Group representations
Exercise sheet 4

`https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html`

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Email your answers to mascotn@tcd.ie by Monday April 3rd, 12:00.

Exercise 1 *The character table of A_4 (100 pts)*

Let $G = A_4$ be the alternating group of even permutations on 4 objects.

1. (20 pts) Let V_4 be the Klein subgroup of A_4 consisting of the double transpositions and of the identity. Prove that V_4 is normal in A_4, and that A_4/V_4 is cyclic.

2. (10 pts) Prove that (123) and (132) are **not** conjugate in A_4.

3. (50 pts) Determine the character table of A_4.

   You may want to define $\omega = e^{2\pi i/3}$; note that $\omega^2 = \overline{\omega} = -\omega - 1$.

4. (10 pts) Deduce that V_4 is the derived subgroup of A_4.

5. (10 pts) Determine the decomposition into irreducible representations of the restriction to A_4 of each the five irreducible representations of S_4.

6. (Bonus question, 0 pts) We admit that the group of rotations of $\mathbb{R}^3$ that leave the regular tetrahedron invariant is isomorphic to A_4 via the permutations induced on the 4 vertices, whence a representation of A_4 of degree 3. Write down the decomposition (over $\mathbb{C}$) of this representation into irreducible representations.

Solution 1

1. By the conjugacy principle, conjugacy in S_n preserves cycle decomposition patterns. Therefore, V_4 is normal in S_4, and *a fortiori* in A_4. The quotient A_4/V_4 has order $\#A_4/\#V_4 = 12/4 = 3$, which is prime; therefore any non-trivial element has order 3 by Lagrange, and is therefore a generator.

2. The conjugacy principle shows that (123) and (132) are conjugate in S_4 by any permutation satisfying $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$. In fact, there exists a unique such permutation, since these identities force $4 \mapsto 4$. However, this permutation, which is a transposition, does not lie in A_4. The only way to remedy this would be to write (132) as (321) or as (213), but this does not actually fix the problem (because we are cyclically permuting $1, 3, 2$, and thus changing the conjugating permutation by a 3-cycle, which is even, and therefore does not affect its odd-ness).

3. By playing around with the conjugacy principle (and taking care to only conjugate by even permutations, namely 3-cycles and double transpositions), we find that the conjugacy classes of A_4 are

$$\{\mathrm{Id}\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (214), (341), (432)\}, \{(132), (241), (431), (342)\}.$$

That is 4 of them, so we have 4 irreducible representations. Their degrees $n_1 \leq \cdots \leq n_4$ satisfy $n_1 = 1$ (by $\mathbb{1}$), $n_4 \geq 2$ (as A_4 is not Abelian), and $\sum_{i=1}^4 n_i^2 = \#A_4 = 12$, whence $n_1 = n_2 = n_3 = 1$, $n_4 = 3$.

Pulling back the irreducible representations of the quotient $A_4/V_4 \simeq \mathbb{Z}/3\mathbb{Z}$ gives us the 3 characters of degree 1, which are still irreducible (because they have degree 1, or alternatively, because the projection morphism is surjective):

Class	Id	(12)(34)	(123)	(132)
#	1	3	4	4
$\mathbb{1}$	1	1	1	1
ϕ	1	1	ω	$\overline{\omega}$
$\overline{\phi}$	1	1	$\overline{\omega}$	ω
χ	3	?	?	?

and we can find the missing character χ by the fact that the character of the regular representation 12 0 0 0 is $\mathbb{1} + \phi + \overline{\phi} + 3\chi$, or by invoking exercise 2 of assignment 3, which implies that the permutation representation corresponding to the action of A_4 on 4 objects decomposes as the direct sum of $\mathbb{1}$ and of an irreducible representation of degree 3:

Class	Id	(12)(34)	(123)	(132)
#	1	3	4	4
$\mathbb{1}$	1	1	1	1
ϕ	1	1	ω	$\overline{\omega}$
$\overline{\phi}$	1	1	$\overline{\omega}$	ω
ρ	3	-1	0	0

4. We know that

$$D(A_4) = \bigcap_{\substack{\chi \in \mathrm{Irr}(A_4) \\ \deg \chi = 1}} \operatorname{Ker} \chi = \bigcap_{\substack{\chi \in \mathrm{Irr}(A_4) \\ \deg \chi = 1}} \{g \in A_4 \mid \chi(g) = 1\}.$$

The character table of A_4 shows that this is V_4.

5. Recall the character table of S_4:

Class	Id	(12)	(123)	(1234)	(12)(34)
$\mathbb{1}_{S_4}$	1	1	1	1	1
ε	1	-1	1	-1	1
ψ	2	0	-1	0	2
χ	3	1	0	-1	-1
$\chi\varepsilon$	3	-1	0	1	-1

Restricting these characters to A_4, we find

Class	Id	(12)(34)	(123)	(132)
#	1	3	4	4
$\mathbb{1}_{S_4}$	1	1	1	1
ε	1	1	1	1
ψ	2	2	-1	-1
χ	3	-1	0	0
$\chi\varepsilon$	3	-1	0	0

from which we immediately see that $\mathbb{1}_{S_4}$ and ε restrict to $\mathbb{1}$, and that χ and $\chi\varepsilon$ restrict to ρ (unsurprisingly, ε becomes trivial on A_4...). We also determine (either by inner products or by spotting this directly from $\omega + \overline{\omega} = -1$) that ψ decomposes as $\phi \oplus \overline{\phi}$.

Note: This can serve as a reminder of the fact that the restriction of an irreducible representation may no longer be irreducible.

6. Double transpositions of the vertices may be obtained by rotating the tetrahedron by 180° about a line joining perpendicularly the midpoints of two opposite edges. In a suitable basis, such a rotation has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

whence trace $1-1-1=-1$. 3-cycles are obtained by rotating the tetrahedron by 120° about a line joining a vertex to the midpoint of the opposite face. In a suitable basis, such a rotation has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(120^\circ) & -\sin(120^\circ) \\ 0 & \sin(120^\circ) & \cos(120^\circ) \end{pmatrix},$$

whence trace $1 + 2\cos(120^\circ) = 0$.

The character of this tetrahedron representation of A_4 is thus $3, -1, 0, 0$, so this representation is isomorphic to ρ.

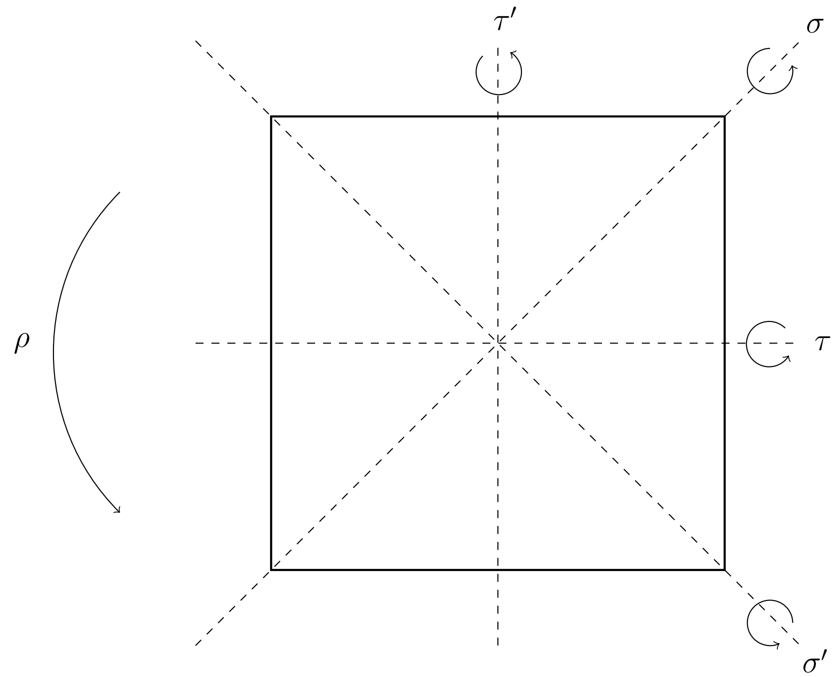
Note: Knowing in advance this interpretation of A_4 was yet another way to fill in the character table of A_4.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they is not worth any points, and you do not have to submit them. However, I *strongly* suggest you can try to solve them, as this is *excellent practice for the exam.* You are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 2 *The character table of D_8*

1. Let G be a non-Abelian group with 8 elements. Determine the degrees of the irreducible representations of G, and deduce that G must have exactly 5 conjugacy classes, and that the quotient $G/D(G)$ of G by its derived subgroup must have order 4.

2. Let $G = D_8$ be the group of symmetries of the square, most of whose elements we name as follows:



   Determine the derived subgroup $D(G)$ and the structure of the quotient $G/D(G)$.

3. Determine the character table of D_8.

Solution 2

1. Let $n_1 \leq n_2 \leq \cdots \leq n_r$ be the degrees of the irreducible representations of G, so that the number of conjugacy classes in G is r. We know that $\sum_{i=1}^{r} n_i^2 = \#G = 8$; and also that $n_1 = 1$ (because of the existence of the trivial representation $\mathbb{1}$) and that $n_r \geq 2$ since G is not Abelian. This only leaves the possibility $r = 5$, $n_1 = n_2 = n_3 = n_4 = 1$, $n_5 = 2$.

   In particular, since the irreducible representations of G of degree 1 may be identified with the irreducible representations of $G/D(G)$, we deduce that

$$\#\big(G/D(G)\big) = \sum_{\chi \in \mathrm{Irr}(G/D(G))} (\deg \chi)^2 = n_1^2 + n_2^2 + n_3^2 + n_4^2 = 4.$$

2. By the previous question, $D(G)$ is a subgroup of G of index 4, and therefore of order 2, so it contains only one non-trivial element. We observe (e.g. thanks to the "conjugacy principle") that

$$\tau\rho\tau^{-1}\rho^{-1} = (\tau\rho\tau^{-1}\rho^{-1})\rho^{-1} = \rho^{-1}\rho^{-1} = \rho^2 \neq \mathrm{Id},$$

whence $D(G) = \{\mathrm{Id}, \rho^2\}$.

The quotient $G/D(G)$ is Abelian by definition of $D(G)$, and of order 4, so it is isomorphic to either C_4 or $C_2 \times C_2$ (where C_n denotes the cyclic group of order n, which is often[1] written $\mathbb{Z}/n\mathbb{Z}$). But every element of $G/D(G)$ has order at most 2, since the only elements of order > 2 in$G$ are ρ and ρ^{-1}, which become of order 2 in $G/D(G)$ as $\rho^2 = \rho^{-2} \in D(G)$. Therefore

$$G/D(G) \simeq C_2 \times C_2.$$

3. We have shown in the lectures that the conjugacy classes of G are

$$\{\mathrm{Id}\}, \{\rho^2\}, \{\rho, \rho^{-1}\}, \{\sigma, \sigma'\}, \{\tau, \tau'\}.$$

Conjugates in G become equal in the Abelian quotient $G/D(G)$, so

$$G/D(G) = \{\mathrm{Id} = \rho^2, \sigma = \sigma', \tau = \tau', \rho = \rho^{-1}\}.$$

We get 4 irreducible characters of degree 1 (including $\mathbb{1}$) of G by pulling back those of $G/D(G) \simeq C_2 \times C_2$, which we have determined in class:

Class	Id	ρ^2	ρ	σ	τ
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1

We know that $\# \mathrm{Irr}(G) = \#$ conj. classes $= 5$, so there is one missing irreducible character, whose degree is 2 according to the first question. We can find this missing character χ by using the fact that the regular representation, whose character is 8 0 0 0 0, decomposes as $\mathbb{1} + \phi + \psi + \phi\psi + 2\chi$, or by checking that the degree 2 representation of G afforded by G acting on a square is irreducible since the dot product of its character with itself is 1.

Class	Id	ρ^2	ρ	σ	τ
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1
χ	2	-2	0	0	0

Exercise 3 *The character table of Q_8*

Let $Q_8 = \{1, -1, I, -I, J, -J, K, -K\}$ be the (Hamiltonian) quaternionic group, whose multiplication is defined by the rules

$$\text{For all } x, y \in Q_8, (-x)y = x(-y) = -(xy), \text{and} x1 = 1x = x,$$

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = K = -JI, \quad JK = I = -KJ, \quad KI = J = -IK.$$

[1] I personally prefer to write C_n for the group., $\mathbb{Z}/n\mathbb{Z}$ for the ring, and, if n is prime, $\mathbb{F}_n$ for the field.

1. By the first question of the previous exercise, Q_8 has exactly 5 conjugacy classes. Check that these classes are $\{1\}$, $\{-1\}$, $\{I, -I\}$, $\{J, -J\}$, and $\{K, -K\}$.

2. Determine the centre Z of Q_8.

3. Prove that Q_8/Z is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

4. Determine the character table of Q_8. Any comments?

5. It is standard to realise Q_8 as a group of complex matrices by identifying I with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, J with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and K with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ (since these matrices satisfy the relations defining the group law of Q_8, as you may check if you wish). How would you interpret this in terms of the character table of Q_8?

Solution 3

1. Since we know that there must be exactly 5 conjugacy classes, it suffices to check that $-I$ is conjugate to I, that $-J$ is conjugate to J, and that K is conjugate to $-K$. This follows from $JIJ^{-1} = JI(-J) = (-K)(-J) = KJ = -I$, and similarly for J and K.

2. The centre is made up of the elements that sit alone in their conjugacy class, so $Z = \{1, -1\}$.

3. Since $Z = \{1, -1\}$, we have that $Q_8/Z = \{\pm 1, \pm I, \pm J, \pm K\}$ is now Abelian, and $(\pm I)^2 = (\pm J)^2 = (\pm K)^2 = -1 = \pm 1$ so all the nontrivial elements of Q_8/Z have order 2. It follows that we have an isomorphism
$$\begin{array}{rcl} (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & Q_8/Z \\ (a, b) & \longmapsto & \pm I^a J^b. \end{array}$$

4. We can inflate the 4 irreducible characters of degree 1 of $Q_8/Z \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ to characters of Q_8 (which are still irreducible, because they have degree 1, or alternatively because the projection map is surjective):

Class	1	-1	$\pm I$	$\pm J$	$\pm K$
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1

Since there are 5 conjugacy classes, we are missing one character, which is of degree 2 by virtue of $\sum_{\chi \in \mathrm{Irr}(Q_8)} (\deg \chi)^2 = \# Q_8$, and whose values we can recover from the character of the regular representation, or alternatively, by orthgonality of columns:

Class	1	-1	$\pm I$	$\pm J$	$\pm K$
#	1	1	2	2	2
$\mathbb{1}$	1	1	1	1	1
ϕ	1	1	1	-1	-1
ψ	1	1	-1	1	-1
$\phi\psi$	1	1	-1	-1	1
χ	2	-2	0	0	0

And... this is the same table as for D_8! And yet, the groups D_8 and Q_8 are NOT isomorphic, e.g. because because D_8 has 5 elements of order 2 and 2 of order 4 whereas Q_8 has 6 elements of order 4 and 2 of order 2. This shows that two groups can have the same character table without being isomorphic!

NB the reason why D_8 and Q_8 have the same character table has mostly been exposed in the first question of the previous exercise, as it showed that non-Abelian groups of order 8 must have a lot in common.

5. This realisation of Q_8 by complex matrices is by definition a faithful representation of Q_8 of degree 2. By looking at the trace of the matrices, we see that this representation as character χ, so it is the unique irreducible representation of Q_8 of degree 2.

Exercise 4 *When s**t gets real*

Nicolas M., lecturer at a college somewhere in Europe, has been pestering some of his students for writing things such as

$$(\chi \mid \chi) = \frac{1}{\#G} \sum_{g \in G} \chi(g)^2$$

when instead they should have written

$$(\chi \mid \chi) = \frac{1}{\#G} \sum_{g \in G} |\chi(g)|^2.$$

He admits that most of the characters in his lectures were real-valued. But how come?

1. Let G be a finite group. Prove that every character of G is real-valued if and only if every $g \in G$ is conjugate to its inverse.

   Hint: Recall that $\chi(g^{-1}) = \overline{\chi(g)}$.

2. Let $n \in \mathbb{N}$. Prove that every character of S_n is real-valued. What about A_4?

Solution 4

1. The formula $\chi(g^{-1}) = \overline{\chi(g)}$ was proved in the lectures, when we proved that $\rho(g)$ is always diagonalisable with eigenvalues which are $\#G$-th roots of 1.

   Suppose that g is conjugate to g^{-1} for all $g \in G$. Then for all χ,

$$\overline{\chi(g)} = \chi(g^{-1}) = \chi(g)$$

since χ is a class function. It follows that χ is real-valued.

Conversely, suppose that every χ is real-valued, and let $g \in G$. Then

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(g^{-1})} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(g)^2$$

must be a nonnegative real number, which cannot be zero because of the presence of $\mathbb{1} \in \mathrm{Irr}(G)$. By the second orthogonality relations, this proves that g and g^{-1} are conjugate.

Note: This is an avatar of a more general phenomenon, which results from the application of the Galois theory of cyclotomic fields to character values; see for example the chapter on group representations in J.-P. Serre's book on finite groups.

2. Let $g \in S_n$. Then g^{-1} has the same cycle decomposition shape as g (with the order inside each of the cycles reversed), so g and g^{-1} are conjugate in S_n. It follows that the characters of S_n are real-valued.

   On the other hand, question 2. of the first exercise explains why some of the characters of A_4 are not real-valued.