

Group representations

Exercise sheet 4

<https://www.maths.tcd.ie/~mascotn/teaching/2025/MAU34104/index.html>

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Submit your answers in class or to mismet@tcd.ie by Monday 24 March, 09:00.

Exercise 1 *Group theory thanks to representation theory (40 pts)*

Let G be a non-Abelian group of order 8. Prove that G has exactly 5 conjugacy classes, and that the Abelianisation $G^{\text{ab}} = G/D(G)$ of G must be of order 4.

Hint: This exercise is part of an assignment on group representations!

Solution 1

We know that the number of conjugacy classes agrees with the number of irreducible representations, and that

$$\sum_{\chi \in \text{Irr}(G)} (\deg \chi)^2 = \#G = 8.$$

Furthermore, since G is not Abelian, we must have $\deg \chi \geq 2$ for at least one $\chi \in \text{Irr}(G)$; and in fact it must be $\deg \chi = 2$ exactly since $3^2 > 8$. On the other hand, the sum must also contain a term 1^2 because of the trivial character.

So we look for decompositions of 8 as sums of squares, where at least one term is 1^2 and at least one term is 2^2 . Removing these terms, we reduce the problem to finding decompositions of $8 - 1^2 - 2^2 = 3$ as sums of squares; and since $2^2 > 3$, the only possibility is $3 = 1^2 + 1^2 + 1^2$, whence $8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$. That's 5 terms, so we have proved that necessarily $\#\text{Irr}(G) = 5$, whence the first part of result.

Next, we also have

$$\#G^{\text{ab}} = \sum_{\chi \in \text{Irr}(G^{\text{ab}})} (\deg \chi)^2.$$

But as G^{ab} is Abelian, all the terms are 1^2 , so $\#G^{\text{ab}} = \#\chi \in \text{Irr}(G^{\text{ab}})$. Finally, we know that the irreducible representations of G^{ab} correspond bijectively to the irreducible representations of G of degree 1, so $\#G^{\text{ab}} = 4$ by the above.

Exercise 2 *The character table of A_4 (60 pts)*

Let $G = A_4$ be the alternating group of even permutations on 4 objects.

1. (12 pts) Let V_4 be the Klein subgroup of A_4 consisting of the double transpositions and of the identity. Prove that V_4 is normal in A_4 , and that A_4/V_4 is cyclic.

2. (6 pts) Prove that (123) and (132) are **not** conjugate in A_4 .
3. (30 pts) Determine the character table of A_4 .
You may want to define $\omega = e^{2\pi i/3}$; note that $\omega^2 = \bar{\omega} = -\omega - 1$.
4. (6 pts) Deduce that V_4 is the derived subgroup of A_4 .
5. (6 pts) Determine the decomposition into irreducible representations of the restriction to A_4 of each the five irreducible representations of S_4 .
6. (Bonus question, 0 pts) We admit that the group of rotations of \mathbb{R}^3 that preserve a regular tetrahedron invariant is isomorphic to A_4 via the permutations induced on the 4 vertices, whence a representation of A_4 of degree 3. Write down the decomposition (over \mathbb{C}) of this representation into irreducible representations.

Solution 2

1. By the conjugacy principle, conjugacy in S_n preserves cycle decomposition patterns. Therefore, V_4 is normal in S_4 , and *a fortiori* in A_4 . The quotient A_4/V_4 has order $\#A_4/\#V_4 = 12/4 = 3$, which is prime; therefore any non-trivial element has order 3 by Lagrange, and is therefore a generator.
2. The conjugacy principle shows that (123) and (132) are conjugate in S_4 by any permutation satisfying $1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2$. In fact, there exists a unique such permutation, since these identities force $4 \mapsto 4$. However, this permutation, which is a transposition, does not lie in A_4 . The only way to remedy this would be to write (132) as (321) or as (213), but this does not actually fix the problem (because we are cyclically permuting 1, 3, 2, and thus changing the conjugating permutation by a 3-cycle, which is even, and therefore does not affect its odd-ness).
3. By playing around with the conjugacy principle (and taking care to only conjugate by even permutations, namely 3-cycles and double transpositions), we find that the conjugacy classes of A_4 are

$$\{\text{Id}\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (214), (341), (432)\}, \{(132), (241), (431), (342)\}.$$

That is 4 of them, so we have 4 irreducible representations. Their degrees $n_1 \leq \dots \leq n_4$ satisfy $n_1 = 1$ (by $\mathbb{1}$), $n_4 \geq 2$ (as A_4 is not Abelian), and $\sum_{i=1}^4 n_i^2 = \#A_4 = 12$, whence $n_1 = n_2 = n_3 = 1, n_4 = 3$.

Pulling back the irreducible representations of the quotient $A_4/V_4 \simeq \mathbb{Z}/3\mathbb{Z}$ gives us the 3 characters of degree 1, which are still irreducible (because they have degree 1, or alternatively, because the projection morphism is surjective):

Class	Id	(12)(34)	(123)	(132)
#	1	3	4	4
$\mathbb{1}$	1	1	1	1
ϕ	1	1	ω	$\bar{\omega}$
$\bar{\phi}$	1	1	$\bar{\omega}$	ω
χ	3	?	?	?

and we can find the missing character χ by the fact that the character of the regular representation $12\ 0\ 0\ 0$ is $\mathbb{1} + \phi + \bar{\phi} + 3\chi$, or by invoking exercise 2 of assignment 3, which implies that the permutation representation corresponding to the action of A_4 on 4 objects decomposes as the direct sum of $\mathbb{1}$ and of an irreducible representation of degree 3:

Class	Id	(12)(34)	(123)	(132)
#	1	3	4	4
$\mathbb{1}$	1	1	1	1
ϕ	1	1	ω	$\bar{\omega}$
$\bar{\phi}$	1	1	$\bar{\omega}$	ω
ρ	3	-1	0	0

4. We know that

$$D(A_4) = \bigcap_{\substack{\chi \in \text{Irr}(A_4) \\ \deg \chi = 1}} \text{Ker } \chi = \bigcap_{\substack{\chi \in \text{Irr}(A_4) \\ \deg \chi = 1}} \{g \in A_4 \mid \chi(g) = 1\}.$$

The character table of A_4 shows that this is V_4 .

5. Recall the character table of S_4 :

Class	Id	(12)	(123)	(1234)	(12)(34)
$\mathbb{1}_{S_4}$	1	1	1	1	1
ε	1	-1	1	-1	1
ψ	2	0	-1	0	2
χ	3	1	0	-1	-1
$\chi\varepsilon$	3	-1	0	1	-1

Restricting these characters to A_4 , we find

Class	Id	(12)(34)	(123)	(132)
#	1	3	4	4
$\mathbb{1}_{S_4}$	1	1	1	1
ε	1	1	1	1
ψ	2	2	-1	-1
χ	3	-1	0	0
$\chi\varepsilon$	3	-1	0	0

from which we immediately see that $\mathbb{1}_{S_4}$ and ε restrict to $\mathbb{1}$, and that χ and $\chi\varepsilon$ restrict to ρ (unsurprisingly, ε becomes trivial on A_4 ...). We also determine (either by inner products or by spotting this directly from $\omega + \bar{\omega} = -1$) that ψ decomposes as $\phi \oplus \bar{\phi}$.

Note: This can serve as a reminder of the fact that the restriction of an irreducible representation may no longer be irreducible.

6. Double transpositions of the vertices may be obtained by rotating the tetrahedron by 180° about a line joining perpendicularly the midpoints of two opposite edges. In a suitable basis, such a rotation has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

whence trace $1 - 1 - 1 = -1$. 3-cycles are obtained by rotating the tetrahedron by 120° about a line joining a vertex to the midpoint of the opposite face. In a suitable basis, such a rotation has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(120^\circ) & -\sin(120^\circ) \\ 0 & \sin(120^\circ) & \cos(120^\circ) \end{pmatrix},$$

whence trace $1 + 2 \cos(120^\circ) = 0$.

The character of this tetrahedron representation of A_4 is thus $3, -1, 0, 0$, so this representation is isomorphic to ρ .

Alternatively, we could argue from the degrees that the only possible decompositions are $\mathbb{1} + \phi + \bar{\phi}$ and ρ . However, the former has nontrivial kernel $\text{Ker } \mathbb{1} \cap \text{Ker } \phi \cap \text{Ker } \bar{\phi} = \bigcap_{\deg \chi=1} \text{Ker } \chi = D(A_4)$, so we can rule it out as this tetrahedron representation is faithful by construction.

Note: Knowing this tetrahedron interpretation of A_4 in advance was yet another way to fill in the character table of A_4 .

These were the only mandatory exercises, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* suggest you can try to solve them, as this is *excellent practice for the exam*. You are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 3 *The character table of $C_2 \times C_2$*

In this exercise, we write¹ C_2 for the group $\mathbb{Z}/2\mathbb{Z}$, and we consider representations of the group

$$G = C_2 \times C_2 = \{(x, y) \mid x \in C_2, y \in C_2\}.$$

1. Let χ be an irreducible character of G . Prove that $\chi(g) \in \{+1, -1\}$ for all $g \in G$.
2. Write down the character table of G .
3. Is any of the irreducible representations of G faithful?
4. Does there exist faithful representations of G ? If so, what is the smallest possible degree of such a representation?

¹For clarity, I personally prefer to write C_n for the group, $\mathbb{Z}/n\mathbb{Z}$ for the ring, and, if n is prime, \mathbb{F}_n for the field.

Solution 3

1. First of all, G is Abelian. As χ is irreducible by assumption, this forces $\deg \chi = 1$, so χ is actually a group morphism $G \rightarrow \mathbb{C}^\times$. But $g^2 = 1_G$ for all $g \in G$, whence

$$\chi(g)^2 = \chi(g^2) = \chi(1_G) = 1.$$

2. Again, G is Abelian, so it has $\#G = 4$ conjugacy classes and therefore 4 irreducible characters. Bearing in mind that these characters happen to be morphisms (because degree 1) to ± 1 (by the previous question), we find

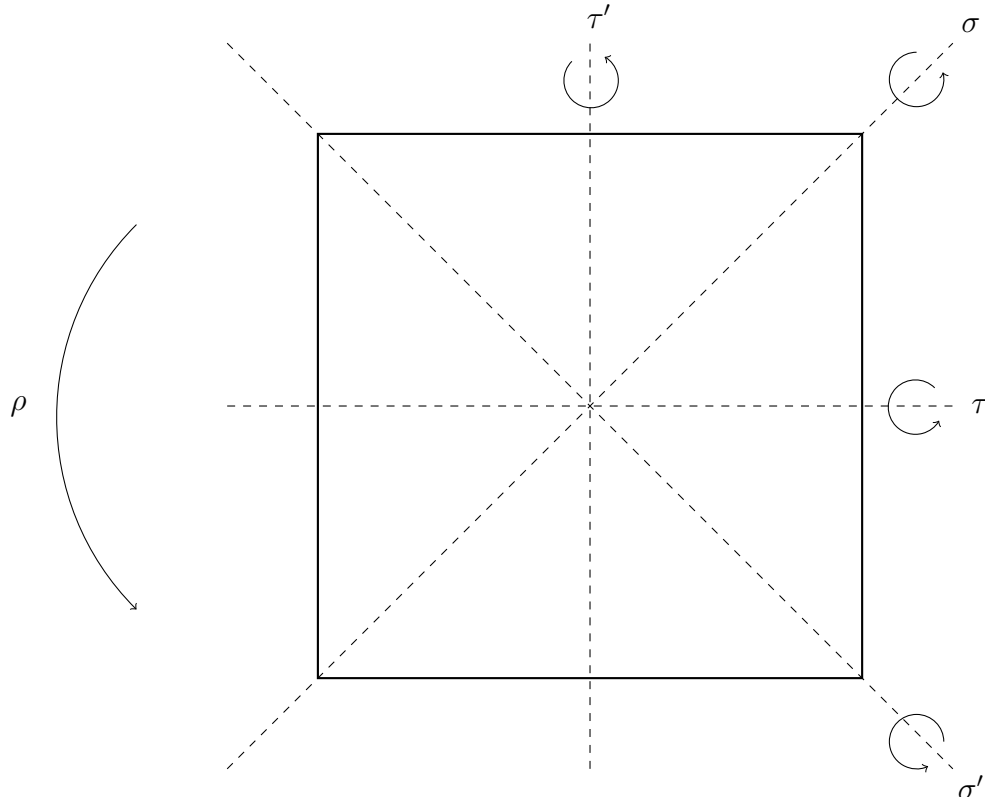
	(1, 1)	(h, 1)	(1, h)	(h, h)
$\mathbb{1}$	1	1	1	1
χ_L	1	-1	1	-1
χ_R	1	1	-1	-1
χ_D	1	-1	-1	1

where we have written $C_2 = \{1, h\}$. (We could also have constructed χ_L and χ_R by inflating the nontrivial irreducible character of C_2 by the left and right projections of G onto C_2 , and χ_D by inflating from G/D where D is the diagonal subgroup $\{(x, x) \mid x \in C_2\}$. We could also have invoked the fact that a product of characters is a character.)

3. No. More specifically, we see from the character table that $\text{Ker } \mathbb{1} = G$, $\text{Ker } \chi_L = \{(1, x) \mid x \in C_2\}$, $\text{Ker } \chi_R = \{(x, 1) \mid x \in C_2\}$, and $\text{Ker } \chi_D = D$.

Exercise 4 The character table of D_8

1. Let $G = D_8$ be the group of symmetries of the square, most of whose elements we name as follows:



Determine the derived subgroup $D(G)$ and the structure of the quotient $G/D(G)$.

- Determine the character table of D_8 .

Solution 4

- By Exercise 1, $D(G)$ is a subgroup of G of index 4, and therefore of order 2, so it contains only one non-trivial element. We observe (e.g. thanks to the “conjugacy principle”) that

$$\tau\rho\tau^{-1}\rho^{-1} = (\tau\rho\tau^{-1}\rho^{-1})\rho^{-1} = \rho^{-1}\rho^{-1} = \rho^2 \neq \text{Id},$$

whence $D(G) = \{\text{Id}, \rho^2\}$.

The quotient $G/D(G)$ is Abelian by definition of $D(G)$, and of order 4, so it is isomorphic to either C_4 or $C_2 \times C_2$ (where C_n denotes the cyclic group of order n , which is often written $\mathbb{Z}/n\mathbb{Z}$). But every element of $G/D(G)$ has order at most 2, since the only elements of order > 2 in G are ρ and ρ^{-1} , which become of order 2 in $G/D(G)$ as $\rho^2 = \rho^{-2} \in D(G)$. Therefore

$$G/D(G) \simeq C_2 \times C_2.$$

- We have shown in the lectures that the conjugacy classes of G are

$$\{\text{Id}\}, \{\rho^2\}, \{\rho, \rho^{-1}\}, \{\sigma, \sigma'\}, \{\tau, \tau'\}.$$

Conjugates in G become equal in the Abelian quotient $G/D(G)$, so

$$G/D(G) = \{\text{Id} = \rho^2, \sigma = \sigma', \tau = \tau', \rho = \rho^{-1}\}.$$

We get 4 irreducible characters of degree 1 (including $\mathbb{1}$) of G by inflating those of $G/D(G) \simeq C_2 \times C_2$, which we determined in Exercise 3:

Class	Id	ρ^2	ρ	σ	τ
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1

We know that $\#\text{Irr}(G) = \#\text{conj. classes} = 5$, so there is one missing irreducible character, whose degree is 2 according to the first question. We can find this missing character χ by using the fact that the regular representation, whose character is 8 0 0 0 0, decomposes as $\mathbb{1} + \phi + \psi + \phi\psi + 2\chi$, or by checking that the degree 2 representation of G afforded by G acting on a square is irreducible since the dot product of its character with itself is 1.

Class	Id	ρ^2	ρ	σ	τ
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1
χ	2	-2	0	0	0

Exercise 5 *The character table of Q_8*

Let $Q_8 = \{1, -1, I, -I, J, -J, K, -K\}$ be the (Hamiltonian) quaternionic group, whose multiplication is defined by the rules

$$\text{For all } x, y \in Q_8, (-x)y = x(-y) = -(xy), \quad x1 = 1x = x,$$

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = K = -JI, \quad JK = I = -KJ, \quad KI = J = -IK.$$

1. By Exercise 1, Q_8 has exactly 5 conjugacy classes. Check that these classes are $\{1\}$, $\{-1\}$, $\{I, -I\}$, $\{J, -J\}$, and $\{K, -K\}$.
2. Determine the centre Z of Q_8 .
3. Prove that Q_8/Z is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
4. Determine the character table of Q_8 . Any comments?
5. It is standard to realise Q_8 as a group of 2×2 complex matrices by identifying I with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, J with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and K with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ (since these matrices satisfy the relations defining the group law of Q_8 , as you may check if you wish). How would you interpret this in terms of the character table of Q_8 ?

Solution 5

1. Since we know that there must be exactly 5 conjugacy classes, it suffices to check that $-I$ is conjugate to I , that $-J$ is conjugate to J , and that K is conjugate to $-K$. This follows from $JIJ^{-1} = JI(-J) = (-K)(-J) = KJ = -I$, and similarly for J and K .
2. The centre is made up of the elements that sit alone in their conjugacy class, so $Z = \{1, -1\}$.
3. Since $Z = \{1, -1\}$, we have $Q_8/Z = \{\pm 1, \pm I, \pm J, \pm K\}$ which is now Abelian, and $(\pm I)^2 = (\pm J)^2 = (\pm K)^2 = -1 = \pm 1$ so all the nontrivial elements of Q_8/Z have order 2. It follows that we have an isomorphism

$$\begin{aligned} (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) &\longrightarrow Q_8/Z \\ (a, b) &\longmapsto \pm I^a J^b. \end{aligned}$$

4. We can inflate the 4 irreducible characters of degree 1 of $Q_8/Z \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ to characters of Q_8 (which are still irreducible, because they have degree 1, or alternatively because the projection map is surjective):

Class	1	-1	$\pm I$	$\pm J$	$\pm K$
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1

Since there are 5 conjugacy classes, we are missing one character, which is of degree 2 by virtue of $\sum_{\chi \in \text{Irr}(Q_8)} (\deg \chi)^2 = \#Q_8$, and whose values we can recover from the character of the regular representation, or alternatively, by orthogonality of columns:

Class	1	-1	$\pm I$	$\pm J$	$\pm K$
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1
χ	2	-2	0	0	0

And... this is the same table as for D_8 ! And yet, the groups D_8 and Q_8 are NOT isomorphic, e.g. because D_8 has 5 elements of order 2 and 2 of order 4 whereas Q_8 has 6 elements of order 4 and 2 of order 2. This shows that two groups can have the same character table without being isomorphic!

NB the fact that D_8 and Q_8 have the same character table is explained by Exercise 1 plus the fact that the final character of degree 2 can be determined from the character of the regular representation.

5. This realisation of Q_8 by complex matrices is by definition a faithful representation of Q_8 of degree 2. By looking at the trace of the matrices, we see that this representation has character χ , so it is the unique irreducible representation of Q_8 of degree 2.

Exercise 6 *To be or not to be real*

Nicolas M., lecturer at a college somewhere in Europe, has been trying to hammer into his students that they must not write things such as

$$(\chi | \chi) = \frac{1}{\#G} \sum_{g \in G} \chi(g)^2$$

as this is incorrect since characters are complex-valued, the correct formula being

$$(\chi | \chi) = \frac{1}{\#G} \sum_{g \in G} |\chi(g)|^2.$$

However, he admits that it may be misleading that most of the characters in his lectures were real-valued. But how come?

1. Let G be a finite group. Prove that every character of G is real-valued if and only if every $g \in G$ is conjugate to its inverse.
Hint: Recall that $\chi(g^{-1}) = \overline{\chi(g)}$.
2. Let $n \in \mathbb{N}$. Prove that every character of S_n is real-valued. What about A_4 ?

Solution 6

1. The formula $\chi(g^{-1}) = \overline{\chi(g)}$ was proved in the lectures, when we proved that $\rho(g)$ is always diagonalisable with eigenvalues which are $\#G$ -th roots of 1.

Suppose that g is conjugate to g^{-1} for all $g \in G$. Then for all χ ,

$$\overline{\chi(g)} = \chi(g^{-1}) = \chi(g)$$

since χ is a class function. It follows that χ is real-valued.

Conversely, suppose that every χ is real-valued, and let $g \in G$. Then

$$\sum_{\chi \in \text{Irr}(G)} \chi(g) \overline{\chi(g^{-1})} = \sum_{\chi \in \text{Irr}(G)} \chi(g)^2$$

must be a nonnegative real number, which cannot be zero because of the presence of $\mathbb{1} \in \text{Irr}(G)$. By the second orthogonality relations, this proves that g and g^{-1} are conjugate.

Note: This is an avatar of a more general phenomenon, which results from the application of the Galois theory of cyclotomic fields to character values; see for example the chapter on group representations in J.-P. Serre's book on finite groups.

2. Let $g \in S_n$. Then g^{-1} has the same cycle decomposition shape as g (with the order inside each of the cycles reversed), so g and g^{-1} are conjugate in S_n . It follows that the characters of S_n are real-valued.

On the other hand, question 2. of the first exercise explains why some of the characters of A_4 are not real-valued.