Group representations
Exercise sheet 5

https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html

Version: March 31, 2023

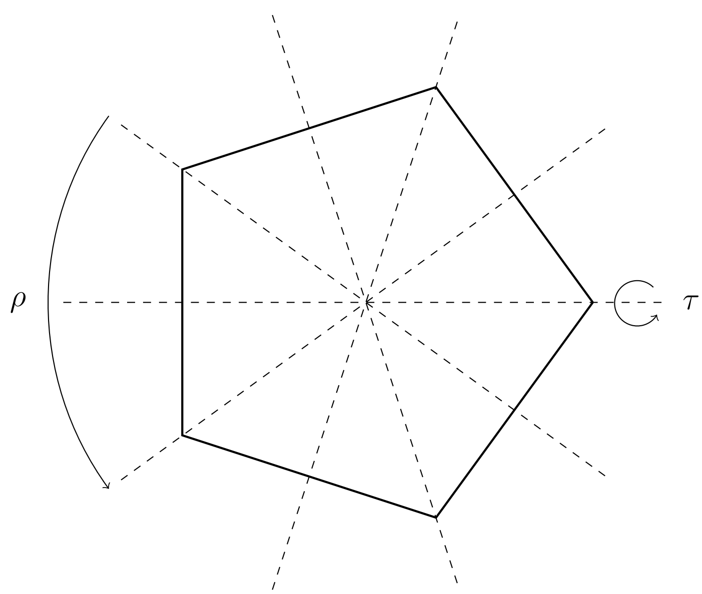
Email your answers to mascotn@tcd.ie by Tuesday April 11th, 12:00.

Exercise 1 *The character table of D_{2n}, n odd (100 pts)*

In this exercise, we fix an *odd* integer $n = 2m + 1 \geq 3$, and we consider the dihedral group $G = D_{2n}$ of transformations of the plane $\mathbb{R}^2$ that leave a regular n-gon invariant.

This group has $2n$ elements, n of which (including the identity) are rotations, whereas the other n are axial symmetries across lines joining a vertex to the midpoint of the opposite edge. We call ρ the rotation of angle $2\pi/n$, which has this order n, and τ one of the axial symmetries.

The following picture illustrates the situation in the case $n = 5$:



We write H for the subgroup of G generated by ρ; H is therefore a cyclic group isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Since $\tau\rho\tau^{-1} = \rho^{-1}$, H is actually a normal subgroup of G. Its index is $[G : H] = 2$, so the quotient G/H is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We recall that the irreducible representations of H are all of degree 1; they are naturally indexed by $y \in \mathbb{Z}/n\mathbb{Z}$ so that their characters are $\chi_y : \rho^x \mapsto e^{2\pi i x y/n}$. Observe that

$$\mathbb{Z}/n\mathbb{Z} = \{-m, \cdots, -1, 0, 1, 2, \cdots, m\};$$

you may find this useful later.

1. (5 pts) Explain why the conjugacy classes of G are as follows:
   - $\{\text{Id}\}$,
   - $\{\rho, \rho^{-1}\}$,
   - $\{\rho^2, \rho^{-2}\}$,
   - $\vdots$
   - $\{\rho^m, \rho^{-m}\}$,
   - { All the n axial symmetries }.

2. (10 pts) Use the fact that $G/H \simeq \mathbb{Z}/2\mathbb{Z}$ to write down two irreducible characters of degree 1 of G.

3. Let $y \in \mathbb{Z}/n\mathbb{Z}$, let $\chi_y : \rho^x \mapsto e^{2\pi i x y/n}$ be the corresponding irreducible character of H, and let $\psi_y = \operatorname{Ind}_H^G \chi_y$.

   (a) (20 pts) Write down the values of the character ψ_y.
   You may want to use the identity $e^{it} + e^{-it} = 2\cos t$.

   (b) (10 pts) Determine the decomposition of $\operatorname{Res}_H^G \psi_y$ into irreducible characters of H.

   (c) (25 pts) Use Frobenius reciprocity to determine whether ψ_y is irreducible.
   Your answer may depend on y*!*

4. (20 pts) Sketch the character table of G.

5. (10 pts) Determine the centre of G and the derived subgroup of G.

Solution 1

1. We already know the conjugating by τ takes ρ to ρ^{-1}, and therefore ρ^x to ρ^{-x} for all $x \in \mathbb{Z}$. Besides, the rotations commute since H is Abelian. Thus conjugating a rotation by a symmetry inverses it, whereas conjugating it by a rotation leaves it invariant. Since these are all the elements of G, it follows that the conjugacy class of ρ^x is indeed $\{\rho^x, \rho^{-x}\}$. Finally, the axial symmetries are all conjugate to each other by rotations, in view of the conjugacy principle.

2. We can inflate all the representations of $G/H \simeq \mathbb{Z}/2\mathbb{Z}$ into representations of G. Doing so for the two irreducible representations of degree 1 of $\mathbb{Z}/2\mathbb{Z}$, which are $\mathbb{1}$ and $\mathbb{Z}/2\mathbb{Z} \simeq \{\pm 1\}$, we obtain the following characters of G:

Class	Id	ρ	$\cdots$	ρ^x	$\cdots$	ρ^m	τ
#	1	2	$\cdots$	2	$\cdots$	2	n
$\mathbb{1}$	1	1	$\cdots$	1	$\cdots$	1	1
ε	1	1	$\cdots$	1	$\cdots$	1	-1.

   These characters are irreducible because they are inflated from irreducible characters (any G-invariant subspace would also be a G/H-invariant subspace), or simply because they are of degree 1.

3. (a) Let $g \in G$. If $g = \mathrm{Id}$, we see directly that $\psi_y(g) = \deg \psi_y = [G : H] \deg \chi_y = 2$. Else, the second formula for induced characters (which also applies if $g = \mathrm{Id}$ of course!) tells us that

$$\mathrm{Ind}_H^G \chi_y(g) = \frac{[G:H]}{\#cc_G(g)} \sum_j \#cc_H(h_j)\chi_y(h_j),$$

where $cc_G(g) \cap H = \coprod_j cc_H(h_j)$.

If $g = \rho^x$ is a nontrivial rotation, then $cc_G(g) = \{\rho^x, \rho^{-x}\} = cc_H(\rho^x) \coprod cc_H(\rho^{-x})$ since the conjugacy classes of the Abelian group H are all singletons, whence

$$\psi_y(\rho^x) = \frac{2}{2}\big(1\chi_y(\rho^x) + 1\chi_y(\rho^{-x})\big) = e^{2\pi i xy/n} + e^{-2\pi i xy/n} = 2\cos(2\pi xy/n).$$

If g is a symmetry, then $cc_G(g)$ consists only of symmetries, so $cc_G(g) \cap H = \emptyset$, whence $\psi_y(g) = 0$ (an empty sum is 0).

(b) We spot directly that for all $x \in \mathbb{Z}/n\mathbb{Z}$,

$$\psi_y(\rho^x) = e^{2\pi i xy/n} + e^{-2\pi i xy/n} = \chi_y(\rho^x) + \chi_{-y}(\rho^x).$$

As the ρ^x cover H, it follows that $\mathrm{Res}_H^G \psi_y = \chi_y + \chi_{-y}$.

(c) By Frobenius reciprocity,

$$(\psi_y \,|\, \psi_y)_G = (\mathrm{Id}_H^G \chi_y \,|\, \psi_y)_G = (\chi_y \,|\, \mathrm{Res}_H^G \psi_y)_H = (\chi_y \,|\, \chi_y + \chi_{-y})_H = (\chi_y \,|\, \chi_y)_H + (\chi_y \,|\, \chi_{-y})_H$$

in view of the previous question and of the linearity of the dot product. We always have $(\chi_y \,|\, \chi_y)_H = 1$ by irreducibility of χ_y, whereas $(\chi_y \,|\, \chi_{-y})$ is 1 if $y = -y$ in $\mathbb{Z}/n\mathbb{Z}$, and 0 else. As $\mathbb{Z}/n\mathbb{Z} = \{-m, \cdots, -1, 0, 1, 2, \cdots, m\}$, we see that $y = -y$ only when $y = 0$; therefore ψ_y is irreducible unless $y \neq 0$.

NB for $y = 0$, question 3a shows that $\psi_y = \mathbb{1} + \varepsilon$.

4. We know that the number of irreducible characters of G is the number of conjugacy classes of G, which is $m + 1$ by the first question. We have already found 2 irreducible representations of degree 1 in question 2, so we need $m - 1$ more. By the previous question, the characters $\psi_1, \cdots, \psi_m$ are irreducible; besides, they are pairwise distinct since they have distinct restrictions to H according to question 3b. We have therefore found all irreducible characters of G:

Class	Id	ρ	$\cdots$	ρ^x	$\cdots$	ρ^m	τ
#	1	2	$\cdots$	2	$\cdots$	2	n
$\mathbb{1}$	1	1	$\cdots$	1	$\cdots$	1	1
ε	1	1	$\cdots$	1	$\cdots$	1	-1
ψ_1	2	$2\cos(2\pi/n)$	$\cdots$	$2\cos(2x\pi/n)$	$\cdots$	$2\cos(2m\pi/n)$	0
$\vdots$	$\vdots$			$\vdots$			$\vdots$
ψ_y	2	$2\cos(2y\pi/n)$	$\cdots$	$2\cos(2xy\pi/n)$	$\cdots$	$2\cos(2my\pi/n)$	0
$\vdots$	$\vdots$			$\vdots$			$\vdots$
ψ_m	2	$2\cos(2m\pi/n)$	$\cdots$	$2\cos(2xm\pi/n)$	$\cdots$	$2\cos(2m^2\pi/n)$	0.

NB it is clear from question 3a that $\psi_y = \psi_{-y}$.

5. The centre of G is the set of elements which sit alone in their conjugacy class; it is therefore reduced to $\{\mathrm{Id}\}$.

   The derived subgroup of G is

$$D(G) = \bigcap_{\deg \phi = 1} \operatorname{Ker} \phi = \operatorname{Ker} \mathbb{1} \cap \operatorname{Ker} \varepsilon = \operatorname{Ker} \varepsilon = H.$$

This was the only mandatory exercise, that you must submit before the deadline. The following exercise is not mandatory; it is not worth any points, and you do not have to submit it. However, you can try to solve it for practice, and you are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 2 *The character table of D_{2n}, n even*

Redo the same exercise in the case where $n = 2m$ is even.

You may want to consult the previous assignment for inspiration.

Solution 2

The group $G = D_{2n}$ still has $2n$ elements (as its name indicates), n of which (including the identity) are rotations, while the other n are axial symmetries. However, these symmetries now fall into two categories: those whose axis joins opposite vertices, and those whose axis join middle points of opposite edges, resulting in two conjugacy classes of size m by the conjugacy principle. Let us call σ and τ representatives of these two classes.

Let ρ be the rotation of angle $2\pi/n$; it spans a cyclic subgroup $H \simeq \mathbb{Z}/n\mathbb{Z}$ of G. Symmetries (of either kind) still conjugate any rotation to its inverse, so the conjugacy class of ρ^x is still $\{\rho^x, \rho^{-x}\}$; however this time $\rho^{-x} = \rho^x$ not only when $x = 0$, but also when $x = m$ (and indeed $\rho^m = -\mathrm{Id}$ is central). This results in two conjugacy classes of size 1, and $m-1$ of size 2. In total, we thus get $m+3$ conjugacy classes.

As previously, the subgroup H is normal; however, the above shows that its subgroup $K \simeq \mathbb{Z}/m\mathbb{Z}$ generated by ρ^2 is also normal[1]. The quotient G/K has order $[G : K] = 4$, and indeed its elements are the identity, the two kinds of symmetries, and the class of ρ. Since the square of these elements are all trivial, we conclude as in the previous assignment that $G/K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whence 4 irreducible characters of G of degree 1:

[1] In the case when n is odd, the subgroup generated by ρ^2 is the whole of H, since 2 is then invertible in $\mathbb{Z}/n\mathbb{Z}$.

Class	Id	ρ	$\cdots$	ρ^x	$\cdots$	ρ^{m-1}	$-\operatorname{Id}$	σ	τ
#	1	2	$\cdots$	2	$\cdots$	2	1	m	m
$\mathbb{1}$	1	1	$\cdots$	1	$\cdots$	1	1	1	1
ξ	1	-1	$\cdots$	$(-1)^x$	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
η	1	-1	$\cdots$	$(-1)^x$	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	1	-1
$\xi\eta$	1	1	$\cdots$	1	$\cdots$	1	1	-1	-1.

Let now $y \in \mathbb{Z}/n\mathbb{Z} \simeq H$, and $\psi_y = \operatorname{Ind}_H^G \chi_y$ as before. The second formula for induced characters shows that $\psi_y(\rho^x) = 2\cos(2\pi xy/n)$, whereas $\psi_y(g) = 0$ if g is a symmetry (of either kind).

In particular, $\operatorname{Res}_H^G \psi_y = \chi_y + \chi_{-y}$, so Frobenius reciprocity shows that ψ_y is irreducible iff. $y \neq -y$, which happens unless $y = 0$ or $y = m$. For thoroughness, we observe that $\psi_0 = \mathbb{1} + \xi\eta$ and that $\psi_m = \xi + \eta$, and also that $\psi_{-y} = \psi_y$. Anyway, the ψ_y for $y \in \{1, \cdots, m-1\}$ are $m-1$ pairwise distinct irreducible characters of degree $[G : H] \deg \chi_y = 2$. With the 4 characters of degree 1 found above, that's a total of $m+3$ irreducible characters, so we have found all of them. The character table of D_{2n} is therefore

Class	Id	ρ	$\cdots$	ρ^x	$\cdots$	ρ^{m-1}	$-\operatorname{Id}$	σ	τ
#	1	2	$\cdots$	2	$\cdots$	2	1	m	m
$\mathbb{1}$	1	1	$\cdots$	1	$\cdots$	1	1	1	1
ξ	1	-1	$\cdots$	$(-1)^x$	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
η	1	-1	$\cdots$	$(-1)^x$	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	1	-1
$\xi\eta$	1	1	$\cdots$	1	$\cdots$	1	1	-1	-1
ψ_1	2	$2\cos(2\pi/n)$	$\cdots$	$2\cos(2x\pi/n)$	$\cdots$	$-2\cos(2\pi/n)$	-2	0	0
$\vdots$	$\vdots$			$\vdots$					$\vdots$
ψ_y	2	$2\cos(2\pi y/n)$	$\cdots$	$2\cos(2xy\pi/n)$	$\cdots$	$(-1)^y 2\cos(2y\pi/n)$	-2	0	0
$\vdots$	$\vdots$			$\vdots$					$\vdots$
ψ_{m-1}	2	$-2\cos(2\pi/n)$	$\cdots$	$(-1)^x 2\cos(2x\pi/n)$	$\cdots$	$2\cos(2\pi/n)$	-2	0	0.

For thoroughness, we note the identities $\psi_y\xi = \psi_y\eta = \psi_{m-y}$, whence $\psi_y\xi\eta = \psi_y$.

This character table shows that $Z(G) = \{\pm \operatorname{Id}\}$ and that $D(G) = K$. Note that if m is odd, neither contains the other.