\zeta, v|^{e^s - 1}(\zeta, v).$$

It easily follows that:

$$\phi_s \circ D_{\alpha_1} = D_{\alpha_1^{e^s}} \circ \phi_s.$$

To this end, let

$$f(\zeta, v) = 2v \log|\zeta, v|$$

be a contact potential function. By Lemma 3.2, it follows that

$$\xi_f = \Big(\zeta \log|\zeta, v| + \frac{i\zeta v}{2(|\zeta|^2 + iv)}\Big) Z + \Big(\bar{\zeta} \log|\zeta, v| - \frac{i\bar{\zeta} v}{2(|\zeta|^2 - iv)}\Big) \bar{Z} + 2v \log|\zeta, v| T$$

generates a contact flow. Rewriting in terms of the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial v}\}$ and identifying $\mathbf{R}^3$ with $\mathbf{C} \times \mathbf{R}$ we obtain:

$$\xi_f = \big(\zeta \log|\zeta, v|, 2v \log|\zeta, v|\big) + \Big(\frac{i\zeta v}{2(|\zeta|^2 + iv)}, \frac{-|\zeta|^4 v}{|\zeta^4| + v^2}\Big) = X + Y.$$

Observing that the gradient of $r(\zeta, v) = |\zeta|^4 + v^2$ is given by

$$\nabla r = (4\zeta|\zeta|^2, 2v)$$

it follows that X is transverse to the level sets of r and Y is tangent. The flow for X can be directly computed:

$$\Phi_s(\zeta, v) = |\zeta, v|^{e^s - 1}(\zeta, v).$$

Consequently, $\Phi_s \circ D_\alpha = D_{\alpha^{e^s}} \circ \Phi_s$. Furthermore, a computation shows that $[X, Y] = 0$. Thus, if Ψ denotes the flow for ξ_f, then $\Psi_s \circ D_\alpha = D_{\alpha^{e^s}} \circ \Psi_s$ as well.

For a more direct, though less geometric argument, differentiate $\Psi_s \circ D_\alpha = D_{\alpha^{e^s}} \circ \Psi_s$ to obtain:

$$\xi_f\big(D_\alpha(\zeta, v)\big) = \dot{D}_{\alpha^{e^s}}(\zeta, v) + (D_{\alpha^{e^s}})_* \xi_f(\zeta, v).$$

Since $\Psi_s \circ D_\alpha = D_{\alpha^{e^s}} \circ \Psi_s$ holds for $s = 0$, $\Psi_s \circ D_\alpha = D_{\alpha^{e^s}} \circ \Psi_s$ holds everywhere if and only if the preceding condition on the derivative holds. A straightforward calculation verifies this to be the case.

In summary, we have shown:

Proposition 4.1. *Any two purely hyperbolic Hopf manifolds* $\mathscr{H} \setminus \{0\}/\langle D_{\alpha_1}\rangle$ *and* $\mathscr{H} \setminus \{0\}/\langle D_{\alpha_2}\rangle$, *are quasiconformally equivalent.*

In the same way, it is possible to show any loxodromic quotient is quasiconformally equivalent to a purely hyperbolic quotient. Set:

$$g(\zeta, v) = 2|\zeta|^2 \log|\zeta|.$$

Using g as a contact potential function gives a contact vector field

$$\xi_g = (i\zeta \log|\zeta|, 0) + \left(\frac{i}{2}\zeta, -|\zeta|^2\right)$$

whose flow Ψ satisfies:

$$\Psi_s \circ D_\alpha = D_{e^{is\log(\alpha)}\alpha} \circ \Psi_s.$$

Thus Ψ conjugates a loxodromic action into a purely hyperbolic one.

Corollary 4.2. *Any two loxodromic Hopf manifolds* $\mathscr{H} \setminus \{0\}/\langle D_{\alpha_1}\rangle$ *and* $\mathscr{H} \setminus \{0\}/\langle D_{\alpha_2}\rangle$ *are quasiconformally equivalent.*

5. Parabolic quotients

In this section, it will be more convenient to think of the underlying set of $\mathscr{H}^3$ as $\mathbf{R}^3$ instead of $\mathbf{C} \times \mathbf{R}$. Accordingly, let $T_{(1,0,0)}$ and $T_{(0,0,1)}$ denote left translation by $(1 + 0i, 0)$ and $(0 + 0i, 1)$ in the Heisenberg group. Observe that $T_{(0,0,1)}$ is central while $T_{(1,0,0)}$ is not. Let G_1 denote the cyclic subgroup of $\mathrm{Sim}(\mathscr{H})$ generated by $T_{(1,0,0)}$ and let G_2 be the cyclic subgroup generated by $T_{(0,0,1)}$. Denote by Z_i the cylinder $\mathscr{H}/G_i$. Topologically, $Z_1 \cong Z_2 \cong \mathbf{R}^2 \times S^1$.

Theorem 5.1. *There exists no quasiconformal homeomorphism between* Z_1 *and* Z_2.

The proof depends on global distortion inequalities, and explicit computations for the module of two curve families. Define

$$D_r = B_r \cap \{(x, y, t) : -\tfrac{1}{2} \le x \le \tfrac{1}{2}\}$$

where B_r is the gauge ball of radius r. Next, define Γ_1 to be the set of rectifiable curves in Z_1 homotopic to a generator of $\pi_1(Z_1)$. In particular, a rectifiable curve must be tangent to the contact plane [2]. Since $\{(x, y, t) : -\frac{1}{2} \le x \le \frac{1}{2}\}$ is a fundamental domain for G_1, it is easy to identify D_r with a subset of Z_1. Then define

$$\Gamma_r = \{\gamma \in \Gamma_1 : \gamma \subset D_r\}.$$

That is, Γ_r consists of paths in $\mathscr{H}$ beginning at some point $(-\frac{1}{2}, y, v)$ and ending at $T_{(1,0,0)}(-\frac{1}{2}, y, v)$ and which are contained in the "cylinder" consisting of all the translates of D_r by G_1.

Now suppose a quasiconformal map $f\colon Z_1 \longrightarrow Z_2$ exists. Then f extends to a q.c. map $f\colon \mathscr{H} \longrightarrow \mathscr{H}$ which commutes with the group action. That is:

$$f \circ T_{(1,0,0)} = T_{(0,0,1)} \circ f.$$

Without loss of generality we may assume f has been normalized so that $f(0) = 0$.

Now consider $B_r \subset \mathscr{H}$. By the global distortion estimate in Proposition 3.3, there exists a constant c_0 such that for all r, there is some r' for which

$$B_{r'} \subset f(B_r) \subset B_{c_0 r'}.$$

Suppose r is an integer. Then since

$$f \circ T_{(r,0,0)} = T_{(0,0,r)} \circ f$$

the point $(0,0,r)$ lies on $\partial f(B_r)$. Remembering the inhomogeneity in the definitions of the metric d, it follows that:

$$r' \leq \sqrt{r} \leq c_0 r'. \tag{1}$$

Now we state two propositions which we shall soon prove.

Proposition 5.2. *There exists a constant c_1 such that for all r, $M(\Gamma_r) \geq c_1 r^2$.*

Proposition 5.3. *There exists a constant c_2 such that for all r, $M\big(f(\Gamma_r)\big) \leq c_2 (r')^2$.*

From these statements we easily derive a contradiction. Since f is quasiconformal, there is a constant K such that

$$M(\Gamma_r) \leq K M\big(f(\Gamma_r)\big).$$

From Propositions 5.2 and 5.3, it follows that

$$c_1 r^2 \leq K c_2 (r')^2.$$

On the other hand, from (1), we know that $r' \leq \sqrt{r}$. Thus,

$$r^2 \leq K c_2 / c_1 r$$

for all r which cannot happen. Theorem 5.1 follows.

Figure 1.

Proof of 5.2. Consider the following curves for fixed s, v, where $a = \sqrt{2/\pi}$:

$$\begin{aligned}
\alpha_{(s,v)}(t) &= (-\tfrac{1}{2} + t, s, v + 2st), & t \in [0, \tfrac{1}{2} + a\sqrt{s}\,] \\
\beta_{(s,v)}(t) &= \big(a\sqrt{s}\cos(t), s + a\sqrt{s}\sin(t), v + s(1 + 2a\sqrt{s}\cos(t) - 2a^2 t)\big), & t \in [0, \pi] \\
\gamma_{(s,v)}(t) &= \big(t - a\sqrt{s}, s, v + s(2t - 2a\sqrt{s} - 3)\big), & t \in [0, \tfrac{1}{2} + a\sqrt{s}].
\end{aligned}$$

Set $\eta_{(s,v)} = \alpha \star \beta \star \gamma$. Then $\eta_{(s,v)}$ is an element of Γ_r provided $0 \le s \le \pi/8$ and $-\frac{1}{2}r^2 \le v \le \frac{1}{2}r^2$, and r is large. Moreover, the α, β and γ paths foliate certain regions of D_r, and in fact the mapping $(t, s, v) \mapsto \eta_{(s,v)}(t)$ is a diffeomorphism on these sets. Consequently, these paths define a new coordinate system on a subset of D_r.

More specifically, let A_i denote the regions indicated in Figure 1. Set $B_i = A_i \times [-\frac{1}{2}r^2, \frac{1}{2}r^2]$. Then B_1, B_2 and B_3 are foliated by the α, β and γ curves, respectively. The mapping $(s, t, v) \mapsto \alpha_{(s,v)}(t)$ is a diffeomorphism, and thus defines new coordinates on B_1, and similarly with the other curve families. On B_1 and B_3, one computes in (t, s, v) coordinates that:

$$\mathrm{dvol} = dx \wedge dy \wedge dz = dt \wedge ds \wedge dv.$$

On B_2,

$$\mathrm{dvol} = dx \wedge dy \wedge dz = \big(\tfrac{1}{2}a^2 + a\sqrt{s}\sin(t)\big)\, dt \wedge ds \wedge dv.$$

Also, $|\alpha'_{(s,v)}(t)| = |\gamma'_{(s,v)}(t)| = 1$ and $|\beta'_{(s,v)}(t)| = a\sqrt{s}$, where $|\cdot|$ denotes the gauge length of a tangent vector. (As an aside, this shows that $\eta_{(s,v)}$ has length $O(\sqrt{s}\,)$. This implies the reverse inequality in Proposition 5.2 and shows that $M(D_r; \Gamma_r)$ is $O(r^2)$.)

Now let σ be an admissible function for Γ_r. Observe that trivially,

$$3\int_{D_r}\sigma^4 d\mathrm{vol} \geq 3\int_{\cup B_i}\sigma^4 d\mathrm{vol} \geq \sum\int_{B_i}\sigma^4 d\mathrm{vol}.$$

Now expand each summand as a triple integral:

$$\int_{B_1}\sigma^4\, d\mathrm{vol} = \int_{-\frac{1}{2}r^2}^{\frac{1}{2}r^2} dv \int_0^{\pi/8} ds \int_0^{\frac{1}{2}+a\sqrt{s}} \sigma^4(\alpha_{(s,v)}(t))\, dt.$$

Since $|\alpha'_{(s,v)}(t)| = 1$, Hölder's inequality with $p = 4$ and $q = 4/3$ gives:

$$(\tfrac{1}{2}+a\sqrt{s}\,)^3 \int_0^{\frac{1}{2}+a\sqrt{s}} \sigma^4\big(\alpha_{(s,v)}(t)\big)\, dt \geq \left(\int_0^{\frac{1}{2}+a\sqrt{s}} \sigma\big(\alpha_{(s,v)}(t)\big)|\alpha'_{(s,v)}(t)|\, dt\right)^4.$$

Thus,

$$\int_{B_1}\sigma^4 d\mathrm{vol} \geq \int_{-\frac{1}{2}r^2}^{\frac{1}{2}r^2} dv \int_0^{\pi/8} ds\, (\tfrac{1}{2}+a\sqrt{s}\,)^{-3}\left(\int_0^{\frac{1}{2}+a\sqrt{s}} \sigma\big(\alpha_{(s,v)}(t)\big)|\alpha'_{(s,v)}(t)|\, dt\right)^4.$$

The analogous statement holds for B_3. For B_2, recall that $|\beta'_{(s,v)}(t)| = a\sqrt{s}$ so now Hölder's inequality gives:

$$\left(\int_0^{\pi/8}(a\sqrt{s}\,)^{4/3}\, dt\right)^3\left(\int_0^{\pi/8}\sigma^4\big(\beta_{(s,v)}(t)\big)\, dt\right) \geq \left(\int_0^{\pi/8}\sigma\big(\beta_{(s,v)}(t)\big)|\beta'_{(s,v)}(t)|\, dt\right)^4.$$

Simplifying and further approximating gives:

$$\int_0^{\pi/8}\sigma^4\big(\beta_{(s,v)}(t)\big)\, dt \geq a^{-4}s^{-2}(\pi/8)^{-3}\left(\int_0^{\pi/8}\sigma\big(\beta_{(s,v)}(t)\big)|\beta'_{(s,v)}(t)|\, dt\right)^4.$$

Since $a^{-4}s^{-2}(\pi/8)^{-3} \geq 2^{13}\pi^3 > 1$ we obtain altogether that

$$\begin{aligned}
\int_{D_r}\sigma^4 d\mathrm{vol} &\geq 1/3\int_{-\frac{1}{2}r^2}^{\frac{1}{2}r^2} dv \int_0^{\pi/8}\left(\int_\alpha\sigma\right)^4 + \left(\int_\beta\sigma\right)^4 + \left(\int_\gamma\sigma\right)^4 \\
&\geq 1/9\int_{-\frac{1}{2}r^2}^{\frac{1}{2}r^2} dv \int_0^{\pi/8}\left(\int_\eta\sigma\right)^4 ds \\
&\geq \pi r^2/72
\end{aligned}$$

since by definition $\int_\eta \sigma \geq 1$. The result now follows. □

Proof of 5.3. Since f commutes with the group action, $f(D_r)$ is entirely contained in a fundamental domain, and the "faces" of D_r are mapped into the boundary of a fundamental domain. Since ∂D_r has Lebesgue measure 0, and because quasiconformal mappings are absolutely continuous in measure [3], $m\big(f(\partial D_r)\big) = 0$ as well. Next, we need a lemma.

Lemma 5.4. *Let $L(x,y)$ denote the vertical line passing through $(x,y,0)$. Then for almost every (x,y), $m\big(L(x,y)\cap f(D_r)\big) \le 1$.*

Proof. Since $m\big(f(\partial D_r)\big) = 0$, for almost every (x,y), $L(x,y)\cap f(\partial D_r)$ has measure zero as well, by Fubini's theorem. Thus it suffices to consider $L(x,y)$ intersected with the image of the interior of D_r. Suppose the intersection has measure greater than 1. Since the intersection is open, there must exist a finite, disjoint collection of open intervals contained in the intersection whose measure is also greater than 1. Since $f(D_r)$ is contained in a fundamental domain, this cannot happen. □

Returning to the proof of Proposition 5.3, let l_0 denote the Carnot distance from $(0,0,0)$ to $(0,0,1)$. Since the Carnot metric is invariant under translation, it is clear that $\operatorname{len}(\gamma) \ge l_0$ for every path $\gamma \in f(\Gamma_r)$. It follows that $\sigma = 1/l_0$ is an admissible function for $f(\Gamma_r)$. Consequently, using the lemma, one sees that

$$M\big(f(\Gamma_r)\big) \le \int_{f(D_r)} \frac{1}{l_0}\, d\mathrm{vol} \le \int_{p(f(D_r))} \frac{1}{l_0}\, dx\, dy$$

where p denotes vertical projection onto the xy-plane. Since $f(D_r) \subset B_{c_0 r'}$, certainly $p\big(f(D_r)\big) \subset p(B_{c_0 r'})$. Thus,

$$M\big(f(\Gamma_r)\big) \le \int_{p(f(D_r))} \frac{1}{l_0} \le \frac{\pi c_0^2 (r')^2}{l_0}.$$

This finishes the proof. □

References

[1] Korányi, A., and H.M. Reimann: Quasiconformal mapping on the Heisenberg group. - Inv. Math. 80, 1985, 309–338.

[2] Korányi, A.: Geometric aspects of analysis on the Heisenberg group. - Topics in Modern Harmonic Analysis, Instituto Nazionale di Ala. Mathematica, Roma, 1983.

[3] Korányi, A., and H.M. Reimann: Foundations for the theory of quasiconformal mappings on the Heisenberg group. - To appear.

[4] Pansu, P.: Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. - Ann. Math. 129, 1989, 1–60.

Received 17 December 1992