Group representations
Exercise sheet 1

https://www.maths.tcd.ie/~mascotn/teaching/2025/MAU34104/index.html

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Submit your answers in class or to mismet@tcd.ie by Wednesday February 12 noon.

Exercise 1 *(Sub)representations of degree 1 (60 pts)*

Let G be a group, K a field, and $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of G over K.

1. (10 pts) Prove that if ρ is of degree 1, then ρ is irreducible and indecomposable.

2. (20 pts) We no longer assume that $\deg \rho = 1$. Let $W \subseteq V$ be a subspace of dimension 1, so that $W = Kv = \{\lambda v \mid \lambda \in K\}$ for some $0 \neq v \in V$. Prove that W is a subrepresentation of V if and only if v is an eigenvector of $\rho(g)$ for all $g \in G$.

3. (30 pts) Suppose finally that $K = \mathbb{C}$, that G is Abelian, and that $\dim V$ is finite and nonzero (but not necessarily 1). Prove that V admits at least one subrepresentation of degree 1.

   Hint: Prove that if $T, U : V \longrightarrow V$ are linear transformations which commute with each other, then for each eigenvalue λ of T, the λ-eigensubspace of T is stable by U. Then induct on $\dim V$. You may want to treat separately the case where all the $\rho(g)$ act as scalar matrices.

Solution 1

1. ρ being a representation of degree 1 means that $\dim V = 1$, so V has no subspaces apart from $\{0\}$ and itself. In particular, ρ does not have any subrepresentations apart from $\{0\}$ and V, which proves that ρ is irreducible.

   Since irreducibility always implies indecomposability, it follows that ρ is also indecomposable.

2. If W is a subrepresentation, then it is stable by the $\rho(g)$ for all $g \in G$. In particular, for all $g \in G$, we have $\rho(g)(v) \in W$ since $v \in W$, so $\rho(g)(v) = \mu v$ for some $\mu \in K$ since $W = \{\lambda v \mid \lambda \in K\}$ (but note that μ will, in general, depend on g). Thus v is an eigenvector of $\rho(g)$.

   Conversely, suppose that v is a common eigenvector of all the $\rho(g)$ say $\rho(g)(v) = \mu_g v$ for all $g \in G$, and let $w \in W$. Then $w = \lambda v$ for some $\lambda \in K$, so for all $g \in G$, $\rho(g)(w) = \lambda \rho(g)(v) = \lambda \mu_g v \in W$ since $\rho(g)$ is linear. Thus the subspace W is stable under all the $\rho(g)$, and is thus a subrepresentation.

3. We induct on $\dim V$. If $\dim V = 1$, we can just take the subrepresentation to be V itself, so we are done.

   Now suppose $\dim V > 1$. If all the $\rho(g)$ act as scalars V, then any nonzero $v \in V$ is a common eigenvector of the $\rho(g)$, so we are done. Otherwise, pick

$g_0 \in G$ such that $\rho(g_0)$ does not act on V as a scalar. Since we are working over $K = \mathbb{C}$, the characteristic polynomial of $\rho(g_0)$ (which exists since $\dim V$ is finite) has at least one root $\lambda \in K = \mathbb{C}$. Let $W = \operatorname{Ker}(\rho(g_0) - \lambda \operatorname{Id}_V) = \{v \in V \mid \rho(g_0)(v) = \lambda v\}$ be the corresponding eigensubspace. Then W is nonzero since λ is an eigenvalue of $\rho(g_0)$, and W is not the whole of V either since $\rho(g_0)$ does not act as a scalar by assumption. So $0 < \dim W < \dim V$.

We are going to prove that W is in fact a subrepresentation. This will conclude the proof, since the induction hypothesis applied to W will provide us with a sub-subrepresentation of degree 1. So let $g \in G$ and $v \in W$; we must prove that $\rho(g)(v) \in W$ still. Using the fact that $\rho(g_0)(v) = \lambda v$, that ρ is a group morphism, that $gg_0 = g_0 g$ as G is Abelian, and that $\rho(g)$ is linear, we find that

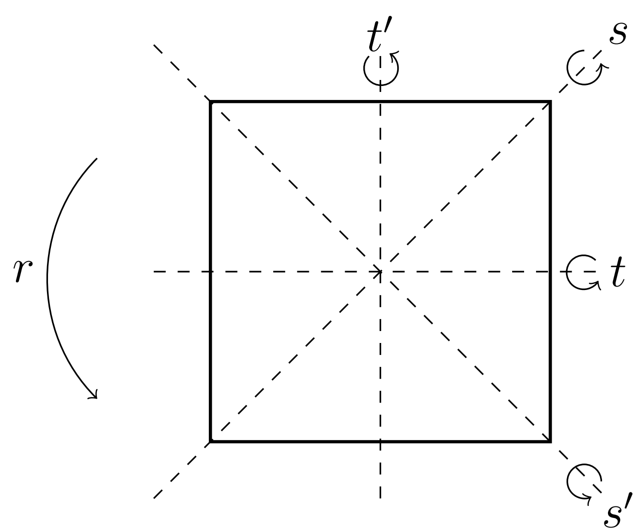
$$\rho(g_0)\big(\rho(g)(v)\big) = \rho(g_0 g)(v) = \rho(g g_0)(v) = \rho(g)\big(\rho(g_0)(v)\big) = \rho(g)(\lambda v) = \lambda \rho(g)(v),$$

which shows that $\rho(g)(v)$ also lies in the W, the λ-eigenspace of $\rho(g_0)$, so we are done.

Remark: This shows that over $\mathbb{C}$, representations of Abelian groups are never irreducible if they are not of degree 1. We will see later that if G is finite, then every representation of G over $\mathbb{C}$ actually decomposes into a direct sum of representations of degree 1.

Exercise 2 *The dihedral group D_8 (40 pts)*

Let $G = D_8$ be the group of transformations of the plane $\mathbb{R}^2$ that preserve a square. It consists of 8 elements: Id, the rotations r (by $\pi/2$), r^2, r^3, (note $r^4 = \text{Id}$), and the reflections s, s' and t, t', as shown on the figure below:



It thus comes naturally with a representation $\rho : D_8 \longrightarrow \mathrm{GL}(\mathbb{R}^2)$ of D_8 of degree 2 over $\mathbb{R}$.

1. (5 pts) For each of the 8 elements $g \in D_8$, write down the matrix of $\rho(g)$ with respect to the (fixed) basis of $\mathbb{R}^2$ of your choice (specify which basis you pick).

2. (10 pts) Deduce that ρ is faithful.

3. (15 pts) Use the results of Exercise 1 to prove that ρ is irreducible.
4. (10 pts) We can also view ρ as a representation $\rho_{\mathbb{C}}$ over $\mathbb{C}$ instead of $\mathbb{R}$ (keep the same matrices, but view them as complex matrices rather than real ones). Is $\rho_{\mathbb{C}}$ irreducible, i.e. is ρ still irreducible over $\mathbb{C}$?

   Hint: Use the results of Exercise 1 again, but work slightly harder.

Solution 2

1. With respect to the standard basis of $\mathbb{R}^2$ (e_1 pointing to the right, e_2 pointing up, same magnitude as e_1), we have

$$\rho(\mathrm{Id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(r^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(r^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(t') = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(s') = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

2. We see from the previous question that the $\rho(g)$ are all distinct. (Even better: we see that $\operatorname{Ker} \rho$ is reduced to Id.)
3. Since ρ is of degree 2, any non-trivial (i.e. neither 0 nor the whole thing) subrepresentation must be of degree 1. By Exercise 1, finding such a representation thus amounts to finding a common eigenvector of the $\rho(g)$ for $g \in D_8$. However, the rotation $\rho(r)$ has clearly no eigenlines (remember we are working over $\mathbb{R}$!), so no such subrepresentation can exist, so ρ is irreducible (and therefore indecomposable).
4. Over $\mathbb{C}$, the argument that $\rho(r)$ has no eigenlines breaks down. In fact, every $\rho(g)$ has eigenlines over $\mathbb{C}$ (but in the case of $g = r$, they are imaginary, so we could not use them over $\mathbb{R}$). However, as explained in the previous question, for a subrepresentation to exist we would need a *common* eigenline for all the $\rho(g)$. Now, $\rho(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ only admits the horizontal and vertical axes as eigenlines (these are its eigenspaces for its respective eigenvalues ± 1), but neither of these axes is an eigenline for $\rho(s)$ for example (and also not for $\rho(r)$, whose eigenlines are not real as we have already mentioned). So again, there are no common eigenlines, so ρ is still irreducible over $\mathbb{C}$.

   Remark: This proves that the assumption that G is Abelian cannot be removed in the last question of Exercise 1.

These were the only mandatory exercises, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* recommend you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 3 *Representation morphisms form a subspace*

Let G be a group, K a field, and let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two representations of G over K.

Recall that the set $\mathrm{Hom}(V_1, V_2)$ of all linear transformations from V_1 to V_2 has a vector space structure, with addition and scalar multiplications defined pointwise (that is to say if $T, U \in \mathrm{Hom}_(V_1, V_2)$ and $\lambda \in K$, then $T + U$ is defined as $(T + U)(v_1) = T(v_1) + U(v_1)$ for all $v_1 \in V_1$, and λT is defined as $(\lambda T)(v_1) = \lambda\big(T(v_1)\big)$ for all $v_1 \in V_1$).

Prove that the subset $\mathrm{Hom}_G(V_1, V_2)$ consisting of linear transformations which are representation morphisms is a subspace of $\mathrm{Hom}(V_1, V_2)$.

Solution 3

We must prove that whenever $T, U \in \mathrm{Hom}_G(V_1, V_2)$ and $\lambda \in K$, $T + U$ and λT also lie in $\mathrm{Hom}_G(V_1, V_2)$.

First of all, recall that $\mathrm{Hom}_G(V_1, V_2)$ is defined as the set of $T \in \mathrm{Hom}(V_1, V_2)$ such that for all $g \in G$ and $v_1 \in V_1$,

$$T\big(\rho_1(g)(v_1)\big) = \rho_2(g)\big(T(v_1)\big).$$

So let $T, U \in \mathrm{Hom}_G(V_1, V_2)$, and let $v_1 \in V_1$ and $g \in G$. Then

$$\begin{aligned}
(T + U)\big(\rho_1(g)(v_1)\big) &= T\big(\rho_1(g)(v_1)\big) + U\big(\rho_1(g)(v_1)\big) \text{ by definition of } T + U \\
&= \rho_2(g)\big(T(v_1)\big) + \rho_2(g)\big(U(v_1)\big) \text{ as } T, U \in \mathrm{Hom}_G(V_1, V_2) \\
&= \rho_2(g)\big(T(v_1) + U(v_1)\big) \text{ as } \rho_2(g) : V_2 \to V_2 \text{ is linear for all } g \in G \\
&= \rho_2(g)\big((T + U)(v_1)\big) \text{ by definition of } T + U,
\end{aligned}$$

which proves that $T + U \in \mathrm{Hom}_G(V_1, V_2)$. Similarly, if $\lambda \in K$, $T \in \mathrm{Hom}_G(V_1, V_2)$, $v_1 \in V_1$, and $g \in G$, then

$$\begin{aligned}
(\lambda T)\big(\rho_1(g)(v_1)\big) &= \lambda\Big(T\big(\rho_1(g)(v_1)\big)\Big) \text{ by definition of } \lambda T \\
&= \lambda\Big(\rho_2(g)\big(T(v_1)\big)\Big) \text{ as } T \in \mathrm{Hom}_G(V_1, V_2) \\
&= \rho_2(g)\Big(\lambda\big(T(v_1)\big)\Big) \text{ as } \rho_2(g) : V_2 \to V_2 \text{ is linear for all } g \in G \\
&= \rho_2(g)\big((\lambda T)(v_1)\big) \text{ by definition of } \lambda T.
\end{aligned}$$

Exercise 4 *Decomposition over* $\mathbb{R}$

In this exercise, we consider over $K = \mathbb{R}$ two representations of $G = S_3$ constructed in the lectures, namely the permutation representation

$$\mathrm{Perm} : S_3 \longrightarrow \mathrm{GL}_3(K)$$

induced by $S_3 \circlearrowright \{1, 2, 3\}$, and

$$\triangleleft : S_3 \longrightarrow \mathrm{GL}_2(K), \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

obtained by labelling the vertices of an equilateral triangle by $\{1, 2, 3\}$.

1. Prove that $\triangleleft$ is irreducible. Is $\triangleleft$ indecomposable?
2. Prove that $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$ as representations of S_3.

Solution 4

1. Since $\deg \triangleleft = 2$, if it were reducible, then it would have a subrepresentation of degree 1.

   We know that $(123) \in S_3$ acts on the representation space $\mathbb{R}^2$ of $\triangleleft$ by a rotation of angle $2\pi/3$; therefore, it does not stabilise any vector line, so $\triangleleft$ is irreducible.

   Here is a more algebraic[1] way to say this: by Exercise 4, a subrepresentation of $\triangleleft$ of degree 1 would be spanned by a common eigenvector for all $g \in S_3$, yet the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ of (123) has characteristic polynomial $x^2 + x + 1$, which has negative discriminant, so (123) has no eigenvector over $\mathbb{R}$.

   Here is another proof, which shows that $\triangleleft$ is actually irreducible even over $\mathbb{C}$: the matrix $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ clearly has eigenvalues 1 and -1, with respective eigenvectors $v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_- = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; but since since neither is an eigenvector of (123), there is no common eigenvector for all S_3.

   Anyway, we conclude that $\triangleleft$ is irreducible, and therefore also indecomposable.

2.
   - Analysis: Let us define $\sigma = (123)$, $\tau = (12)$. The representation space of $\mathbb{1}$ is $\mathbb{R}$, with trivial action of S_3; let u be a vector which spans it. The representation space of $\triangleleft$ is $\mathbb{R}^2$; let (e_1, e_2) be its standard basis, on which the matrix of σ is $S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and that of τ is $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Finally, the representation space of Perm is $\mathbb{R}^3$, on which S_3 acts by permuting the coordinates.

     Suppose we have a representation morphism $f : \mathbb{1} \oplus \triangleleft \longrightarrow \mathrm{Perm}$, and define $v = f(u)$, $f_1 = f(e_1)$, $f_2 = f(e_2) \in \mathbb{R}^3$.

[1] I do not mean to imply that this more algebraic way is better. In fact, I personally find the geometric argument more convincing!

Since $gu = u$ for all $g \in S_3$, we must also have $gv = v$ for all $g \in S_3$; therefore v is a multiple of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

We also observe that $\tau e_1 = e_1$, so $\tau f_1 = f_1$, which means that $f_1 = \begin{pmatrix} a \\ a \\ b \end{pmatrix}$ for some $a, b \in \mathbb{R}$. Furthermore, we have $\sigma e_2 = -e_1$, so $e_2 = -\sigma^{-1} e_1$, whence $f_2 = -\sigma^{-1} f_1 = \begin{pmatrix} -a \\ -b \\ -a \end{pmatrix}$. Besides, from the fact that T is triangular, we see that it has the eigenvalue -1, and we compute that the corresponding eigenspace is the span of $e_1 - 2e_2$, so that $\tau(e_1 - 2e_2) = -(e_1 - 2e_2)$ whence $\begin{pmatrix} a+2b \\ 3a \\ b+2a \end{pmatrix} = \tau(f_1 - 2f_2) = -(f_1 - 2f_2) = \begin{pmatrix} -3a \\ -a-2b \\ -b-2a \end{pmatrix}$; this forces $b = -2a$.

- Synthesis: Let $f : \mathbb{1} \oplus \triangleleft \longrightarrow \mathrm{Perm}$ be the linear transformation defined by

$$f(u) = v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad f(e_1) = f_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad f(e_2) = f_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Since $\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 9 \neq 0 \in \mathbb{R}$, we see that this linear transformation is invertible. It remains to check if it is a representation morphism, that is to say if $f(gw) = gf(w)$ for all $w \in \mathbb{1} \oplus \triangleleft$ and $g \in S_3$. Clearly, it is sufficient to check this for w ranging over the basis (u, e_1, e_2) of $\mathbb{1} \oplus \triangleleft$, and also for $g = \sigma$ and τ since σ and τ generate S_3.

By construction, we have $f(gu) = f(u) = v = gv$ for all $g \in S_3$.

We also have $f(\tau e_1) = f(e_1) = f_1 = \tau f_1$ by construction, and we check that $f(\sigma e_1) = f(-e_1 + e_2) = -f_1 + f_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \sigma f_1$.

And finally, we check that $f(\tau e_2) = f(e_1 - e_2) = f_1 - f_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \tau f_2$, and we have $f(\sigma e_2) = f(-e_1) = -f_1 = \sigma f_2$ by construction.

In conclusion, the answer is **yes**, and we have exhibited an isomorphism of representations $f : \mathbb{1} \oplus \triangleleft \simeq \mathrm{Perm}$.

Remark: we can summarise the verification calculations that we have just performed by saying that the respective matrices of σ and τ on the basis (v, f_1, f_2) of $\mathbb{R}^3$ are

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

This way, we really "see" the representation isomorphism and the decomposition of Perm *at work.*

You may also have noticed that we had no special reason to take $a = 1$, nor to decide that v is not another multiple of $(1, 1, 1)$. Indeed, any other choice also "works". This demonstrates the existence of automorphisms of the representation $\mathbb{1} \oplus \triangleleft \simeq$ Perm (and it can be shown that this is actually all of its automorphisms).

Finally, we note that it is possible to "see" the decomposition Perm $\simeq \mathbb{1} \oplus \triangleleft$ purely geometrically: imagine an orthonormal frame in $\mathbb{R}^3$ with 3 mutually orthogonal unit vectors shooting from the origin, and a tilted plane resting on the tip of these 3 vectors. Drawing line segments joining these 3 vector tips, we get an equilateral triangle in this plane; and the line spanned by the vector $(1, 1, 1)$ goes through the origin and the centre of this triangle, where it shoots perpendicularly through this plane. As S_3 permutes these 3 vectors (by definition of Perm*), the vertices of the triangle are also permuted, but the plane remains globally invariant, whence the subrepresentation $\triangleleft$, whereas the points of the line spanned by $(1, 1, 1)$ remain all fixed, whence the subrepresentation $\mathbb{1}$.*

The following exercise has been included for those of you who wish to try their skills in more "exotic" situations. It is not meant to be as profitable for your understanding of the material of this module as the previous exercises. You are still welcome to try to solve it for practice, and to email me if you have questions about it. The solution will be also made available with the solution to the other exercises.

Exercise 5 *Decomposition over $\mathbb{Z}/p\mathbb{Z}$*

Redo Exercise 4, but with $K = \mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{R}$, where $p \in \mathbb{N}$ is prime. Is $\triangleleft$ still irreducible? Indecomposable? Do we still have Perm $\simeq \mathbb{1} \oplus \triangleleft$?
Your answers may depend on the value of p; explore all cases!

Solution 5

1. We can still argue that if $\triangleleft$ is reducible, then it has a subrepresentation of degree 1, and thus a common eigenvector for $S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.
   The eigenvalues of the triangular matrix T are ± 1. If $p \neq 2$, they are disctinct, so T is diagonalisable, with eigenspaces spanned respectively by $v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_- = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; thus the subrepresentation must be one of these. We compute that $Sv_+ = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is not collinear to $v_=$ for any p because of its second component, so it never spans a subrepresentation; and that $Sv_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If Sv_- were collinear to v_-, then by the first coefficient we would actually have

$Sv_- = v_-$, which happens iff. $p = 3$. So for $p \geq 5$, $\triangleleft$ is still irreducible and in particular indecomposable, whereas for $p = 3$, $\triangleleft$ is reducible, but still indecomposable as we have proved that it has only one subrepresentation of degree 1 (we would need two of them to decompose $\triangleleft$).

Finally, if $p = 2$, then the only eigenvalue of T is $1 = -1$, so T is not diagonalisable (else T would be conjugate to the identity, and therefore equal to the identity). its egeinspace is still spanned by v_+, and as Sv_+ is not collinear to v_+, we conclude that $\triangleleft$ is still irreducible (and in particular, indecomposable) for $p = 2$.

2. No matter what the value of p is, the computations done in the Synthesis part in the answer to the previous exercise show that the map f defined there is still a morphism of representations from $\mathbb{1} \oplus \triangleleft$ to Perm. Besides, it is an isomorphism provided that $\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 9 \neq 0$, i.e. that $p \neq 3$. So the answer is still yes if $p \neq 3$.

   Let us now suppose that $p = 3$, and let us take a look again at $v_- = e_1 - 2e_2 = e_1 + e_2$. We still have $Tv_- = -v_-$, but now that $p = 3$, we also have $Sv_- = v_-$. Therefore, if Perm were isomorphic to $\mathbb{1} \oplus \triangleleft$, it would contain a nonzero vector which is negated by (12) and thus of the form $\begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}$ with $0 \neq c \in \mathbb{Z}/3\mathbb{Z}$, and fixed by (123) which is a contradiction. So for $p = 3$, the answer is no!

   Actually, we can prove that Perm is indecomposable when $p = 3$. This immediately implies that Perm $\not\simeq \mathbb{1} \oplus \triangleleft$. Although of course this was not required by the exercise, this has the advantage of answering the question in a less "magic" way than the previous approach, which throws in v_- without really explaining where the idea comes from.

   First of all, observe that

$$x^3 = x \text{ for all } x \in \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2 = -1\}. \tag{1}$$

   If Perm decomposed, it would be either into a direct sum of two representations of degrees 1 and 2, or three of degree 1. Putting two of them together in the latter case, we may assume that Perm decomposes as $L \oplus P$ with $\deg L = 1$, $\deg P = 2$.

   Let $S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ be the matrix of (123) on the standard basis of Perm. Its characteristic polynomial is $x^3 - 1$, so its only eigenvalue is 1 by (1), and we find that the corresponding eigenspace is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. As L must be spanned by an eigenvector of S, it must be this eigenspace.

Now that L is determined, let us focus on P. It is a hyperplane, so of the form $P = \operatorname{Ker} l$ for some linear form $l \neq 0$ (i.e. it is defined by a single linear equation, which is not the trivial equation $0 = 0$), which is unique up to scaling. Let $\begin{pmatrix} a & b & c \end{pmatrix}$ be the matrix of l (i.e. $l(x, y, z) = ax+by+cz$). Since P is stable by S_3 and in particular by (123), we must have $\operatorname{Ker} l = P = \operatorname{Ker}(lS)$, so $lS = \begin{pmatrix} a & b & c \end{pmatrix}$ is proportional to l. This means that there exists $0 \neq \lambda \in \mathbb{Z}/3\mathbb{Z}$ such that $lS = \lambda l$, whence $b = \lambda a$, $c = \lambda b$, $a = \lambda c$. This implies $a = \lambda^3 a$, $b = \lambda^3 b$, $c = \lambda^3 c$; since a, b, c are not all 0, we must have $\lambda^3 = 1$, whence $\lambda = 1$ in view of (1). Thus $a = b = c$, so P must be the hyperplane defined by $x + y + z = 0$. But then L and P are not in direct sum since $L \subset P$, absurd.