Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Mathematics

JS/SS Maths/TP/TJH **Semester 2, 2020–2021**

MAU34104 Group representations

Thursday 20 May 2021 **Online exam** **12:00PM – 6:00PM**

Dr. Nicolas Mascot
mascotn@tcd.ie

Instructions that apply to all take-home exams:

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material which is directly linked from the module's Blackboard page or from the module's webpage, if it has one. You may not use any other resources except where your examiner has specifically indicated in the "Additional instructions" section below. Similarly, you may only use software if its use is specifically permitted in that section. You are not allowed to collaborate, seek help from others, or provide help to others.

2. If you have any questions about the content of this exam, you may seek clarification from the lecturer using the e-mail address provided. You are not allowed to discuss this exam with others. You are not allowed to send exam questions or parts of exam questions to anyone or post them anywhere.

3. Unless otherwise indicated by the lecturer in the "Additional instructions" section, solutions must be submitted through Blackboard in the appropriate section of the module webpage by the deadline listed above. You must submit a single pdf file for each exam separately and sign the following declaration in each case. It is your responsibility to check that your submission has uploaded correctly in the correct section.

Additional instructions for this particular exam:

All the representations considered in this exam are over the field $\mathbb{C}$ of the complexes.

This exam is made up of **three** Questions. Attempt **all three** Questions.
Typesetting your answers in LaTeX is allowed, but will not result in any bonus marks.
The use of non-programmable calculators is allowed.

Plagiarism declaration: I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar which are available through https://www.tcd.ie/calendar.

Signature: ______________________________

Question 1 *Homotheties and representations (30 pts)*

Let G be a finite group, let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a complex representation of G of degree $d \in \mathbb{N}$, and let χ be the character of ρ. A *homothety* of V is a linear transformation of V of the form $\lambda \, \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$; its matrix is therefore the diagonal matrix $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ with respect to any basis of V.

1. (5 pts) Let $g \in G$. Prove that $\rho(g)$ is a homothety of V if and only if $|\chi(g)| = d$.
2. (3 pts) Let $Z(\chi) = \{g \in G \mid \rho(g)$ is a homothety of $V\}$. Prove that $Z(\chi)$ is a normal subgroup of G.
3. (10 pts) The *centre* of G is the normal subgroup $Z(G)$ of G defined as
$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$
Prove that if ρ is irreducible, then $Z(G) \subseteq Z(\chi)$.

   Hint: Schur.
4. (12 pts) Prove that $Z(G) = \bigcap_{\psi \in \mathrm{Irr}(G)} Z(\psi)$.

   Hint: Let $z \in \bigcap_{\psi \in \mathrm{Irr}(G)} Z(\psi)$ and $g \in G$, and consider the element $h = zgz^{-1}g^{-1} \in G$.

Solution 1

1. Let $n = \#G$. As explained in the lectures, by Lagrange, $g^n = \mathrm{Id}$, so $\rho(g)^n = \mathrm{Id}$, and therefore $\rho(g)$ is diagonalisable, and its eigenvalues are n-th roots of unity, say
$$\rho(g) = \begin{pmatrix} \zeta^{m_1} & & \\ & \ddots & \\ & & \zeta^{m_d} \end{pmatrix}$$
with respect to a suitable basis of V, where $\zeta = e^{2\pi i/n}$ and $m_1, \cdots m_d \in \mathbb{Z}/n\mathbb{Z}$. In particular,
$$\chi(g) = \mathrm{Tr}\, \rho(g) = \sum_{k=1}^{d} \zeta^{m_k}.$$

If $\rho(g) = \lambda \operatorname{Id}$ is a homothety of ration $\lambda \in \mathbb{C}$, then $\zeta^{m_k} = \lambda$ for all k, so $|\lambda| = 1$ and $|\chi(g)| = |d\lambda| = d$.

Conversely, suppose that $|\chi(g)| = d$, say $\chi(g) = de^{i\theta}$ for some $\theta \in \mathbb{R}$. Then

$$d = e^{-i\theta}\chi(g) = e^{-i\theta}\sum_{k=1}^{d} e^{2\pi i m_k/n} = \sum_{k=1}^{d} e^{i(2\pi m_k/n - \theta)}.$$

The real part of the RHS is $\sum_{k=1}^{d}\cos(2\pi m_k/n - \theta)$, which is $\leq d$ with equality iff. all the cosines are 1, iff. $2\pi m_k/n \equiv \theta \bmod 2\pi$ for all k, iff. $\zeta^{m_k} = e^{i\theta}$ for all k. But then

$$\rho(g) = \begin{pmatrix} e^{i\theta} & & \\ & \ddots & \\ & & e^{i\theta} \end{pmatrix} = e^{i\theta}\operatorname{Id}$$

is a homothety.

NB the detailed analysis of these inequalities is not required to get full marks; saying without justification that $|\chi(g)| < d$ unless all the eigenvalues have the same argument is perfectly sufficient.

2. $Z(\chi)$ is the preimage by ρ of the normal subgroup $\{\lambda \operatorname{Id}, \lambda \in \mathbb{C}^\times\}$ of $\mathrm{GL}(V)$.

   Alternatively, it is a subgroup as the inverse image of the subgroup of homotheties (or by direct inspection), and this subgroup is normal because the previous question shows that it is defined in terms of the class function χ.

3. Let $z \in Z(G)$, and let $T = \rho(z) \in \mathrm{GL}(V)$. For all $v \in V$ and $g \in G$, we have

   $$\rho(g)Tv = \rho(g)\rho(z)v = \rho(gz)v = \rho(zg)v = \rho(z)\rho(g)v = T\rho(g)v,$$

   so T is actually a representation morphism from V to itself. If V is irreducible, T is a homothety by Schur's lemma.

4. The previous question shows that $Z(G) \leq \bigcap_{\psi \in \mathrm{Irr}(G)} Z(\psi)$; it remains to prove the opposite inclusion.

Let $z \in \bigcap_{\psi \in \mathrm{Irr}(G)} Z(\psi)$ and $g \in G$, and consider the element $h = zgz^{-1}g^{-1} \in G$. Let $\psi \in \mathrm{Irr}(G)$, and let ρ_ψ be a representation of character ψ; then $\rho_\psi(z)$ is a homothety, so $\rho_\psi(h) = \rho_\psi(z)\rho_\psi(g)\rho_\psi(z)^{-1}\rho_\psi(g)^{-1} = \mathrm{Id}_V$ since homoetheties are central. Therefore, $h \in \operatorname{Ker} \rho_\psi$ for all $\psi \in \mathrm{Irr}(G)$. Let now R be the regular representation. By Maschke, it decomposes into a direct sum of copies of the ρ_ψ for various ψ, so $h \in \operatorname{Ker} R$. Bu R is faithful, so $h = \mathrm{Id}$, which means that $zg = gz$. As this holds for all $g \in G$, we conclude that $z \in Z(G)$.

Question 2 *The affine group* $\mathrm{AGL}(1,5)$ *(44 pts)*

Let $\mathbb{F}_5$ be the field $\mathbb{Z}/5\mathbb{Z} = \{0,1,2,3=-2,4=-1\}$, and let G be the set of polynomials $f_{a,b} = ax+b$ with $a,b \in \mathbb{F}_5$, $a \neq 0$. We equip G with the law given by composition:

$$(f_{a,b} \cdot f_{a',b'})(x) = f_{a,b}\big(f_{a',b'}(x)\big).$$

We admit that G is a group, whose identity is $f_{1,0}$ and whose conjugacy classes are:

- $\{f_{1,0}\}$,
- $\{f_{1,b} \mid b \neq 0\}$,
- $\{f_{2,b} \mid b \in \mathbb{F}_5\}$,
- $\{f_{-2,b} \mid b \in \mathbb{F}_5\}$,
- $\{f_{-1,b} \mid b \in \mathbb{F}_5\}$.

1. (1 pt) Prove that the map

$$\begin{aligned} \lambda : G &\longrightarrow \mathbb{F}_5^\times \\ f_{a,b} &\longmapsto a \end{aligned}$$

is a group morphism.

2. (8 pts) Deduce the existence of four pairwise non-isomorphic irreducible representations of G of degree 1, and write down their characters.

   Hint: $\mathbb{F}_5^\times$ is cyclic and generated by 2.

3. (5 pts) Determine the number of irreducible representations of G, and their degrees.

4. (12 pts) Determine the character table of G.

5. (8 pts) We have a natural action of G on $\mathbb{F}_5$ defined by $f_{a,b} \cdot x = f_{a,b}(x)$. Determine up to isomorphism the decomposition into irreducible representations of the permutation representation $\mathbb{C}[\mathbb{F}_5]$ attached to this action of G on $\mathbb{F}_5$.

Question continues on the next page

6. (10 pts) The element $f_{-1,0}$ of G has order 2, and thus generates a subgroup $H = \{f_{1,0}, f_{-1,0}\}$ of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\varepsilon : H \longrightarrow \mathrm{GL}_1(\mathbb{C})$ be the degree 1 representation of H such that $\varepsilon(f_{-1,0}) = -\mathrm{Id}_{\mathbb{C}}$. Determine up to isomorphism the decomposition into irreducible representations of the induced representation $\mathrm{Ind}_H^G(\varepsilon)$.

Solution 2

1. We have
$$f_{a,b}\big(f_{a',b'}(x)\big) = a(a'x + b') + b = aa'x + (ab' + b)$$
so $\lambda(f_{a,b} \circ f_{a'b'}) = aa' = \lambda(f_{a,b})\lambda(f_{a',b'})$; besides $\lambda(\mathrm{Id}) = \lambda(f_{1,0}) = 1$.

2. We can use the morphism λ to view any representation of $\mathbb{F}_5^\times$ as a representation of G. Moreover, since λ is clearly surjective, irreducible representations of $\mathbb{F}_5^\times$ yield irreducible representations of G.

   As mentioned in the hint, $\mathbb{F}_5^\times = \langle 2 \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ is cyclic, so it has four irreducible representations of degree one, whose characters are the $\psi_j : 2^x \mapsto i^{jx}$ for $j \in \mathbb{Z}/4\mathbb{Z}$. The corresponding irreducible representations of G are clearly still of degree 1, and their characters are the $\chi_j : f_{a,b} \mapsto \psi_j\big((\lambda(f_{a,b})\big) = i^{ja}$ for $j \in \mathbb{Z}/4\mathbb{Z}$. These characters are pairwise distinct, so these representations are pairwise non-isomorphic.

3. The number of irreducible representations of G is the number of conjugacy classes of G, which is five. Let $n_1 \leq \cdots \leq n_5$ be their degrees; by the previous question, $n_1 = \cdots = n_4 = 1$. We also know that $\sum_{j=1}^5 n_j^2 = \#G$, which is 20 as can be seen form the explicit decomposition of G into conjugacy classes, or directly on the definition of G. It follows that $n_5 = 4$.

4. Let ϕ be the irreducible character of G of degree 4 whose existence was established at the previous question. We know that the character of the regular representation of G, which is $\sum_{\psi \in \mathrm{Irr}(G)} (deg\psi)\psi = \chi_0 + \chi_1 + \chi_2 + \chi_3 + 4\phi$, assumes the value $\#G = 20$ at $f_{1,0}$ and 0 everywhere else; since the χ_j are known, this allows us to determine ϕ. We find the following character table:

Class	$f_{1,0}$	$f_{1,*}$	$f_{2,*}$	$f_{2^2,*}$	$f_{2^3,*}$
$\#$	1	4	5	5	5
$\mathbb{1} = \chi_0$	1	1	1	1	1
χ_1	1	1	i	-1	$-i$
χ_2	1	1	-1	1	-1
χ_3	1	1	$-i$	-1	i
ϕ	4	-1	0	0	0

5. Let P be the character of this permutation representation. We know that for each $g \in G$, $P(g)$ is the number of fixed points of g. $f_{1,0}$ is the identity and therefore fixes all five elements of $\mathbb{F}_5$, $f_{1,*}$ is a translation and fixes none, and finally, if $a \neq 1$, then $f_{a,b}(x) = ax + b$ has a unique fixed point, namely $b/(1-a) \in \mathbb{F}_5$. In conclusion, the values of P are $5, 0, 1, 1, 1$.

   We have $P = \mathbb{1} + \phi$, either by dot products or by direct inspection; therefore this permutation representation decomposes into the direct sum of one copy of the trivial representation and of one copy of the irreducible representation of character ϕ.

   NB this question can also be solved by invoking the double transitivity of this action of G, which implies that $\mathbb{C}[\mathbb{F}_5]$ decomposes into the direct sum of $\mathbb{1}$ and of an irreducible representation of degree $5 - 1 = 4$, as seen in one of the homework assignments.

6. Note that we can identify the degree 1 representation ε with it s character.

   Let $\psi \in \operatorname{Irr}(G)$. The number of copies of the representation of character ψ in the decomposition of $\operatorname{Ind}_H^G(\varepsilon)$ is $(\operatorname{Ind}_H^G(\varepsilon) \mid \psi)_G$, which agrees with

$$(\varepsilon \mid \operatorname{Res}_H^G \psi)_H = \frac{1}{\#H} \sum_{h \in H} \varepsilon(h)\overline{\psi(h)} = \frac{\overline{\psi(f_{1,0})} - \overline{\psi(f_{-1,0})}}{2}$$

   by Frobenius reciprocity. As the conjugacy class of $f_{-1,0}$ is $f_{2^2,*}$, this yields

$$\operatorname{Ind}_H^G(\varepsilon) = \chi_1 + \chi_3 + 2\phi$$

   which means that $\operatorname{Ind}_H^G(\varepsilon)$ decomposes as the direct sum of one copy of the representation of character χ_1, one copy of the representation of character χ_3, and two copies of the representation of character ϕ.

Question 3 *The determinant of the character table (26 pts)*

Let G be a finite group of order $n \in \mathbb{N}$. We arrange the conjugacy classes $C_1, \cdots, C_r$ and the irreducible characters $\chi_1, \cdots, \chi_r$ of G arbitrarily, and we view the character table of G as an $r \times r$ complex matrix X. In other words, for all $1 \leq i, j \leq r$, the i, j-coefficient $X_{i,j}$ of X is $\chi_i(g_j)$, where g_j is any element of C_j.

1. (a) (6 pts) Prove that the complex conjugate of a character is a character.

   (b) (6 pts) Prove that the complex conjugate of an **irreducible** character is an **irreducible** character.

   (c) (4 pts) Deduce that $\overline{\det X} = \pm \det X$. What does this imply about the complex number $\det X$?

2. (a) (8 pts) Let D be the diagonal matrix whose j, j-coefficient is $\#C_j$ for all $1 \leq j \leq r$. Express the orthogonality relations in terms of the matrices X and D.

   (b) (4 pts) Deduce the value of $|\det X|$ in terms of n and of the $\#C_j$.

Solution 3

1. (a) If χ is the character of a representation ρ, then $\overline{\chi}$ is the character of the dual representation $\mathrm{Hom}(\rho, \mathbb{1})$.

   (b) Let $\chi \in \mathrm{Irr}(G)$. As χ is irreducible, we have $(\chi \,|\, \chi) = 1$; therefore

$$(\overline{\chi} \,|\, \overline{\chi}) = \frac{1}{n} \sum_{g \in G} \bar{\chi}(g)\overline{\bar{\chi}(g)} = \frac{1}{n} \sum_{g \in G} \chi(g)\overline{\chi(g)} = (\chi \,|\, \chi) = 1,$$

   which shows that $\overline{\chi}$ is also irreducible.

   (c) The previous question shows that the complex-conjugate $\overline{X}$ of X can be obtained from X by permuting the rows of X in some way. The properties of the determinant thus ensure that

$$\overline{\det X} = \det \overline{X} = \pm \det X,$$

   which means that $\det X$ is either real (if the sign is $+$) or purely imaginary (if the sign is $-$).

2. (a) The orthogonality relations state that for all $\chi, \psi \in \mathrm{Irr}(G)$,

$$(\chi \,|\, \psi) = \frac{1}{n} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

is 1 if $\psi = \chi$ and 0 else. For each $j \leq r$, pick $g_j \in C_j$. As character are class functions, we have that

$$(\chi \,|\, \psi) = \frac{1}{n} \sum_{g \in G} \chi(g) \overline{\psi(g)} = \frac{1}{n} \sum_{j=1}^{r} \#C_j \chi(g_j) \overline{\psi(g_j)} = \frac{1}{n} \sum_{j=1}^{r} X_{\chi,j} D_{j,j} \overline{{}^t X_{j,\psi}}$$

is the χ, ψ coefficient of the matrix

$$\frac{1}{n} X D \overline{{}^t X}.$$

Therefore, the orthogonality relations are equivalent to the statement that $\frac{1}{n} X D \overline{{}^t X}$ is the identity matrix.

(b) Taking the determinant of

$$n I_r = X D \overline{{}^t X},$$

where I_r is the identity matrix of size $r \times r$, we get that

$$n^r = \det(X D \overline{{}^t X}) = \det(X) \det(D) \overline{\det(X)}.$$

But as D is diagonal, we have $\det(D) = \prod_{j=1}^{r} \#C_j$, whence

$$|\det X|^2 = \det(X) \overline{\det(X)} = n^r / \prod_{j=1}^{r} \#C_j,$$

so finally

$$|\det X| = \sqrt{n^r / \prod_{j=1}^{r} \#C_j} = \sqrt{\prod_{j=1}^{r} \frac{n}{\#C_j}}.$$

NB $n/\#C_j$ is the order of the centraliser of g_j in G; however mentioning this fact is not required to get full marks to this question.