Group representations
Exercise sheet 3

`https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html`

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Email your answers to mascotn@tcd.ie by Thursday March 16, 12:00.

Exercise 1 *An enigmatic group (100 pts)*

As you were walking down the street, you found lying on the ground a finite group G, consisting of 11 conjugacy classes denoted by $C_1, C_2, \cdots, C_{11}$, as well as two (not necessarily irreducible) representations of G whose characters are given by the following data:

Conj. class	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
Size	1	15	40	90	45	120	144	120	90	15	40
ϕ	6	2	0	0	2	2	1	1	0	-2	3
ψ	21	1	-3	-1	1	1	1	0	-1	-3	0

(the first line lists the conjugacy classes of G; the second line gives the sizes of these conjugacy classes; the last two lines give the values of these two characters).

1. (1 pt) Which conjugacy class does the identity of G lie in?
2. (8 pt) Determine the degree of ϕ, and that of ψ.
3. (1 pt) Determine $\#G$.
4. (30 pts) Is the representation of character ϕ irreducible? If not, determine the shape of its decomposition into a direct sum of irreducible representations, including the degree of each irreducible constituent.
5. (30 pts) Prove that G admits an irreducible representation of degree 16. Write down its character. Finally, give an element of $\mathbb{C}[G]$ which acts as a projector onto the corresponding isotypical component.
6. (30 pts) Prove that G admits a representation of degree 25 which decomposes into the direct sum of 4 pairwise non-isomorphic irreducible representations.

Solution 1

1. Let 1_G be the identity of G. Then $g1_Gg^{-1} = gg^{-1} = 1_G$ for all $g \in G$, so the identity sits alone in its conjugacy class. Since C_1 is the only conjugacy class of size 1, it must be the conjugacy class of 1_G.

2. The degree of a character is the value that it assumes on 1_G (because the trace of an identity matrix is the size of that matrix), so $\deg \phi = \phi(1_G) = 6$ and $\deg \psi = \psi(1_G) = 21$ by the previous question.

3. Since conjugacy classes partition G, we have

$$\#G = 1 + 15 + 40 + 90 + 45 + 120 + 144 + 120 + 90 + 15 + 40 = 720.$$

4. We compute that $(\phi \mid \phi) = \dfrac{1}{\#G} \displaystyle\sum_{g \in G} \phi(g)\overline{\phi(g)} = \dfrac{1}{720} \sum_{j=1}^{11} \#C_j |\phi(C_j)|^2 = 2$. As $(\phi \mid \phi) \neq 1$, this shows that this representation is not irreducible.

   More precisely, if this representation decomposes as $\bigoplus_{i=1}^k \rho_i^{\oplus n_i}$ with the ρ_i irreducible and pairwise non-isomorphic, then $\sum_{i=1}^k n_i^2 = 2$, whence $k = 2$ and $n_1 = n_2 = 1$, which means that this representation is the direct sum of two non-isomorphic irreducible subrepresentations.

   Besides, we observe that $(\phi \mid \mathbb{1}) = \frac{1}{720} \sum_{j=1}^{11} \#C_j \phi(C_j) 1 = 1$, so one of these irreducible subrepresentations is the trivial representation $\mathbb{1}$, whose degree is $\deg \mathbb{1} = 1$; therefore the other irreducible subrepresentation has degree $\deg \phi - \deg \mathbb{1} = 6 - 1 = 5$.

5. Given a character χ, let us write ρ_χ for the representation of character χ from now on (this is legitimate since χ determines ρ_χ up to isomorphism).

   We compute that $(\psi \mid \psi) = 2$, so that ρ_ψ decomposes into the direct sum of two non-isomorphic subrepresentations by the same argument as in the previous question. Using the fact that the degree 5 irreducible character found in the previous question is $\xi = \phi - \mathbb{1}$, we also observe that $(\psi \mid \xi) = 1$, so one of the irreducible subrepresentations of ρ_ψ is ρ_ξ; therefore the decomposition of ρ_ψ is $\rho_\xi \oplus \rho_\eta$ where $\eta = \psi - \xi = \psi - \phi + \mathbb{1}$ is an irreducible character of degree $\deg \psi - \deg \xi = 21 - 5 = 16$.

   The formula $\eta = \psi - \phi + \mathbb{1}$ allows us to write down the values of η:

Conj. class	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
η	16	0	-2	0	0	0	1	0	0	0	-2

   Finally, the projection onto the isotypical component of η is

$$\begin{aligned} \frac{\deg \eta}{\#G} \sum_{g \in G} \overline{\eta(g)} e_g &= \frac{16}{720} \sum_{j=1}^{11} \left(\overline{\eta(C_j)} \sum_{g \in C_j} e_g \right) \\ &= \frac{1}{45} \left(16 \sum_{g \in C_1} e_g - 2 \sum_{g \in C_3} e_g + 1 \sum_{g \in C_7} e_g - 2 \sum_{g \in C_{11}} e_g \right) \\ &= \frac{16}{45} e_{1_G} - \frac{2}{45} \sum_{g \in C_3} e_g + \frac{1}{45} \sum_{g \in C_7} e_g - \frac{2}{45} \sum_{g \in C_{11}} e_g. \end{aligned}$$

6. We know that the character of the representation $\operatorname{Hom}(\rho_\xi, \rho_\xi)$ is $\xi\overline{\xi} = \xi^2$ as $\xi = \phi - \mathbb{1}$ is real-valued. This character has degree $\deg \xi^2 = \xi^2(1_G) = \xi(1_G)^2 = 5^2 = 25$ (alternatively, the underlying space is the space of linear

transformations from the 5-dimensional underlying space of ρ_ξ to itself, which is isomorphic to the space of 5×5 matrices and has thus dimension 5^2). If this representation decomposes as $\bigoplus_{i=1}^k \rho_i^{\oplus n_i}$ with the ρ_i irreducible and pairwise non-isomorphic, then $\displaystyle\sum_{i=1}^k n_i^2 = (\xi^2 \mid \xi^2) = 4$, so that either $k = 4$ and $n_i = 1$ for all i, or $k = 1$ and $n_1 = 2$. However, the latter case is impossible, as we would have $25 = \deg \xi^2 = \deg \rho_1^{\oplus 2} = 2 \deg \rho_1$, which is absurd since 25 is odd.

This is the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* recommend you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *Decomposition of permutation representations*

Let G be a finite group acting on a finite set X having at least two elements, and let $g \in G$, $x \in X$. We make the following definitions:

- The *orbit* of x is $G \cdot x = \{h \cdot x, \ h \in G\} \subseteq X$.
- The *fixed set* of g is $\operatorname{Fix} g = \{x \in X \mid g \cdot x = x\} \subseteq X$.

Observe that the orbits form a partition (= disjoint union) of X. In this exercise, we *admit* Burnside's formula, which states that the number of orbits is

$$\frac{1}{\#G} \sum_{g \in G} \# \operatorname{Fix} g.$$

We say that the action of G on X is *transitive* if there is only one orbit, i.e. if for all $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$. We say that it is *doubly transitive* if for all $x, x', y, y' \in X$ such that $x \neq x'$ and $y \neq y'$, there exists $g \in G$ which achieves simultaneously $g \cdot x = y$ and $g \cdot x' = y'$ (the conditions $x \neq x'$ and $y \neq y'$ are there because this obviously would not be possible if $x = x'$ but $y \neq y'$).

Finally, let π be the permutation representation corresponding to $G \circlearrowright X$. We *admit* (cf. the first lecture of chapter 4) that its character is given by $\chi_\pi(g) = \# \operatorname{Fix} g$ for all $g \in G$.

1. Prove that there exists a representation ρ of G such that $\pi \simeq \mathbb{1} \oplus \rho$.

2. Prove that ρ has no subrepresentation isomorphic to $\mathbb{1}$ iff. the action of G on X is transitive.

3. Suppose that the action of G on X is transitive. Prove that ρ is irreducible iff. the action of G on X is actually doubly transitive.

   Hint: It may help to define an action of G on the set $X \times X$ of (ordered) pairs of elements of X by the formula $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Note that this action cannot be transitive, since the diagonal

   $$\Delta = \{(x, x) \mid x \in X\} \subsetneq X \times X$$

   is clearly stable under G.

Solution 2

1. We have

   $$(\chi_\pi \mid \mathbb{1}) = \frac{1}{\#G} \sum_{g \in G} \chi_\pi(g) \overline{\mathbb{1}(g)} = \frac{1}{\#G} \sum_{g \in G} \# \operatorname{Fix} g > 0,$$

   so π contains at least one copy of $\mathbb{1}$. The existence of ρ follows from Maschke.

   Remark: Since G permutes the points of X, the vector $\sum_{x \in X} e_x \in \mathbb{C}[X]$ is clearly fixed by every element of G. It therefore spans a one-dimensional subspace of $\mathbb{C}[X]$ which is a copy of $\mathbb{1}$. Pointing this out was anohter way to answer this question.

2. Let χ_ρ be the character of ρ; it is given by $\chi_\rho = \chi_\pi - \mathbb{1}$. Thus

$$(\chi_\rho \,|\, \mathbb{1}) = (\chi_\pi - \mathbb{1} \,|\, \mathbb{1}) = (\chi_\pi \,|\, \mathbb{1}) - (\mathbb{1} \,|\, \mathbb{1}) = -1 + \frac{1}{\# G} \sum_{g \in G} \# \operatorname{Fix} g$$

by the previous question and because $\mathbb{1}$ is irreducible (degree 1). By the first question, this is 1 less than the number of orbits. We conculde that if the action is transitive, then there is only one orbit, so $(\chi_\rho \,|\, \mathbb{1}) = 0$, so ρ does not contain any copy of $\mathbb{1}$; and conversely, if the action is not transitive, then there are at least two orbits, so $(\chi_\rho \,|\, \mathbb{1}) > 0$, so ρ contains at least a copy of $\mathbb{1}$.

Remark: For each orbit $o \subseteq X$, the vector $f_o = \sum_{x \in o} e_x \in \mathbb{C}[X]$ is fixed by every element of G, since o is closed under the action of G by definition of an orbit. We therefore get one copy of $\mathbb{1}$ in π per orbit o; this is another way to answer one direction of the question. The converse direction can also be attacked with the same idea: each copy of $\mathbb{1}$ is a one-dimensional space spanned by a vector which is fixed by every element of G, so we have a bijection between copies of $\mathbb{1}$ in π and nonzero fixed vectors up to scaling. Since an element $g \in G$ turns a generic vector $\sum_{x \in X} \lambda_x e_x \in \mathbb{C}[X]$ into $\sum_{x \in X} \lambda_x e_{g \cdot x} = \sum_{x \in X} \lambda_{g^{-1} \cdot x} e_x$, such a vector is fixed by g iff. $\lambda_{g^{-1} \cdot x} = \lambda_x$ for all $x \in X$. As a result, this vector is fixed by every element of G if and only if λ_x is constant on each orbit, which is equivalent to this vector being a linear combination of the f_o. Therefore the maximal number of copies of $\mathbb{1}$ in a direct sum decomposition π is the number of orbits (or, if you prefer, the isotypical component for $\mathbb{1}$ of π is exactly the span of the f_o), so the multiplicity of $\mathbb{1}$ in π is the number of orbits. This is essentially the same idea as the determination of the centre of the group algebra $\mathbb{C}[G]$ in chapter 4.

3. First of all, ρ is irreducible iff. $(\chi_\rho \,|\, \chi_\rho) = 1$.

   By sesquilinearity of the dot product,

$$\begin{aligned}(\chi_\rho \,|\, \chi_\rho) &= (\chi_\pi - \mathbb{1} \,|\, \chi_\pi - \mathbb{1}) = (\chi_\pi \,|\, \chi_\pi) - (\chi_\pi \,|\, \mathbb{1}) - (\mathbb{1} \,|\, \chi_\pi) + (\mathbb{1} \,|\, \mathbb{1}) \\ &= \frac{1}{\# G} \sum_{g \in G} |\# \operatorname{Fix} g|^2 - 2(\chi_\pi \,|\, \mathbb{1}) + 1 = -1 + \frac{1}{\# G} \sum_{g \in G} (\# \operatorname{Fix} g)^2\end{aligned}$$

   since $(\mathbb{1} \,|\, \chi_\pi) = \overline{(\chi_\pi \,|\, \mathbb{1})} = \overline{1} = 1$ as G acts transitively on X, and as $(\mathbb{1} \,|\, \mathbb{1}) = 1$ by irreducibility of $\mathbb{1}$.

   Consider now the action of G on $X \times X$ suggested in the hint. An element $g \in G$ fixes a pair $(x, y) \in X \times X$ iff. g fixes both x and y, whence $\operatorname{Fix}_{X \times X} g = \operatorname{Fix}_X g \times \operatorname{Fix}_X g$. Therefore, the value at g of the character ψ of the permutation representation attached to $G \circlearrowright X \times X$ is

$$\psi(g) = \# \operatorname{Fix}_{X \times X} g = \#(\operatorname{Fix}_X g \times \operatorname{Fix}_X g) = \chi_\pi(g)^2;$$

   therefore $(\chi_\rho \,|\, \chi_\rho)$ is the number of orbits of G on $X \times X$ minus 1. In conclusion, if G acts doubly transitively, then there are exactly two orbits on $X \times X$, namely the diagonal and its complement, so $(\chi_\rho \,|\, \chi_\rho) = 1$ and ρ is irreducible; conversely, if G does not act doubly transitively, then there are more than two orbits on $X \times X$, so $(\chi_\rho \,|\, \chi_\rho) > 1$ and ρ is not irreducible.

Remark: This implies in particular that for all $n \geq 2$ (resp. $n \geq 4$), the symmetric group S_n (resp. the alternating group A_n) has an easy-to-compute irreducible character of degree $n-1$.

Unlike the previous questions, I do not think that there is a way to solve this question without relying on inner products.

For the sake of completeness, here is a proof of Burnside's formula: Observe that for each $x \in X$, the map

$$\begin{array}{ccc} G & \longrightarrow & G \cdot x \\ g & \longmapsto & g \cdot x \end{array}$$

is $\#S_x$-to-one, so that $\#G \cdot x = \#G/\#S_x$ (orbit-stabiliser formula). Consider the function

$$\begin{array}{rcl} f : G \times X & \longrightarrow & \{0,1\} \\ (g,x) & \longmapsto & \left\{ \begin{array}{l} 1, \ \textit{if } g \cdot x = x \\ 0, \ \textit{if } g \cdot x \neq x, \end{array} \right. \end{array}$$

then we have that

$$\frac{1}{\#G} \sum_{g \in G} \# \operatorname{Fix} g = \frac{1}{\#G} \sum_{g \in G} \sum_{x \in X} f(g,x) = \frac{1}{\#G} \sum_{x \in X} \sum_{g \in G} f(g,x) = \frac{1}{\#G} \sum_{x \in X} \#S_x = \sum_{x \in X} \frac{\#S_x}{\#G}$$

$$= \sum_{x \in X} \frac{1}{\#(G \cdot x)} = \sum_{o \ \textit{orbit}} \sum_{x \in o} \frac{1}{\#(G \cdot x)} = \sum_{o \ \textit{orbit}} \sum_{x \in o} \frac{1}{\#o} = \sum_{o \ \textit{orbit}} \#o \frac{1}{\#o} = \sum_{o \ \textit{orbit}} 1$$

is the number of orbits. We can also get an alternative, group-representation proof of this formula by combining the two approaches to the second question!